On Upper and Lower Bounded Spaces

DECLARATION

The work provided in this thesis, unless otherwise referenced, is the researcher's own work, and has not been submitted elsewhere for any other degree or qualification.

Student's name:

Signature:

Date:

اسم الطالب: لبنى توفيق محمد الأستاذ
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By
LubnaT.Elostath

Supervised By
Dr.HishamB.Mahdi

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On Upper and Lower Bounded Spaces

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- د. هشام بشير مهدي
- د. أحمد عبد الرؤوف المبوج
- د. محمد جمال عقيلان

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أ.د. فؤاد علي العجز
قُلْ هَلْ يَسْتَوِي الَّذِينَ يَعْلَمُونَ
وَالَّذِينَ لَا يَعْلَمُونَ
[اليوم: 9]
To

My parents
My Husband
My children
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Abstract

In this thesis, we study a special types of Alexandroff spaces, called upper bounded and lower bounded $T_0$ A-spaces. We prove that UB $T_0$ A-spaces and g-Artinian $T_0$ A-spaces are incomparable and containing Artinian $T_0$ A-spaces in the regain of intersection. We introduce A-dcts as a proper subclass of UB $T_0$ A-space. We give a characterization of g-closed set in LB $T_0$ A-spaces, and we find new relations between various kinds of g-closed sets in UB $T_0$ A-space. The main purpose of this thesis is generalizing many results, which are proved previously on Artinian $T_0$ A-space, to UB $T_0$ A-space.
Introduction

The topology as a branch of mathematics has many applications in practical life. Nowadays, topology plays a big role in computer science, and image processing. The kind of topological space that served this topic called digital topology which is a class of Alexandroff space. Alexandroff spaces were first introduced by P.Alexandroff in 1937 [42] under the name of Discrete Raume (discrete space). In general, a topological space \((X, \tau)\) satisfies the property "a finite intersection of open sets is open and an arbitrary intersection of open sets need not be open". An Alexandroff space (or minimal neighborhood space) \(X\) is a space in which the arbitrary intersection of open sets is open.

We focus on Alexandroff spaces that satisfy the separation axiom \(T_0\). We use their specialization orders in proofs to illustrate the results and the concepts. The importance of this study comes from the fact that we can characterize topological properties just by looking at the corresponding poset.

A \(T_0\) Alexandroff space where its corresponding poset satisfies the ACC (resp. DCC) is considered in [12] and called Artinian (resp. Noetherian) \(T_0\) A-space. In [5] Rose and et al. introduce new two spaces, called Upper bounded (resp. Lower bounded) \(T_0\) A-space as, a class of \(T_0\) Alexandroff space which is larger than Artinian (resp. Noetherian) \(T_0\) A-space. The aim of this thesis is to study Upper bounded and Lower bounded \(T_0\) A-space.

This thesis consists of four chapters. In Chapter one, we give preliminaries that will be used in the remainder of the thesis. Section one contains basic definitions and theorems.
In the second section, we give an introduction of $T_\alpha$ Alexandroff spaces and their related definitions. In the third section, we analyze defections and the results of Artinian $T_\alpha$ Alexandroff spaces as in caption to generalizing them.

In Chapter two, we study Upper bounded that satisfying every chain of points in the corresponding poset $(X, \leq)$ is bounded above. And Lower bounded (LB)$T_\alpha$ Alexandroff spaces that satisfying every chain of points is bounded below. It includes four sections. In the first section we talk about the definition of UB $T_\alpha$ A-spaces and give some examples for this spaces and proved some main results and characterize some of generalized open set on UB $T_\alpha$ A-space. In second section we study definition and some theorems about Generalized Artinian $T_0$ space. In third section we study the relationship between Generalized Artinian $T_0$ space, and UB $T_\alpha$ A-space. we introduce a new space called A-dcts space. In this section also we construct figure shows the relation between three spaces UB $T_\alpha$ A-space, Artinian $T_\alpha$ A-space and A-dcts space. In fourth section we talk about $\tau_a$ on UB $T_\alpha$ A-space.

Chapter three talking about various kinds of g-closed sets. This topic has been studied extensively in recent years by many topologists. More precisely, they studied several new properties of these sets. Many of these new properties are separation axioms which are weaker than $T_1$ and used in computer science, and digital topology.

In the first section, we firstly give definitions of many kinds of g-closed sets and their general relations. We study them in UB $T_\alpha$ A-space and we characterize these sets in the corresponding poset. In the second section we study $cl^*$ and $\tau^*$ on LB $T_\alpha$ A-space. We characterize g-closed set and $cl^*$ on LB $T_\alpha$ A-space. We prove that $\tau^*$ is again $T_\alpha$ A-space, then we give a description of the specialization order $\leq^*$. In the third section, we study the relation between $\alpha$-open sets in $\tau$ and g-closed set in the dual topology $\tau^d$. We prove that a set $A$ is $\alpha$-open in $\tau$ if and only if $A$ is g-closed set in $\tau^d$.

Separation axioms are studied in Chapter four, which is consisting of three sections.
In the first section, we deal with lower septation axioms such as $T_{\frac{1}{2}}, T_{\frac{1}{3}},$ and $T_{\frac{1}{4}}$. In the second section, we study new separation axioms depending on g-closed sets. This section includes a comprehensive theorem of many equivalent points. In the third section, we study the submaximality, sg-submaximal, and extremally disconnected properties. We characterize the extremally disconnected property in UB $T_0$ A-spaces.

Finally, we give a conclusion page contains a summary of our work and what we are looking for to make in the future to improve this work and develop it.
Chapter 1

Preliminaries

1.1 Introduction to Topological Spaces

We are going to introduce, first a structure on a set $X$ by using a collection of subsets of $X$. For this, we shall soon set forth a set of axioms which a collection of subsets must obey in order to fall within the circle of our studies. Any collection of subsets of $X$ satisfying these axioms will be called a topology of $X$.

**Definition 1.1.1.** [49] Let $X$ be a nonempty set. Then a topology $\tau$ on $X$ is a collection of subsets of $X$ satisfying the following axioms:

(1) $X$ and $\emptyset$ belong to $\tau$.

(2) Any finite intersection of elements of $\tau$ belongs to $\tau$.

(3) If $\{U_\alpha : \alpha \in \Delta\}$ is an indexed family of sets, each of which belongs to $\tau$, then $\bigcup_{\alpha \in \Delta} U_\alpha$ belongs to $\tau$.

A topological space $(X, \tau)$ is a set $X$ together with a topology $\tau$ on $X$. The members of $\tau$ are called open subsets of $X$. The shortened notation the "topological space $X$" will also be used when no confusion arises concerning the topology on $X$. 

4
Definition 1.1.2. If $X$ is a topological space and $E \subseteq X$, we say $E$ is closed if $X - E$ is open.

Theorem 1.1.3. If $\mathcal{F}$ is the collection of all closed sets in a topological space $X$, then

1. $X$ and $\emptyset$ belong to $\mathcal{F}$.
2. any intersection of members of $\mathcal{F}$ belongs to $\mathcal{F}$.
3. any finite union of members of $\mathcal{F}$ belongs to $\mathcal{F}$.

Definition 1.1.4. If $X$ is a topological space and $E \subseteq X$, the closure of $E$ in $X$ is denoted by $CL(E)$ (briefly $\overline{E}$) and it is given by the set,

$$
\overline{E} = Cl(E) = \bigcap \{K : K \text{ is a closed set containing } E\}.
$$

When confusion is possible as to what space the closure is to be taken in, we will write $Cl_X(E)$.

Remark 1.1.5. By part (2) of Theorem 1.1.3, the intersection of all closed sets containing $E$ is closed, that is; $cl(E)$ (or $\overline{E}$) is closed and it is the smallest closed set containing $E$.

Where the precise meaning of smallest is that $K_1 \leq K_2$ if and only if $K_1 \subseteq K_2$. Moreover, a subset $A$ of a space $X$ is closed if and only if $\overline{A} = A$.

Definition 1.1.6. The Kuratowski closure axioms are a set of axioms which can be used to define a topological structure on a set. They are equivalent to the more commonly used open set definition.

A closure space $(X, cl)$ is a set $X$ together with a function $Cl : P(X) \rightarrow P(X)$. The function is called the closure operator or Kuratowski closure if it satisfy the following properties for all $A, B$ in $P(X)$:

1. $A \subseteq cl(A)$.
(2) \( cl(A) = cl(cl(A)) \).

(3) \( cl(A \cup B) = cl(A) \cup cl(B) \).

(4) \( cl(\emptyset) = \emptyset \).

We can construct the topology \( \tau \) on \( X \) by taking all subsets that satisfy \( A \subseteq X \), \( cl(A^c) = A^c \).

**Definitions 1.1.7.** Let \((X, \tau)\) be a topological space and \( A \subseteq X \).

(1) A point \( x \in X \) is an interior point of \( A \) if there exists an open set \( U \) containing \( x \) such that \( U \subseteq A \). The set of all interior points of \( A \) is called the interior of \( A \) and is denoted by \( A^o \) (or \( Int(A) \)).

(2) A point \( x \in X \) is an exterior point of \( A \) if there exists an open set \( U \) containing \( x \) such that \( U \cap A = \emptyset \). The set of all exterior points of \( A \) is called the exterior of \( A \) and is denoted by \( Ext(A) = (X - A)^o \).

(3) A point \( x \in X \) is a boundary point of a set \( A \) if every open set in \( X \) containing \( x \) contains at least one point of \( A \), and at least one point of \( X - A \). The set of boundary points of \( A \) is called the boundary of \( A \) and is denoted by \( Bd(A) \) (or \( Fr_X(A) \)).

**Theorem 1.1.8.** Let \( A \) and \( B \) be subsets of the topological space \((X, \tau)\). Then

(1) \( \emptyset = \emptyset \), \( \emptyset^o = \emptyset \).

(2) \( \overline{A \cup B} = \overline{A} \cup \overline{B} \), \( (A \cap B)^o = A^o \cap B^o \).

(3) \( \overline{A} = \overline{A} \), \( (A^o)^o = A^o \).

(4) If \( A \subseteq B \), then \( \overline{A} \subseteq \overline{B} \), and \( A^o \subseteq B^o \).
Definitions 1.1.9. Let $X$ be a topological space and $A \subseteq X$. Then

1. $A$ is dense if $\overline{A} = X$.
2. $A$ is nowhere dense if the interior of the closure of $A$ is empty; that is, $(\overline{A}^o = \emptyset)$.
3. codense if the interior of $A$ equals empty; that is, $(A^o = \emptyset)$.

Definition 1.1.10. [49] Let $(X, \tau)$ be a topological space. A base for $\tau$ is a subcollection $\mathcal{B}$ of $\tau$ satisfies that if $U \in \tau$, then $U$ is a union of members of $\mathcal{B}$. Therefore, a base for $\tau$ is completely determines $\tau$ by arbitrary unions of members of $\mathcal{B}$. Also we see that any topology is a base for itself. So, any topology has at least one base.

Theorem 1.1.11. $\mathcal{B}$ is a base for a topology on $X$ if and only if the following hold:

1. $X = \bigcup_{B \in \mathcal{B}} B$, and
2. whenever $B_1, B_2 \in \mathcal{B}$ with $p \in B_1 \cap B_2$, there is some $B_3 \in \mathcal{B}$ with $p \in B_3 \subseteq B_1 \cap B_2$.

Definitions 1.1.12. If $(X, \tau)$ is a topological space and $x \in X$. Then

1. a neighborhood (abbreviated nhood) of $x$ is a set $U$ which contains an open set $V$ containing $x$. Thus, evidently, $U$ is a nhood of $x$ if and only if $x \in U^o$. The collection $\mathcal{U}_x$ of all nhoods of $x$ is called the nhood system of $x$.

2. A nhood base at $x$ is a sub-collection $\mathcal{B}_x$ taken from the nhood system $\mathcal{U}_x$ having the property that each $U \in \mathcal{U}_x$ contains some $V \in \mathcal{B}_x$.

Definition 1.1.13. If $(X, \tau)$ is a topological space and $A \subseteq X$, the collection $\tau_A = \{G \cap A : G \in \tau\}$ is a topology on $A$, called the relative topology of $A$. This topological space is denoted by $(A, \tau_A)$. The fact that a subset of $X$ is being given this topology is signified by referring to it as a subspace of $X$. When a topology is used on a subset of a topological space without explicitly being described, it is assumed to be the relative topology.
Definitions 1.1.14. Let \((X, \tau)\) be a topological space. Then

1. The space \((X, \tau)\) is a \(T_0\) space if for each pair of distinct points \(x, y \in X\), there is either an open set containing \(x\) but not \(y\) or an open set containing \(y\) but not \(x\).

2. The space \((X, \tau)\) is a \(T_1\) space if for each two distinct points \(x, y \in X\), there exist two open sets \(U\) and \(V\) in \(X\) such that \(x \in U, y \notin U\) and \(x \in V, y \notin V\).

3. The space \((X, \tau)\) is called a \(T_2\) space (or a Hausdorff space) if for each two of distinct points \(x, y \in X\), there exist two disjoint open sets \(U\) and \(V\) in \(X\) such that \(x \in U\), and \(y \in V\).

Remark 1.1.15. Every every \(T_2\)-space is a \(T_1\)-space, and \(T_1\)-space is a \(T_0\)-space.

1.2 \(T_0\)-Alexandroff Spaces

In this section, we study a class of topological spaces called \(T_0\) Alexandroff spaces. In general topological space \((X, \tau)\) satisfies the property that a finite intersection of open sets is open. In addition if an arbitrary intersection of open sets is open, the space is called Alexandroff space. In fact, this property doesn’t hold in all topological spaces. For example, in the standard topology on \(\mathbb{R}\), \(\bigcap_{n \in \mathbb{N}} \left(\frac{1}{n}, \frac{1}{n}\right) = \{0\}\) which is not open in \(\mathbb{R}\). This kind of spaces was first studied in 1937 by P. Alexandroff [42] under the name of Discrete Raïl me (discrete space).The name is not valid now, since a discrete space is a space where the singletons are open.

Notation 1.2.1. [42] Using De Morgan’s low, a space is Alexandroff if and only if an arbitrary union of closed sets is closed.

Lemma 1.2.2. A topological space \((X, \tau)\) is Alexandroff space if and only if each point of \(X\) is contained in a minimal open set.
Proof. \((\Rightarrow)\) Suppose that arbitrary intersection of open sets is open, and for \(x \in X\), let \(\mathcal{B}_x = \{U \in \tau : x \in U\}\). Let \(V(x) = \bigcap_{U \in \mathcal{B}_x} U \in \tau\). Then, \(V(x) \subseteq U\) for all \(U \in \mathcal{B}_x\). Thus, \(V(x)\) is the minimal open set of \(x\).

\((\Leftarrow)\) Suppose that for each \(x \in X\), there is a minimal open set \(V(x)\) containing \(x\). Let \(\{U_\alpha : \alpha \in \Delta\}\) be a collection of open sets in \(X\). Let \(y = \bigcap_{\alpha \in \Delta} U_\alpha\), then, \(y \in U_\alpha\), for all \(\alpha \in \Delta\). So, \(y \in V(y) \subseteq U_\alpha\) for all \(\alpha \in \Delta\). Therefore, \(V(y) \subseteq \bigcap_{\alpha \in \Delta} U_\alpha\). Thus the intersection is open, and hence \(X\) is Alexandroff space.

Note that a collection \(\mathcal{B}_x = \{V(x)\}\) of one set is a minimal nhood base of \(x\).

**Example 1.2.3.** Any finite space is Alexandroff space.

To see this, note that any finite space has finite number of subsets, and consequentially finite number of open sets. So, arbitrary intersections of these finite number of open sets indeed open.

**Example 1.2.4.** It is easy to check that the discrete topology on any non-empty set is Alexandroff.

**Proposition 1.2.5.** [34] An Alexandroff topology \((X, \tau)\) is a \(T_0\)-space if and only if \(x \neq y\) in \(X\) implies \(V(x) \neq V(y)\).

Alexandroff spaces which satisfying the separation axiom \(T_0\) are related to partial ordered sets in 1-1 and onto way.

**Definition 1.2.6.** A relation \(\leq\) on a set \(P\) is called partial order (simply order) on \(P\) if for every \(a, b, c \in P\), we have that:

1. \(a \leq a\) (reflexivity).
2. \(a \leq b\), and \(b \leq a\) implies \(a = b\) (anti-symmetry).
3. \(a \leq b\), and \(b \leq c\) implies \(a \leq c\) (transitivity).
The set \( P \) together with a partial order \( \leq \) is called a **partially ordered set** (briefly a poset). Note that the set \( P \) together with a partial order \( \leq \) is called a **totally ordered set** if for any \( a, b \) in \( P \), either \( a \leq b \) or \( b \leq a \).

**Example 1.2.7.** The set \( \mathbb{N} \) of natural numbers forms a poset under the usual order \( \leq \). Similarly, the set of integers \( \mathbb{Z} \), rationales \( \mathbb{Q} \), and real numbers \( \mathbb{R} \) under the usual order \( \leq \) form posets.

**Example 1.2.8.** Let \( X \) be a set. The set \( \mathcal{P}(X) \) of all subsets of \( X \) under the inclusion \( \subseteq \) forms a poset.

**Diagrammatical representation of a poset**

A poset may be represented by the help of a diagram. To draw the diagram of a poset, we represent each element by a small circle or dot, and any two comparable elements are joined by lines in such a way that if \( a \leq b \) then \( a \) lies below \( b \) in the diagram. Non-comparable elements are not joined. Thus, there will not be any horizontal lines in the diagram of a poset.

**Example 1.2.9.** If \( X = \{1, 2, 3\} \), then the poset \( \mathcal{P}(X) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\} \) under \( \subseteq \) relation is represented by the diagram below:

**Example 1.2.10.** The set \( \{2, 3, 4, 6\} \) under division relation forms a poset with a diagram as below:

**Definition 1.2.11.**

1. Let \( a, b \) be two elements in an ordered set. We say that \( a \) **precedes** \( b \) (or \( a \) is **smaller** than \( b \)) if \( a \leq b \). In this case we say that \( b \) **follows** \( a \) or **larger** than \( a \). Furthermore, we write \( a < b \) if \( a \leq b \), and \( a \neq b \).

2. A subset \( C \) of a poset \( P \) is a **chain** if any two elements of \( C \) are comparable. Thus, if \( C \) is a chain, and \( x, y \in C \) then either \( x \leq y \) or \( y \leq x \).
If a relation $R$ on a set $A$ defines a partial order, then the inverse relation $R^{-1}$ is also a partial order; it is called the inverse order (or the dual order).

**Definition 1.2.12.** Let $A$ be a subset of a poset $X$. Then, the order in $X$ induces an order in $A$ in the following natural way: If $a, b \in A$, then $a \leq b$ as elements in $A$ if and only if $a \leq b$ as elements in $X$. Equivalently, if $R$ is a partial order in $X$, then the relation $R_A = R \cap (A \times A)$ - called the restriction of $R$ to $A$ is a partial order in $A$. The ordered set $(A, R_A)$ is called partially ordered subset of the ordered set $(X, R)$. It should be noted that a chain $C$ as an ordered subset of $X$ is totally ordered. Clearly, if $X$ is totally ordered, then every subset of $X$ will be totally ordered.

Recall that If $X$ be an ordered set and $a, x \in X$. An element $a$ is called maximal if whenever $a \leq x$, then $x = a$; that is, if no element in $X$ follows $a$ except $a$. Similarly, an
element \( a \in X \) is called \textit{minimal} if whenever \( x \leq a \) then \( x = a \), that is; if no element in \( X \) precedes \( a \) except \( a \) itself. We denote the set of all maximal (resp. minimal) elements of an ordered set \( X \) by \( M \) (resp. \( m \)). If \( A \) is any subset of \( X \), we write \( M(A) \) (resp. \( m(A) \)) to denote the set of maximal (resp. minimal) elements of \( A \) under the induced order. If there is an element \( \top \in X \) such that \( x \leq \top \) for all \( x \in X \), then \( \top \) is called \textit{maximum} (or \textit{largest}, or \textit{top}) element. On the other hand, if there is an element \( \bot \in X \) such that \( \bot \leq x \) for all \( x \in X \), then \( \bot \) is called \textit{minimum} (or \textit{smallest}, or \textit{bottom}) element. It should be noted that the set \( M \) of all maximal elements of a poset \( X \) may be an empty set.

\textbf{Definitions 1.2.13.} Let \( A \) be a subset of a poset \( X \).

(a) An element \( u \in X \) is \textit{an upper bound of} \( A \) if \( x \leq u \ \forall \ x \in A \). The \textit{least upper bound} (or the \textit{supremum}) of \( A \) - denoted by \( \sup A \) (or \( \lor A \)) is an upper bound that precedes each upper bound of \( A \). An element \( l \in X \) is \textit{a lower bound of} \( A \) if \( l \leq x \ \forall x \in A \). The \textit{greatest lower bound} (or the \textit{infimum}) of \( A \) - denoted by \( \inf A \) (or \( \land A \)) - is a lower bound of \( A \) that follows each lower bound of \( A \). For a subset \( A \), \( \sup A \), and \( \inf A \) may not exist.

(b) \( A \) is said to be \textit{bounded above} if it has an upper bound, and \textit{bounded below} if it has a lower bound. If \( A \) has both upper, and lower bounds, then it is \textit{bounded}.

\textbf{Example 1.2.14.} Let \( X = \{a, b, c, d, e, f, g\} \) be a set ordered by the following diagram:

Let \( B = \{c, d, e\} \). The elements \( a, b, \) and \( c \) are upper bounds of \( B \), and \( f \) is the only lower bound of \( B \). The element \( g \) is not lower bound of \( B \), since \( g \) doesn’t precede \( d \). Moreover, \( \inf B = f \notin B, \) while \( \sup B = c \in B \)

\textbf{Definition 1.2.15.} A subset \( O \) of a poset \( P \) is a \textit{down set} (or a \textit{lower set}) if, whenever \( x \in O \), and \( y \leq x \), then we have \( y \in O \). On the other hand, a subset \( U \) of a poset \( P \) is an
up set (or an upper set) if, whenever \( x \in U \), and \( x \leq y \), we have \( y \in U \). For \( x \in P \), we define the down set \( \downarrow x = \{ y \in P : y \leq x \} \), and the up set \( \uparrow x = \{ y \in P : x \leq y \} \). For a set \( B \subseteq P \), we define the down set \( \downarrow B = \{ y \in P : y \leq x \text{ for some } x \in B \} \), and the up set \( \uparrow B = \{ y \in P : x \leq y \text{ for some } x \in B \} \). In this case, \( \uparrow x = \uparrow \{ x \} \), and \( \downarrow x = \downarrow \{ x \} \).

If \( A \) is a down set of \( P \), then the complement \( A^c \) is an up set, since if \( a \in A^c \), and \( a \leq b \), by transitive, and definition of \( A^c \), we have \( b \in A^c \).

Example 1.2.16. Let \( \mathbb{R} \) be the set of all real numbers with its usual order. Let \( A, B \subseteq \mathbb{R} \) be such that \( A = [3, \infty) = \uparrow 3 \), and \( B = (-\infty, -1] = \downarrow -1 \), then \( A \) is an up set, and \( B \) is a down set.

Definition 1.2.17. A poset \( P \) satisfies the ascending chain condition (ACC), if for any increasing sequence \( x_1 \subseteq x_2 \subseteq ... \subseteq x_n \subseteq ... \) in \( P \), there exists \( k \in \mathbb{N} \) such that \( x_k = x_{k+1} = ... \). The dual of (ACC) is the descending chain condition (DCC). If a poset satisfies both ACC, and DCC, we say \( P \) is of finite chain condition (FCC).

Example 1.2.18. A collection of subsets of a finite set \( X \) when ordered by inclusion satisfies the ACC, and DCC, so each finite poset is of finite chain condition (FCC).

Definition 1.2.19. For a topological space \((X, \tau)\) we define the specialization pre-order \( \leq_{\tau} \) on \( X \) as follows for any \( x, y \in X \), \( x \leq_{\tau} y \) if and only if \( x \in \overline{y} \).
This gives a pre-order on \( X \) which is reflexive, and transitive. It is anti-symmetric, and hence a partial order if and only if \( X \) is \( T_0 \). On the other hand, if \((X, \leq)\) is a poset, then the collection \( \mathcal{B} = \{ \uparrow x : x \in X \} \) is a base for a topology \( \tau_{\leq} \) on \( X \). This topology is \( T_0 \)-Alexandroff space, and \( \mathcal{B} \) is a minimal base for \( \tau_{\leq} \). Moreover, if \((X, \tau)\) a is \( T_0 \)-Alexandroff space, and if \( \leq_\tau \) is it’s specialization order, then the topology \( \tau_{\leq} \) induced by \( \leq_\tau \) is the original topology itself; that is, \( \tau = \tau_{\leq_\tau} \). If \((X, \leq)\) is a poset, and \( \tau_{\leq} \) is the induced \( T_0 \) Alexandroff topology, then the specialization order \( \tau_{\leq} \) is the original order \( \leq \); that is, \( \leq = \leq_{\tau_{\leq}} \).

We are interested in Alexandroff spaces that satisfy the separation axiom \( T_0 \). We use their specialization orders in proofs to illustrate the results and the concepts. The importance of this study comes from the fact that we can characterize topological properties just by looking at its specialization order. For example, if we define a topological space \((X, \tau)\) to be \( T_2 \) if any singleton is either open or closed, then in the \( T_0 \) A-space, \( X \) is if \( T_2 \) each element in the corresponding poset - the space \( X \) after being ordered by the specialization order \( \leq_{\tau} \) is either maximal or minimal; that is, the graph of its corresponding poset contains two rows; the row of the maximal elements, and the row of the minimal elements.

Now, let us go forward to study this class of topological spaces. The following characterizes open sets by the concepts of posets.

**Theorem 1.2.20.** [12] If \((X, \tau(\leq))\) is a \( T_0 \) A-space then a subset \( A \) of \( X \) is open if and only if it is an up set with respect to the specialization order; that is, \( A \) is open if and only if \( A = \uparrow A \). And \( A \) is closed if and only if it is a down set; that is, \( A \) is closed if and only if \( A = \downarrow A \).

**Example 1.2.21.** Let \( X = \{a, b, c, d\} \) with the order \( a \leq b, a \leq c, \) and \( d \leq c \) as shown in the following figure below:

Then the \( T_0 \) Alexandroff topology is \( \tau = \{\emptyset, X, \{a, b, c\}, \{b\}, \{c\}, \{d, c\}, \{b, c, d\}, \{b, c\}\} \).
with the minimal base $\mathcal{B} = \{\{a, b, c\}, \{b\}, \{c\}, \{d, c\}\}$. The set $A = \{a, b, d\}$ is down set which is not up, so it is closed, and not open. Note that $A^c = \{c\} \in \tau$.

**Proposition 1.2.22.** Let $(X, \tau(\leq))$ be a $T_\sigma$-Alexandroff space, and let $A \subseteq X$. Then

(a) $\forall x \in X$, $\overline{\{x\}} = \downarrow x$.

(b) $A^o = \{x \in A : \uparrow x \subseteq A\}$.

(c) $\overline{A} = \bigcup_{x \in A} \downarrow x$.

Note that boundary of a given subset $A$ is the set $\text{bd}(A) = \overline{A} \setminus A^o = \overline{A} \cap \overline{A^c}$, so it is a closed set, $\text{bd}(A) = \bigcup \{\downarrow x : x \in A\} \setminus \{\uparrow x : x \in A\}$, and if $A^o = \emptyset$, then $\text{bd}(A) = \overline{A}$.

**Example 1.2.23.** Let $X = \{a, b, c, d, e, f, r\}$ with order as follow in diagram below:

\[
A = \{a, b, c, d\}, \quad A^o = \{a, c, b, d\}, \quad \overline{A} = \{a, b, c, d, e, f\}, \quad B = \{e, d, a\}, \quad B^o = \{a, d\}, \quad \overline{B} = \{c, f, e, d, a\}.
\]
1.3 Artinian $T_0$ A-Space

In [12] Mahdi and Elatrash introduced Artinian $T_0$ A-space for which every increasing chain in the specialization order is finite. They get specific results on this class such as characterization of externally disconnected, and submaximality. And also they introduced Noetherian $T_0$ A-space, as dual space of the Artinian $T_0$ A-space. So in this section we will study closely this class. Firstly we want to present some definitions that will be used later.

**Definition 1.3.1.** [12] A $T_0$ A-space whose corresponding poset satisfies the ACC is called Artinian $T_0$ A-space. Dually, a $T_0$ A-space whose corresponding poset satisfies the DCC is called Noetherian $T_0$ A-space.

A $T_0$ A-space whose corresponding poset satisfies the ACC and DCC is called $g$-locally finite.

**Notation 1.3.2.** If a space is Artinian $T_0$ A-space, then the dual space with respect to reverse order is Noetherian $T_0$ A-space, and vice versa.

**Definition 1.3.3.** [12] Let $(X, \tau(\leq))$ be a $T_0$ A-space, and let $M$ (resp. $m$) be the set of all maximal (resp. minimal) elements with respect to the corresponding poset. Then for $x \in X$, we define $\hat{x}$, and $\check{x}$ to be the set: $\hat{x} = \uparrow x \cap M$, and $\check{x} = \downarrow x \cap m$.

**Definition 1.3.4.** [12] If $A$ is a subset of a $T_0$ A-space, then we define $M(A)$ (resp. $m(A)$) to be the set of all maximal (resp. minimal) elements of $A$ with respect to the induced order.

**Proposition 1.3.5.** If $(X, \tau(\leq))$ is an Artinian (resp. Noetherian) $T_0$ A-space, then $M \neq \emptyset$ (resp. $m \neq \emptyset$). Moreover, if $A \subseteq X$, then $M(A) \neq \emptyset$ (resp. $m(A) \neq \emptyset$).

**Proof.** From Notation 1.3.2 above, if we prove the proposition on Artinian $T_0$ A-space then by reverse inclusion we can conclude it on the Noetherian $T_0$ A-space. Let $(X, \tau(\leq))$
be an Artinian $T_0$ A-space. Assume to contrary that $M = \emptyset$. Let $x_1 \in X$ then since $x_1 \notin M$, $\exists x_2 \in X$ such that $x_1 < x_2$. Again, since $x_2 \notin M$, $\exists x_3 \in X$ such that $x_2 < x_3$, and so on. We get an infinite increasing seq. $x_1 < x_2 < x_3 < ...$ which contradicts the fact that $(X, \tau(\leq))$ is Artinian $T_0$ A-space. For $A \subseteq X, A \neq \emptyset$, we can use the same argument to show that $M(A) \neq \emptyset$.

**Definition 1.3.6.** Let $X$ be a topological space $(X, \tau)$, and let $A \subseteq X$. Then a point $x \in X$ is called:

1. a *cluster point of* $A$ $(A')$ if $U$ intersects $A \setminus \{x\}, \forall U$ open in $X$.
2. An *isolated point* if $\{x\}$ is open in the subspace $A$.
3. A *pure* if $\{x\}$ is open or closed.
4. A *mixed* if $\{x\}$ is not pure.

The following theorem gives characterizations of some basic topological properties in Artinian $T_0$ A-space.

**Theorem 1.3.7.** [12] Let $(X, \tau_{\leq})$ be an Artinian $T_0$ A-space. Then.

1. $A^0 = \emptyset \iff A \cap M = \emptyset$.
2. $\overline{A} = \bigcup \{ \downarrow x : x \in M(A) \} = \downarrow M(A)$.
3. The subset $A$ is dense if and only if $M \subseteq A$.
4. The subset $A$ is nowhere dense if and only if $M \cap A = \emptyset$.
5. If $|M| = 1$, then any subset is either dense or nowhere dense.
6. $A' = \bigcup \{(\downarrow x) \setminus \{x\} : x \in M(A)\} = (\downarrow M(A)) \setminus M(A)$.
7. Isolated points of the subset $A$ is $M(A)$.
Proof. Items from (1) to (6) proved in [12] I want to prove (7).

Let \((X, \tau(\leq))\) be an Artinian \(T_o\) A-space. Let \(x \in A \subseteq X\) be isolated point, then \(\{x\}\) is open in the subspace \(A\). So \(\{x\}\) is up set in \(A\); that is, \(\uparrow x \cap A = \{x\}\). Hence \(x\) is maximal in \(A\). (Because if not \(\exists y \in A\) such that \(y \geq x\) then \(\uparrow x \cap A \supseteq \{y, x\}\)). That is, \(x \in M(A)\).

On the other hand, let \(x\) be a maximal element in \(A\). Then \(\uparrow x \cap A = \{x\}\). So \(\{x\}\) is open set in the subspace \(A\), and hence \(x\) is isolated point.

\[\square\]

**Definition 1.3.8.** Let \(A\) be a subset of a \(T_o\) A-space \(X\). A set \(A\) is

1. **dense in itself** if \(A\) contains no isolated point.

2. **perfect** if \(A\) is closed, and dense-in-itself.

**Definition 1.3.9.** A space \((X, \tau)\) is called

1. **scattered** if no subset of \(X\) is dense-in-itself.

2. **\(\alpha\)-scattered** if there exists a dense set of isolated points.[4]

Note that a space is scattered if every subspace contains an isolated point.

**Proposition 1.3.10.** [12] Let \((X, \tau(\leq))\) be an Artinian \(T_o\) A-space. Then

1. \(X\) is scattered.

2. \(X\) is \(\alpha\) — scattered.

Proof. (1) Let \((X, \tau(\leq))\) be an Artinian \(T_o\) A-space. Let \(A \subseteq X\) be non empty. Since \(M(A) \neq \emptyset\), then \(A\) is not dense in itself and hence, \(X\) is scattered.

(2) \(X\) is an Artinian, so \(M \neq \emptyset\) which is dense with isolated point. Hence \(X\) is \(\alpha\) — scattered.

\[\square\]
**Definition 1.3.11.** A subset $A$ of a space $(X, \tau)$

1. A *semi-open set* if $A \subseteq \overline{A}$, and a *semi-closed set* if $A^c$ is semi-open, thus $A$ is semi-closed if and only if $\overline{A}^c \subseteq A$. If $A$ is both semi-open and semi-closed then $A$ is called *semi-regular* [7]. The family of all semi-open sets in $X$ is denoted by $SO(X)$.

2. a *preopen set* if $A \subseteq \overline{A}$, and a *preclosed set* if $A^c$ is preopen, thus $A$ is preclosed if and only if $\overline{A}^\circ \subseteq A$. The family of all preopen sets in $X$ is denoted by $PO(X)$.

3. an *\(\alpha\)-open set* if $A \subseteq \overline{A}^\circ$, and an *\(\alpha\)-closed set* if $A^c$ is $\alpha$-open, thus $A$ is $\alpha$-closed if and only if $\overline{A}^\circ \subseteq A$. The family of all $\alpha$-open sets in $X$ is denoted by $\tau_\alpha$.

4. The notation $\text{pcl}(A)$(resp. $\text{scl}(A)$) denoted to the smallest preclosed (resp. semi-closed) subset of $X$ contains $A$.

5. The notation $\text{pint}(A)$(resp. $\text{scl}(A)$) denoted to the largest preopen (resp. semiopen) subset of $X$ is contained in $A$.

In [26], it has been shown that a set is $\alpha$-open if and only if it is semi-open, and preopen.

Njastad [40] proved that $\tau_\alpha$ is a topology on $X$. In general, $SO(X)$, and $PO(X)$ need not be topologies on $X$, and also he showed that $SO(X)$ is a topology if and only if $(X, \tau)$ is extremally disconnected, where the space is *extremally disconnected* if the closure of every open set is open, in this case, $SO(X) = \tau_\alpha$.

**Definition 1.3.12.** A space $(X, \tau(\leq))$ is called

1. *resolvable* if and only if $X = D \cup D^c$ where both $D$, and $D^c$ are dense. A subset $A \subseteq X$ is resolvable if the subspace $(A, \tau_A)$ is resolvable.

2. *irresolvable* if it is not resolvable.

3. *strongly irresolvable* if no nonempty open set is resolvable.
(4) *hereditarily irresolvable*[7] if no nonempty subset is resolvable.

(5) *nodec*[29] if all nowhere dense sets are closed.

(6) *hyperconnected* if every open subset of $X$ is dense, if $X$ is not hyperconnected, then it is hyperdisconnected.

(7) *submaximal*[19] if each dense subset is open.

The following implications satisfy in any topological space:

\[
\text{submaximal} \Rightarrow \text{hereditarily irresolvable} \Rightarrow \text{strongly irresolvable} \Rightarrow \text{irresolvable}.
\]

Let $(X, \tau(\leq))$ be an Artinian $T_0$ A-space. The set $M$ belongs to all dense subsets of $X$, so no disjoint dense subsets exist in $X$, and hence $X$ is surely irresolvable. Moreover, it is strongly irresolvable[12]. We will use the following two theorems.

**Theorem 1.3.13.** [1] For a space $(X, \tau)$, the following are equivalent:

1. $(X, \tau)$ contains an open, dense, and hereditarily irresolvable subspace.
2. Every open ultrafilter on $X$ is a base for an ultrafilter on $X$.
3. $X$ is strongly irresolvable.
4. For each dense subset $D$ of $X$, $D^\circ$ is dense.
5. For $A \subseteq X$ where $A^\circ = \emptyset$, $A$ is nowhere dense.

**Theorem 1.3.14.** [36] For a space $(X, \tau)$, the following are equivalent:

1. $(X, \tau)$ contains an open, dense, and hereditarily irresolvable subspace.
2. $PO(X) \subseteq SO(X)$.
3. $PO(X) = \tau_\alpha$. 20
(4) The space \((X, \tau_\alpha)\) is submaximal.

Now we have the following results on Artinian \(T_\alpha\) A-space which observed in[12].

**Theorem 1.3.15.** [12] Let \((X, \tau(\leq))\) be an Artinian \(T_\alpha\) A-space, and \(A \subseteq X\). Then:

1. if \(A\) is a preopen set then each maximal element in \(A\) belongs to \(M\); that is, \(M(A) \subseteq M\).
2. the set \(A\) is preclosed if and only if \(\downarrow x \subseteq A\) for all \(x \in A \cap M\).
3. then a set \(A\) is semi-open if and only if \(M(A) \subseteq M\).

**Corollary 1.3.16.** [12] Let \((X, \tau(\leq))\) be an Artinian \(T_\alpha\) A-space. Then

1. \(PO(X) \subseteq SO(X)\); that is, if \(A\) is a preopen then it is a semi-open.
2. \(X\) contains an open, dense, and hereditarily irresolvable subspace.
3. every open ultrafilter on \(X\) is a base for an ultrafilter on \(X\).
4. \(X\) is strongly irresolvable.
5. for each dense subset \(D\) of \(X\), \(D^o\) is dense.
6. \((X, \tau_\alpha)\) is submaximal.
7. for \(A \subseteq X\) where \(A^o = \emptyset\), \(A\) is nowhere dense.

**Theorem 1.3.17.** [12] Let \((X, \tau(\leq))\) be an Artinian \(T_\alpha\) A-space, and let \(A\) be a subset of \(X\), then

(a) \(scl(A) \subseteq pcl(A)\).

(b) \(pint(A) \subseteq sint(A)\).
Theorem 1.3.18. [12] Let \((X, \tau(\leq))\) be an Artinian \(T_\circ\) \(A\)-space, and \(A\) is a subset of \(X\). Then

(a) \(\text{pint}(A) = \{x \in A : \hat{x} \subseteq A\}\).

(b) \(\text{sint}(A) = \{x \in A : \hat{x} \cap A \neq \emptyset\}\).

(c) \(\text{pcl}(A) = A \cup \{\downarrow x : x \in A \cap M\}\).

(d) \(\text{scl}(A) = A \cup \{x : \hat{x} \subseteq A\}\).

Theorem 1.3.19. [19] Let \((X, \tau)\) be a topological space, then the following are equivalent:

(1) \(X\) is submaximal.

(2) Every preopen set is open.

The following Theorem characterizes the submaximality condition in a \(T_\circ\) \(A\)-space.

Theorem 1.3.20. Let \((X, \tau(\leq))\) be a \(T_\circ\) \(A\)-space, then all the following are equivalent:

(1) \(X\) is submaximal,

(2) Each element of \(X\) is either maximal or minimal.

(3) \(X\) is nodec.

Theorem 1.3.21. For a topological space \((x, \tau)\), the following conditions are equivalent:

(1) \(X\) is hyperconnected.

(2) Every nonempty preopen subset of \(X\) is dense.

Theorem 1.3.22. [12] Let \((X, \tau(\leq))\) be a \(T_\circ\) \(A\)-space. Then

(1) if \(X\) is linear order poset, then \(X\) is hyperconnected.
(2) if $X$ contains a maximum element, then $X$ is hyperconnected.

**Theorem 1.3.23.** [12] Let $(X,\tau(\leq))$ be an Artinian $T_\circ A$-space, then $X$ is hyperconnected if and only if $X$ contains a top element.

**Proof.** If $|M| \geq 2$, then $\exists x, y \in M$ then, $\{y\}$ is preopen set but not dense, because $M \nsubseteq \{y\}$.

**Theorem 1.3.24.** [12] In Artinian $T_\circ A$-spaces, the following are equivalent:

1. $(X,\tau)$ is extremally disconnected.

2. $PO(X) = SO(X)$.

3. For all $x \in X$, $|\hat{x}| = 1$; that is, $\forall x \in X$ there exists exactly one element $y \in M$ such that $x \leq y$. 

\[ \Box \]
Chapter 2

Upper Bounded $T_0$ A-spaces

2.1 Definitions and Main Properties

In this section, we will talk about a new type of A-spaces called Upper bounded $T_0$ A-space. We introduce the definition of the space followed by some illustrative examples. Furthermore, we shed some light on some differences between Artinian spaces, and Upper spaces. In [5] Rose et.al introduced two new two classes of $T_0$ A-spaces as a generalization of Artinian and Noetherian $T_0$ A-spaces. In fact, the new classes are properly contains Artinian and Noetherian $T_0$ A-spaces respectively.

Definition 2.1.1. [5] A $T_0$ A-space $(X, \tau(\leq))$ is upper bounded (UB) if every chain of points in the corresponding poset $(X, \leq)$ is bounded above. The space is lower bounded (LB) if every chain of points is bounded below. The space is bi-bounded (BB) if it is both UB and LB.

The following example gives a UB $T_0$ A-space which is not Artinian $T_0$ A-space.

Example 2.1.2. let $X = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$, and define $\leq$ on $X$ as dual of the usual order; that is, $\frac{1}{n} \leq \frac{1}{n+1}$, and $\frac{1}{n} \leq 0 \ \forall n \in \mathbb{N}$. Then $(X, \leq)$ is a poset with graph:

\[ \bullet 0 \]
Clearly, $\forall n \in N, V(\frac{1}{n}) = \uparrow \frac{1}{n} = \{\frac{1}{n}, \frac{1}{n+1}, \ldots\} \cup \{0\}$, and $\uparrow 0 = \{0\}$ since $X$ is linear order with maximum, any chain has 0 element as upper bound. So $(X, \tau(\leq))$ is $UB T_0$ A-space.

Since $X$ itself is infinite increasing chain, $X$ is not Artinian. Note that the same set $X$ under usual order is an example of $LB T_0$ A-space which is not Noethrian.

**Proposition 2.1.3.** If $(X, \tau(\leq))$ is a $UB$ (resp. $LB$) $T_0$ A-space then $\forall x \in X, \hat{x} \neq \emptyset$. (resp. $\check{x} \neq \emptyset$.)

**Proof.** Suppose to contrary that $x \in X$ exists with $\hat{x} = \uparrow x \cap M = \emptyset$. So, $\forall y \in \uparrow x, y \notin M$ which implies that $\exists z \in X$ such that $y < z$. Now, $x \in \uparrow x$, so $\exists y_1 \in X$, such that $x < y_1$.

Again, $y_1 \in \uparrow x$, so $\exists y_2 \in X$, such that $y_1 < y_2$. Continue in this way to get a properly increasing chain $x < y_1 < y_2 < \ldots < y_n < \ldots$ with no upper bound, and this contradicts the fact that $X$ is UB. The proof of $\check{x} \neq \emptyset$ is similarly. \hfill \ensuremath{\blacksquare}

In general, the converse of the above theorem is not true. The following example show that if $X$ is a space satisfies the condition that $\hat{x} \neq \emptyset$ (resp. $\check{x} \neq \emptyset$) $\forall x \in X$, then $X$ need not be UB (resp. LB) $T_0$ A-space.

**Example 2.1.4.** Let $X = \mathbb{N}$, let $E$ be the subset of even natural numbers, and let $O$ the subset of odd natural numbers. So $X = O \cup E$. Define a partial order $(\leq)$ on $X$ by the following diagram:
Then clearly $M = E$, and $\forall x \in \mathbb{N}, \hat{x} \neq \emptyset$. Moreover the set $O$ is a chain in $X$ with no upper bound. So $X$ is not $UB T_0$ $A$-space. If we use $(\leq_d)$ to be the dual order (the reverse order) defined above, then $\hat{x}$ is nonempty $\forall x \in X$, but $X$ is not $LB T_0$ $A$-space.

In [5], the authors et al. viewed some differences between the Artinian $T_0$ $A$-space, and the $UB T_0$ $A$-space. Some theorems - which were proved in Artinian $T_0$ $A$-space - are generalized in $UB T_0$ $A$-space. Furthermore the manner of the proofs is similar to those in [12]. This is because the proofs of these theorems depend basically on the set of maximal elements in $X$ which is commonly nonempty in both $UB$, and Artinian $T_0$ $A$-spaces. The following theorems were proved in [5].


**Theorem 2.1.6.** [5] An open subspace of an $UB T_0$ $A$-spaces is an $UB T_0$ $A$-space, and a closed subspace of $LB T_0$ $A$-space is a $LB T_0$ $A$-space.

**Theorem 2.1.7.** [5] Every $UB T_0$ $A$-space is $\alpha$-scattered but is generally not scattered.
Theorem 2.1.8. [5] A set $D$ is dense in $UB_T_0$ A-space if and only if $M \subseteq D$.

Theorem 2.1.9. Let $(X, \tau(\leq))$ be an U.B $T_0$ A-space. Then

(a) $A^o = \emptyset \iff A \cap M \neq \emptyset$.

(b) the subset $A$ is dense if and only if $M \subseteq A$.

(c) the subset $A$ is nowhere dense if and only if $M \cap A = \emptyset$.

(d) if $|M| = 1$, then any subset is either dense or nowhere dense.

Proof. (a) ($\Rightarrow$) If $x \in A \cap M$. Then $\uparrow x = \{x\} \subseteq A$, and $x \in A^o$. Therefore $A^o \neq \emptyset$.

($\Leftarrow$) If $A^o \neq \emptyset$, then $\exists x \in A$ such that $\uparrow x \subseteq A$. So, as $X$ being UB, $\exists z \in M$ such that $z \geq x$, then $z \in A$. Thus $z \in M \cap A$.

(b) ($\Rightarrow$) Suppose that $A$ is dense, and let $x \in M$, then $\uparrow x \cap A \neq \emptyset$. Since $\uparrow x = \{x\}$, so $x \in A$.

($\Leftarrow$) Suppose that $M \subseteq A$. Let $B$ be any non empty open set with $x \in B$, so $\uparrow x \subseteq B$. Since $\hat{x} \neq \emptyset$, and $\hat{x} \subseteq \uparrow x$, we get $\hat{x} \subseteq B$, and so, $\hat{x} \subseteq M \cap B$. Therefore $A \cap B \neq \emptyset$, and $A$ is dense.

(c) ($\Rightarrow$) Suppose that $A$ is nowhere dense; that is, $\overline{A}^o = \emptyset$. Then by part (a) $M \cap \overline{A} = \emptyset$, and hence $M \cap A = \emptyset$.

($\Leftarrow$) If $\overline{A}^o \neq \emptyset$, then $\overline{A} \cap M \neq \emptyset$. Let $x \in \overline{A} \cap M$. Then $\exists z \in A$ such that $x \in \downarrow z$. As $x \in M$, we get $x = z$. Hence $M \cap A \neq \emptyset$.

(d) Let $M = \{T\}$, and let $A$ be a subset of $X$, then either $T \in A$ or $T \notin A$, and by parts (b), and (c) either $A$ is dense or $A$ is nowhere dense.

\[\square\]

Theorem 2.1.10. [5] Let $(X, \tau(\leq))$ be an Upper bounded $T_0$ A-space, then $X$ is $\alpha$-scattered but is generally not scattered.
Let \((X, \tau(\leq))\) be a UB \(T_\circ A\)-space. The set \(M\) belongs to all dense subsets of \(X\), so no disjoint dense subsets exist in \(X\), and hence \(X\) is surely irresolvable. Moreover, it is strongly irresolvable. We will see that UB is hereditarily irresolvable, using the following two theorems, and after characterizing preopen, semi-open, and \(\alpha\) open sets.

**Proposition 2.1.11.** Let \((X, \tau(\leq))\) be a UB \(T_\circ A\)-space. Then \(A\) is preopen if and only if \(\hat{x} \subseteq A \quad \forall x \in A\).

**Proof.** (\(\Rightarrow\)) Let \(A\) be a subset of \(X\). Suppose that \(\exists x \in A\) such that \(\hat{x} \not\subseteq A\). Then \(\exists y \in M\), and \(y \geq x\), but \(y \not\in A\). So \(y \not\in \{\downarrow z : z \in A\} = \overline{A}\) (as \(y \in M\), if \(y \in \downarrow z\), then \(y = z \in A\)). Since \(y \in \uparrow x\), we get \(\uparrow x \not\subseteq \overline{A}\), and so \(x \not\in (\overline{A})^\circ\). Therefor \(A \not\subseteq \overline{A}\).

(\(\Leftarrow\)) Suppose that \(\hat{x} \subseteq A \quad \forall x \in A\). If \(A\) is not preopen, then \(\exists r \in A\) such that \(r \not\subseteq \overline{A}\). So, \(\uparrow r \not\subseteq \overline{A}\), and \(\exists m \geq r\) such that \(m \not\in \overline{A}\). Since \(\hat{r} \subseteq A\), take \(z \in M \cap \uparrow r \subseteq A\). therefore as \(m \leq z\), \(m \in z = \downarrow z \subseteq \overline{A}\) which is a contradiction. \(\Box\)

**Corollary 2.1.12.** Let \((X, \tau(\leq))\) be a UB \(T_\circ A\)-space. Then the set \(A\) is preclosed if and only if \(\downarrow x \subseteq A \quad \forall x \in A \cap M\).

**Proof.** (\(\Rightarrow\)) Suppose that \(A\) is preclosed set, so \(A^c\) is preopen. Now we have that \(\forall x \in A^c\) \(\hat{x} \subseteq A^c\). Equivalently, \(\forall x \not\in A\), \(A \cap \hat{x} = \emptyset\). Suppose to contrary that \(\exists y \in A \cap M\), such that \(\downarrow y \not\subseteq A\). Let \(z \in \downarrow y\), and \(z \not\in A\) then \(\hat{z} \cap A = \emptyset\). Since \(y \in \hat{z}\), then \(\hat{z} \cap A = \emptyset\), which is a contradiction.

(\(\Leftarrow\)) Suppose that \(\downarrow x \subseteq A\) for all \(x \in A \cap M\). Suppose to contrary that \(A\) is not preclosed then \(A^c\) not preopen then \(\exists x \in A^c\) such that \(\hat{x} \not\subseteq A^c\). Let \(z \in \hat{x} \cap A\), so \(z \in M \cap A\). By assumption, \(\downarrow z \subseteq A\). Since \(x \in \downarrow z\), \(x \in A\), which is a contradiction. \(\Box\)

**Proposition 2.1.13.** Let \((X, \tau(\leq))\) be a UB \(T_\circ A\)-space. Then \(A\) is semi open set if and only if \(\hat{x} \cap A \neq \emptyset \quad \forall x \in A\).
Proof. ($\Rightarrow$) Suppose that $A$ is semi-open set ($= A \subseteq \overline{A^o}$), and let $x \in A$. Then $x \in \overline{A^o} = \downarrow A^o$. Let $r \in A^o$ such that $x \in \downarrow r$. Since $r \in A^o$, so $\uparrow r \subseteq A$. Take $y \in \hat{r} \subseteq M$, then we have $y \geq r \geq x$, so $y \in \uparrow r \cap M$. Therefore, $y \in \hat{x}$, and $\hat{x} \cap A \neq \emptyset$.

($\Leftarrow$) If $y \in A$, then $\exists z \in \text{y} \text{\cap A}$. So, $\uparrow z = \{z\} \subseteq A$, and $z \in A^o$. Hence $\downarrow z \subseteq \overline{A^o}$. Since $z_y \geq y$, then $y \in \downarrow z \subseteq \overline{A^o}$. Therefore, $A \subseteq \overline{A^o}$, and $A$ is semi open. 

Corollary 2.1.14. Let $(X, \tau(\leq))$ be a UB $T_\alpha$ A-space, then $PO(X) \subseteq SO(X)$. That is, if $A$ is a preopen, then it is a semi-open.

Proof. Follows directly from Proposition 2.1.11, and Proposition 2.1.13

Remark 2.1.15. In any topological space $(X, \tau)$. If $D$ is dense subset, then $\overline{D}^d = X$, so it is preopen. Further, M.Ganster in [36] showed that the collection $PO(X)$ forms a topology if and only if the intersection of any two dense sets is preopen. In $UB T_\alpha$ A-space, the intersection of any two dense sets is dense, and hence preopen, so $PO(X)$ is a topology on $X$.

If $U$ is preopen then by Corollary 2.1.14, it is semi-open, and hence it is $\alpha$-open. Therefore we have the following results.

Corollary 2.1.16. Let $(X, \tau(\leq))$ be a UB $T_\alpha$ A-space, then $PO(X) = \tau$; that is, a set $A$ is preopen if and only if it is $\alpha$-open.

Corollary 2.1.17. Let $(X, \tau(\leq))$ be an UB $T_\alpha$ A-space. Then

(1) $X$ contains an open, dense, and hereditarily irresolvable subspace.

(2) for $A \subseteq X$ where $A^o = \emptyset$, $A$ is nowhere dense.

(3) $X$ is strongly irresolvable.

(4) for each dense subset $D$ of $X$, $D^o$ is dense.
(5) \((X, \tau_\alpha)\) is submaximal.

Proof. Direct result from Theorem 1.3.14, Theorem 1.3.13, and Corollary 2.1.16. 

**Theorem 2.1.18.** Let \((X, \tau(\leq))\) be a UB \(T_\alpha\) A-space, and let \(A\) be a subset of \(X\). Then

(a) \(\text{scl}(A) \subseteq \text{pcl}(A)\).

(b) \(\text{pint}(A) \subseteq \text{sint}(A)\).

Proof. (a) \(\text{pcl}(A)\) is preclosed set contains \(A\), so it is semi-closed contains \(A\), and hence \(\text{scl}(A) \subseteq \text{pcl}(A)\).

(b) \(\text{pint}(A)\) is preopen set inside \(A\), so it is semi-open in \(A\). Hence \(\text{pint}(A) \subseteq \text{sint}(A)\).

**Theorem 2.1.19.** Let \((X, \tau(\leq))\) be a UB \(T_\alpha\) A-space, and \(A\) is a subset of \(X\). Then

(a) \(\text{pint}(A) = \{x \in A: \hat{x} \subseteq A\}\).

(b) \(\text{sint}(A) = \{x \in A: \hat{x} \cap A \neq \emptyset\}\).

(c) \(\text{pcl}(A) = A \cup \{\downarrow x: x \in A \cap M\}\).

(d) \(\text{scl}(A) = A \cup \{x: \hat{x} \subseteq A\}\).

Proof. (a) Let \(y \in \{x \in A: \hat{x} \subseteq A\}\). Then \(y \in A\), and \(\hat{y} \subseteq A\). But we have \(\{r\} = \hat{r} \subseteq A\) \(\forall r \in \hat{y}\), so \(\hat{y} \subseteq \{x \in A: \hat{x} \subseteq A\}\). By Proposition 2.1.11 \(\{x \in A: \hat{x} \subseteq A\}\) is preopen. If \(U\) is a preopen in \(A\), then any \(x\) in \(U\), \(\hat{x} \subseteq U \subseteq A\). So \(U \subseteq \{x \in A: \hat{x} \subseteq A\}\). Therefore \(\text{pint}(A) = \{x \in A: \hat{x} \subseteq A\}\).

(b) Let \(z \in \{x \in A: \hat{x} \cap A \neq \emptyset\}\), and \(w \in \hat{z} \cap A\). So, \(w \in M \cap A\). Therefore \(w \in \{x \in A: \hat{x} \cap A \neq \emptyset\}\), and \(w \in \{x \in A: \hat{x} \cap A \neq \emptyset\} \cap \hat{z}\). By Proposition 2.1.13, \(\{x \in A: \hat{x} \cap A \neq \emptyset\}\) is semi open contained in \(A\). Now suppose that \(S\) is a semi-open set contained in \(A\). If \(x \in S\), then by Proposition 2.1.13, \(\hat{s} \cap S \neq \emptyset\). Since
$S \subseteq A$, $x \in A$, and $\hat{x} \cap A \neq \emptyset$. Therefore $S \subseteq \{x \in A : \hat{x} \cap A \neq \emptyset\}$, and $\text{sint}(A) = \{x \in A : \hat{x} \cap A \neq \emptyset\}$.

(c) Let $z \in (A \cup \{\downarrow x : x \in A \cap M\}) \cap M$. If $z \in A \cap M$ then $\downarrow z \subseteq A \cup \{\downarrow x : x \in A \cap M\}$. If $z \in \{\downarrow x : x \in A \cap M\} \cap M$, then $\exists x \in A \cap M$ such that $z \in \downarrow x$. In this case, $z = x$ since $z \in M$. So $z \in A \cap M$, and $\downarrow z \subseteq A \cup \{\downarrow x : x \in A \cap M\}$. In both cases, and by Corollary 2.1.12, the set $A \cup \{\downarrow x : x \in A \cap M\}$ is preclosed set contained in $A$. Let $B$ be any preclosed set contains $A$, and let $x \in A \cap M$. So $x \in B$, and $\downarrow x \subseteq B$. Hence $A \cup \{\downarrow x : x \in A \cap M\} \subseteq B$. Therefore $\text{pcl}(A) = A \cup \{\downarrow x : x \in A \cap M\}$.

(d) If $z \notin A \cup \{x : \hat{x} \subseteq A\}$, then $z \notin A$, and $z \notin \{x : \hat{x} \subseteq A\}$. So $\hat{z} \cap A^c \neq \emptyset$. If $r \in \hat{z} \cap A^c$, then $r \notin A$. Since $r \in M$, so $\hat{r} = r$. Hence $r \notin \{x : \hat{x} \subseteq A\}$. Therefore $\hat{z} \notin A \cup \{x : \hat{x} \subseteq A\}$. By Proposition 2.1.13 $(A \cup \{x : \hat{x} \subseteq A\})^c$ is semi-open, and so, $A \cup \{x : \hat{x} \subseteq A\}$ is semi-closed. If $C$ is a semi-closed set contains $A$, and if $y \in \{x : \hat{x} \subseteq A\}$ not in $C$, then $y \in C^c$. So, $\hat{y} \cap C^c \neq \emptyset$. Therefore, there exists $r \in \hat{y} \cap C^c$. This contradicts with $\hat{y} \subseteq A \subseteq C$. So, $A \cup \{x : \hat{x} \subseteq A\} \subseteq C$, and $\text{scl}(A) = A \cup \{x : \hat{x} \subseteq A\}$.

\[ \square \]

**Theorem 2.1.20.** Let $(X, \tau(\leq))$ be an UB $T_0$ $A$-space, then $X$ is hyperconnected if and only if $X$ contains a top element.

**Proof.** ($\Rightarrow$)If $|M| \geq 2$, then $\exists x, y \in M$, and $\{y\}$ is preopen set but not dense. Hence, $X$ is not hyperconnected.

($\Leftarrow$) Let $X$ contains a top element, then $M = \{T\}$, and $\forall x \in X$, $\hat{x} = \{T\}$). Suppose that, $A \subseteq X$ is nonempty preopen set then, $\forall x \in A$, $\hat{x} \subseteq A$, then $\{T\} \subseteq A$; that is, $M \subseteq A$, and $A$ is dense. Then $X$ is hyperconnected. \[ \square \]

**Theorem 2.1.21.** In UB $T_0$ $A$-spaces, the following are equivalent:
(1) \((X, \tau)\) is extremally disconnected.

(2) \(PO(X) = SO(X)\).

(3) For all \(x \in X\), \(|\hat{x}| = 1\); that is, \(\forall x \in X\), there exists exactly one element \(y \in M\) such that \(x \leq y\).

Proof. (1) \(\iff\) (2) \((X, \tau)\) is extremally disconnected if and only if \(SO(X) = \tau_\alpha\). By Corollary 2.1.16, we have \(PO(X) = \tau_\alpha\), then \(PO(X) = SO(X)\).

(2) \(\Rightarrow\) (3) If there exist \(x_o \in X\) such that \(|\hat{x_o}| \geq 2\), then there are two different elements \(y, z \in \uparrow x_o \cap M\), and hence, \(S = \{x_o, y\}\) is semi-open set that is, not preopen.

(3) \(\Rightarrow\) (2) Let \(S\) be a semi-open set. Then \(\hat{x} \cap S \neq \emptyset \forall x \in S\). Since \(|\hat{x}| = 1\), \(\hat{x} \subseteq S\) \(\forall x \in S\), and by proposition 2.1.11, \(S\) is preopen set.

Theorem 2.1.22. If \((X, \tau(\leq))\) is a \(UB\ T_0\ A\)-space, and \(A\) is preopen subset of \(X\), then \(A\) is a \(UB\ T_0\ A\)-space.

Proof. Suppose that \(A\) is preopen subset of \(X\). Then as a subspace, \(A\) is \(T_0\ A\)-space. So, let \(C\) be a chain in \(A\). Then \(\exists s \in M\) such that \(s\) is an upper bound for \(C\). So \(s \in \hat{x}\), \(\forall x \in C\). Since \(A\) is preopen then \(s \in A\). Hence \(C\) is bounded in \(A\), and \(A\) is a \(UB\ T_0\ A\)-space. is upper bounded.

It is worth mentioned that this theorem is stronger than Theorem 2.1.6, which can be concluded as a corollary.

In general not any subspace of \(UB\ T_0\ A\)-space is \(UB\ T_0\ A\)-space. Furthermore semi-open subspace of upper space needn’t be upper space. Since, semi-open sets need not be preopen, the following example shows this fact.

Example 2.1.23. Let \(\mathbb{N}\) the set of natural numbers, and let \(T\) be a number not in \(\mathbb{N}\). Take \(E\), and \(O\) to be respectively, the sets of even, and odd natural numbers. Define a partial order \(\leq\) on the set \(X = \mathbb{N} \cup \{T\}\) by the diagram below:
Then $X$ is a $UB T_0$ $A$-space, and $M = E \cup \{T\}$. Let $A = \mathbb{N}$, so, $\forall n \in A$, $\hat{n} \neq \emptyset$. (Since if $n \in E$, $\hat{n} = \{n\}$, and if $n \in O$, $\hat{n} = \{n + 1, T\}$). By Proposition 2.1.13, $A$ is semi-open. As a subspace of $X$, the set $O$ of odd numbers is a chain with no upper bound in $A$. Hence $A$ is not a $UB T_0$ $A$-space.

2.2 Generalized Artinian $T_0$ $A$-space

Any poset $(P, \leq)$ on a non-empty set $X$ induces two kinds of topologies Scott topology, and $T_0$ $A$-topology. In general, the Scott topology on a poset is $T_0$. Moreover a subspace of a Scott space is Scott, and every Scott open set is Alexandroff open set, but the converse is not true in general. So the Scott topology is coarser than the Alexandroff topology. In [13], Mahdi, and EL-Mabhouh introduced a new type of posets, and they call it $g$-ACC. The class of posets that are $g$-ACC is strictly contains the class of posets that are satisfying ACC. In this class, equality holds of the two topologies; the Scott topology, and the Alexandroff topology.

Definition 2.2.1. [30]

(1) A subset $U$ of a poset $P$ is directed set if for any $x$, and $y$ in $U$ there exists $z$ in $U$ with $z \geq x$, and $z \geq y$.

(2) A subset $U$ of a poset $P$ is Scott open if $U$ is an up set, and for any directed set
S with supremum, if $\bigvee S \in U$, then there exists $s_0 \in S$ such that $s_0 \in U$; that is, $S \cap U \neq \emptyset$.

(3) A subset $F$ of a poset $P$ is called *Scott closed* if its complement is Scott open. So $F$ is *Scott closed* if $F = \downarrow F$ ($F$ is a down set), and if $U$ is a directed set contained in $F$, and $\bigvee U$ exists, then $\bigvee U \in F$.

The collection $\sigma(P)$ of all Scott open sets forms a topology on $P$ called *Scott topology*.

The Scott topology on a poset is $T_0$. Moreover, every Scott open is Alexandroff open; that is, $\sigma(P) \subseteq \tau(\leq)$. The converse is not always true, as the following example shows.

**Example 2.2.2.** Let $X = \mathbb{R}$ with usual order $\leq$. A set $U \subseteq \mathbb{R}$ is Scott open if and only if $u = (a, \infty)$, for some $a \in \mathbb{R}$. Moreover, $[a, \infty) = \uparrow a$ for some $a \in \mathbb{R}$ is Alexandroff open which is not Scott open. To see this the directed set $U = (a - 1, a)$ has $\bigvee U = a \in [a, \infty)$ but $[a, \infty) \cap (a - 1, a) = \emptyset$.

**Definition 2.2.3.** [13] Let $(P, \leq)$ be a poset. Then we define:

(1) $P^{\text{dir}}$ to be the collection of all directed subsets of $P$,

(2) $P^{\text{di}}$ the collection of all directed subsets of $P$ with supremum, and

(3) $P^{d}$ the collection of all subsets of $P$ with maximum element.

Since each set in $P$ with maximum is directed, we have the following implications:

$$P^{d} \subseteq P^{\text{di}} \subseteq P^{\text{dir}}.$$ 

The converse is not true. In fact, one of the following cases holds for a given poset

**Case 1.** $P^{\text{dir}} \neq P^{\text{di}}$, and $P^{\text{di}} \neq P^{d}$. For example, in the real numbers $\mathbb{R}$ with usual order, the set $A = (0, \infty) \in P^{\text{dir}}$ which is not in $P^{\text{di}}$. And the set $B = (0, 1) \in P^{\text{di}}$ which is not in
Case 2. $P_{\text{dir}} = P_{\text{di}}$ while $P_{\text{di}} \neq P_d$. For example, in the set $[0, 1]$ with usual order, any subset of $[0, 1]$ is bounded by 1, so $P_{\text{dir}} = P_{\text{di}}$. But the set $B = (0, 1) \in P_{\text{di}}$ which is not in $P_d$.

Case 3. $P_{\text{dir}} \neq P_{\text{di}}$ while $P_{\text{di}} = P_d$. For example, in the set of natural numbers $\mathbb{N}$ with its usual order, the set of odd numbers is directed set without supremum. If $A$ is a set with supremum $\bigvee A$, then $\bigvee A \in A$, so $P_{\text{di}} = P_d$.

Case 4. $P_{\text{dir}} = P_{\text{di}} = P_d$. For example, take any finite poset.

**Definition 2.2.4.** [11] Let $(P, \leq)$ be a poset, We say that $P$ is directed-complete (briefly dcpo) if $P_{\text{dir}} = P_{\text{di}}$ that is, every directed subset of $P$ has a supremum.

**Definition 2.2.5.** [13] A poset $(P, \leq)$ is called generalized ascending chain condition (briefly $g$–ACC) if $P_{\text{di}} = P_d$, that is, each directed set with supremum has a maximum.

**Theorem 2.2.6.** [13] A poset satisfies ACC if and only if $P$ is $g$-ACC, and dcpo.

**Definition 2.2.7.** [13] Let $(P, \leq)$ be a poset. Then $\sigma(P) = \tau(\leq)$ if and only if $P$ is $g$-ACC.

**Definition 2.2.8.** [13] A $T_0$ A-space $(X, \tau(\leq))$ is called g-Artinian $T_0$ A-space if the corresponding poset $(X, \leq)$ is $g$-ACC.

**Theorem 2.2.9.** [13] If $X$ is an Artinian $T_0$ A-space, then $X$ is a g-Artinian $T_0$ A-space.

### 2.3 Between Upper Bounded and g-Artinian $T_0$ A-spaces

As we show in the last two sections, an Artinian $T_0$ A-space is both g-Artinian, and UB $T_0$ A-spaces. So the class of Artinian $T_0$ A-spaces is contained in the intersection of
the two classes; the class of g-Artinian \( T_0 \) A-spaces, and the class of UB \( T_0 \) A-spaces. In this section, we deal with the answers of some questions about the equality of the two classes, which bigger than the other? What about the intersection of the two classes? Is there any relation between them?

**Definition 2.3.1.** A \( T_0 \) A-space \((X, \tau(\leq))\) is called **Alexandroff directed complete topological space** (briefly, \( A\)-dcts) if the corresponding poset \((X, \leq)\) is dcpo.

**Theorem 2.3.2.** Let \((X, \tau(\leq))\) be a \( T_0 \) Alexandroff space. If \((X, \tau(\leq))\) is \( A\)-dcts then \((X, \tau(\leq))\) is UB \( T_0 \) A-space.

*Proof.* Let \((X, \tau(\leq))\) be an \( A\)-dcts space, and let \( C \) be a chain of points. Then \( C \) is directed set, and hence it has supremum. Therefore \( C \) is bounded above. \( \Box \)

The converse of the above theorem is not always true. In the following examples, we give a UB \( T_0 \) A-space which is not \( A\)-dcts. Moreover, we show that a class of g-Artinian \( T_0 \) A-spaces contains spaces which are not UB \( T_0 \) A-space. And a class of UB \( T_0 \) A-space also contains spaces which are not g-Artinian \( T_0 \) A-spaces. Finally we give an example to show that the class of Artinian \( T_0 \) A-space is a proper subclass of the intersection of the two classes; g-Artinian, and UB \( T_0 \) A-spaces.

**Examples 2.3.3.** (1) let \( X_1 = [0,1] \cup \{2\} \). Define a partial order \( \leq \) on \( X_1 \) as follows:

\[
\forall x, y \in [0,1], x \leq y \text{ in usual way, } \forall x, y \in [0,1), x \leq 2, \text{ and } 1, 2 \text{ are incomparable.}
\]

Since any increasing chain of points is bounded above, \( X \) is UB \( T_0 \) A-space. The set \( A = (0,1) \in X_1^{\text{dir}} \), and since the set of upper bounds of \( A \) is \( \{1,2\} \), \( \lor A \) does not exists. Hence \( A \notin X_1^{\text{d}} \). Therefore \( X \) is not \( A\)-dcts space. Moreover, the set \( B = (0, \frac{1}{2}) \in X_1^{\text{d}} \) with \( \lor B = \frac{1}{2} \notin B \), then \( B \notin X_1^{\text{d}} \) This proves that \( X_1 \) not g-Artinian \( T_0 \) A-spaces.
(2) let $X_2 = \mathbb{N}$ with usual order. If $A \in X^{di}$, then $A$ is finite, and $\bigvee A \in A$. So, $A \in X^{d}$, and $X$ is $g$-Artinian $T_0$-A-spaces. Since $\mathbb{N}$ itself is an increasing chain with no upper bound, $X_2$ is not $UB T_0$ A-space, and hence not Artinian $T_0$ A-space.

(3) let $X_3 = \mathbb{N} \cup \{T_1, T_2\}$. Define a partial order $\leq$ on $X_1$ as follows:

$\forall x, y \in \mathbb{N}$, $x \leq y$ in usual way, $\forall x \in \mathbb{N}$, $\forall y \in \{T_1, T_2\}$, $x \leq y$, and $T_1, T_2$ are incomparable. If $C$ is a chain in $X_3$, then either $T_1$ or $T_2$ is an upper bound of $C$. So $X_3$ is $UB T_0$ A-space. Suppose $U \in X_3^{di}$, then $\bigvee U$ exists. So $T_1$ or $T_2$.

So not belongs to $U$. If $T_1 \in U$ then $\bigvee U = T_1 \in U$. Similarly if $T_2 \in U$. If $T_i \notin U$, $\forall i \in \{1, 2\}$, then $U$ is finite. (otherwise $T_1$, and $T_2$ are the only upper bounds of $U$, so $\bigvee U$ doesn’t exist). Now, $U$ is finite directed set so $\bigvee U \in U$. Therefore $U \in X_3^d$, and $X_3$ is also $g$-Artinian. Clearly, $\mathbb{N}$ itself is an infinite increasing chain, so $X_3$ is not Artinian.

(4) let $X_4 = \mathbb{N} \cup \{T\}$. Define a partial order $\leq$ on $X_1$ as follows:

$\forall x, y \in N$, $x \leq y$ in usual way, and $\forall x \in \mathbb{N}$, $T \geq x$. Since any directed set has a supremum, $X_4$ is $A$-dcts, and hence, it is $UB T_0$ A-space. The set $\mathbb{N}$ is a directed set with supremum $\bigvee N = T \notin \mathbb{N}$, so, $X_4$ is not $g$-Artinian space.

(5) If $X_5 = \{1, 2\}$ with partial order $1 \leq 2$, then $\tau(\leq) = \{\emptyset, X, \{2\}\}$ which is the Sierpenski topology. This topology is Artinian $T_0$ A-space, so it is both $A$-dcts, and $g$-Artinian.

The diagram below shows the relations between $A$-dcts, $UB$, Artinian and $g$-Artinian $T_0$ A-spaces.

2.4 $\tau_\alpha$ on $UB T_0$ A-space

In [19], $\tau_\alpha$ was studied on Artinian $T_0$ A-spaces. In this section, we will generalize this
study to UB $T_0$ A-spaces. Many of our results, and their proofs are mimic to those in [19]. In fact, Artinian, and UB $T_0$ A-spaces have a common property where the set $M$ of all maximal elements is nonempty, and this property essential in the proofs.

Previously, we prove that in UB $T_0$ A-spaces $\tau_\alpha = PO(X)$. And in general, topological space, $\tau_\alpha = PO(X)$ if and only if $\tau_\alpha$ is submaximal.

**Proposition 2.4.1.** If $(X, \tau(\leq))$ is a UB $T_0$ A-space, then the space $(X, \tau_\alpha)$ is an Alexandroff space (necessarily $T_\alpha$).

**Proof.** Let $\{U_\alpha\}_{\alpha \in \Delta}$ be a collection of preopen subsets of $X$, and let $x \in \bigcap_{\alpha \in \Delta} U_\alpha$. Then $\hat{x} \subseteq U_\alpha \forall \alpha \in \Delta$, so $\hat{x} \subseteq \bigcap_{\alpha \in \Delta} U_\alpha$. Hence by proposition 2.1.11 $\bigcap_{\alpha \in \Delta} U_\alpha$ is preopen, and $(X, \tau_\alpha)$ is Alexandroff space. $(X, \tau_\alpha)$ is $T_\alpha$ since $\tau \subseteq \tau_\alpha$, and $\tau$ is $T_\alpha$. \hfill \Box

Now, $\tau_\alpha$ is $T_\alpha$ A-space, so there is a corresponding specialization order $\leq_\alpha$. Moreover $(X, \tau_\alpha)$ is submaximal, so by Theorem 1.3.20, each element is either maximal or minimal with respect to $\leq_\alpha$. In the following theorem, we describe $\leq_\alpha$ depending on the original order $\leq$.

**Theorem 2.4.2.** Let $(X, \tau(\leq))$ be an UB $T_0$ A-space, $x, y \in X$. Then $x \leq_\alpha y$ if and only if $y \in \{x\} \cup \hat{x}$. 

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Proof. \((\Rightarrow)\) Let \(y \in \hat{x}\) then \(y \in \{y\} \cap M\). By Theorem 2.1.19 part (c), \(pcl\{y\} = \downarrow y\), so we get that \(x \in \downarrow y = pcl\{y\} = Cl_\alpha\{y\}\). Hence \(x \leq_\alpha y\). \((\Leftarrow)\)If \(x \leq_\alpha z\), and \(x \neq z\), then \(z \in M\), and \(x \in pcl(z) = \downarrow z\) (If \(z \notin M\), then there is \(w \in M\) such that \(z \leq w\), which implies that \(x <_\alpha z <_\alpha w\) contradicts the fact that \(\tau_\alpha\) is submaximal see Theorem 1.3.20). Therefore \(x \leq z\), and hence \(z \in \hat{x}\). 

From this theorem, \(x\) is a maximal element of \(X\) with respect to the order \(\leq\) if and only if it is maximal element of \(X\) with respect to the order \(\leq_\alpha\), and since \((X, \tau_\alpha)\) is a submaximal \(T_0\) A-space, the graph of the poset \((X, \tau_\alpha)\) consists of two rows, the row of maximal elements, and the row of minimal elements including the case where some elements are both maximal, and minimal.

Notation 2.4.3. For the order \(\leq_\alpha\), we will use the following notations:

\(M_\alpha\) to be the set of all maximal elements,
\(m_\alpha\) to be the set of all minimal elements.

\[\uparrow_\alpha x := \{y \in X : x \leq_\alpha y\}.\]
\[\downarrow_\alpha x := \{y \in X : x \geq_\alpha y\}.\]
\[\uparrow_\alpha A := \bigcup_{x \in A} \uparrow_\alpha x.\]
\[\downarrow_\alpha A := \bigcup_{x \in A} \downarrow_\alpha x.\]
\[\hat{x}_\alpha := (\uparrow_\alpha x) \cap M_\alpha.\]
\[\bar{x}_\alpha := (\downarrow_\alpha x) \cap m_\alpha.\]

Theorem 2.4.4. Let \((X, \tau(\leq))\) be a UB \(T_0\) A-space. Then

1. \(M_\alpha = M\).
2. \(M \cap m = M_\alpha \cap m_\alpha\).
3. \(m_\alpha = (X/M) \cup (m \cap M)\).
4. \(\uparrow_\alpha x = \{x\} \cup \hat{x}, \forall x \in X\).
(5) if \(x \notin M\), \(\downarrow_{\alpha} x = \{x\}\), and if \(x \in M\), \(\downarrow_{\alpha} x = \downarrow x\).

(6) \(\hat{x} = \hat{x}, \forall x \in X\).

The proof is direct from the description of \(\leq_{\alpha}\) in Theorem 2.4.2. We will prove item (5) to show how we treat with the description of \(\leq_{\alpha}\).

Proof. Clearly \(x \in \downarrow_{\alpha} x\). Let \(y \in \downarrow_{\alpha} x\), then \(y \leq_{\alpha} x\) then \(x \in \{y\} \cup \hat{y}\). If \(x \notin M\), then \(x \notin \hat{y}\), \(\forall y \in X\). Then \(x \in \{y\}\), hence \(x = y\), then \(\downarrow_{\alpha} x = \{x\}\).

If \(x \in M\), then if \(y \in \downarrow x\), then \(y \leq x\), \(x \in \hat{y}\), then \(y \leq x\), hence \(\downarrow x \subseteq \downarrow_{\alpha} x\). If \(y \in \downarrow_{\alpha} x\), then \(x \in \hat{y}\), then \(y \in \downarrow x\), then \(\downarrow_{\alpha} x \subseteq \downarrow x\). Then \(\downarrow_{\alpha} x = \downarrow x\). \(\square\)

Definition 2.4.5. [40]

(1) A topological space \((X, \tau)\) is called \(\alpha\)-topology if \(\tau = \tau_{\alpha}\).

(2) Two topologies \(\tau_1, \tau_2\) on \(X\) are called \(\alpha\)-equivalent if \((\tau_1)_\alpha = (\tau_2)_\alpha\).

(3) The class of all \(\alpha\)-equivalent of a topological space \(\tau\) on \(X\) is denoted by \(\alpha(\tau)\).

In any topological space \((X, \tau)\), and since \(\tau_{\alpha\alpha} = \tau_{\alpha}\), we have that \(\tau\), and \(\tau_{\alpha}\) are always \(\alpha\)-equivalent, and \(\tau_{\alpha}\) is \(\alpha\)-topology. In fact, any \(U\) between \(\tau\), and \(\tau_{\alpha}\) is \(\alpha\)-equivalent to \(\tau\), so it belongs to the class \(\alpha(\tau)\) of \(\alpha\)-equivalent of \(\tau\). This class contains \(\tau_{\alpha}\) as a greatest topology, and as the only \(\alpha\)-topology. Although, all topologies of a given \(\alpha(\tau)\) on \(X\) has a common property that \(\tau_{\alpha} = U_{\alpha}, \forall U \in \alpha(\tau)\), there is some of other common properties of all topologies in \(\alpha(\tau)\). We give some of these common properties in the following theorem, and for more information, see[40].

Proposition 2.4.6. [40] Let \((X, \tau)\) be a topological space and \(\alpha(\tau)\) a class of all \(\alpha\)-equivalent of \(\tau\). Then \(\forall U \in \alpha(\tau)\). We have that:

(1) \(SO(X, \tau) = SO(X, U)\); that is, all topologies of \(\alpha(\tau)\) determine the same class of semi-open sets.
(2) \( D(X, \tau) = D(X, \mathcal{U}) \); that is, all topologies of \( \alpha(\tau) \) determine the same class of dense sets.

(3) \( ND(X, \tau) = ND(X, \mathcal{U}) \); that is, all topologies of \( \alpha(\tau) \) determine the same class of nowhere dense sets.

(4) \( RO(X, \tau) = RO(X, \mathcal{U}) \); that is, all topologies of \( \alpha(\tau) \) determine the same class of regular open sets.

(5) If \( \tau \) is extremally disconnected, then all topologies of \( \alpha(\tau) \) is extremally disconnected.

In a special case, when \( X \) is a UB \( T_\alpha \) A-space, we can use theorems, and remarks we got, specially Theorem2.4.4, in proving these results, and some other results to get a completely different proofs in [40].

**Theorem 2.4.7.** A subset \( D \) is dense with respect to the topology \( \tau(\leq) \) if and only if it is dense with respect to the topology \( \tau_\alpha \).

**Proof.** Since \( M_\alpha = M \) so \( M \subseteq A \) if and only if \( M_\alpha \subseteq A \). \( \square \)

**Theorem 2.4.8.** If \( (X, \tau(\leq)) \) is a UB \( T_\alpha \) A-space. Then

(1) \( (X, \tau(\leq)) \) is extremally disconnected if and only if \( (X, \tau_\alpha) \) is extremally disconnected.

(2) \( (X, \tau(\leq)) \) is hyperconnected if and only if \( (X, \tau_\alpha) \) is hyperconnected.

(3) For a subset \( A \) of \( X \), \( S\text{Int}(A) = S\text{Int}_\alpha(A) \), and \( \text{scl}(A) = \text{scl}_\alpha(A) \).

**Proof.** The proofs of parts (1), and (2) follow directly from Theorems 2.1.20, 2.1.21, and the fact that \( x = \hat{x} \), and \( M = M_\alpha \).

(3) By Theorem 2.1.20, \( (X, \tau(\leq)) \) is hyperconnected if and only if \( | M | = 1 \) if and only if \( | M_\alpha | = 1 \) if and only if \( (X, \tau_\alpha) \) is hyperconnected. \( \square \)

**Theorem 2.4.9.** Let \( (X, \tau(\leq)) \) be a UB \( T_\alpha \) A-space, and \( A \subseteq X \). Then \( M(A) \subseteq M_\alpha(A) \).
Proof. Suppose that $x \notin M(A)$. If $x \notin A$ then $x \notin M(A)$. If $x \in A$, then $\exists z \in A$, such that $z \geq x$, and $x \neq z$. By Theorem 2.4.2, $z \in \hat{x}$. Then $z \geq x$, and $x \notin M(A)$. \hfill \Box

Note that if $(X, \tau(\leq))$ is a UB $T_o$ A-space, then $M(A)$ may be empty for some subsets $A \subseteq X$, and this theorem still true in this case.

**Theorem 2.4.10.** Let $(X, \tau(\leq))$ be a UB $T_o$ A-space, and $A \subseteq X$. Then $A$ is semi-open with respect to the space $(X, \tau(\leq))$ if and only if it is semi-open with respect to the space $(X, \tau_a)$; that is, $SO(X, \tau(\leq)) = SO(X, \tau_a)$.

Proof. Since $\hat{x}_\alpha = \hat{x}$, then we get that $A \cap \hat{x} \neq \emptyset$ if and only if $A \cap \hat{x}_\alpha \neq \emptyset$. By Theorem 2.1.13, $A$ is semi-open with respect to the space $(X, \tau(\leq))$ if and only if it is semi-open with respect to the space $(X, \tau_a)$. \hfill \Box

**Theorem 2.4.11.** Let $(X, \tau(\leq))$ be a UB $T_o$ A-space, $A \subseteq X$ Then $A$ is clopen if and only if $A$ is preclopen (= $\alpha$-clopen).

Proof. ($\Rightarrow$) If $A$ is clopen set then it is preclopen (because any open is $\alpha$-open similarly any closed is $\alpha$-closed).

($\Leftarrow$) If $A$ is preclopen then $A$, and $A^c$ are preopen. Assume to contrary that $A$ not open. Then there exists $x, a$ such that $x \in A$, and $a \notin A$, and $a \geq x$. Then by Proposition 2.1.11 $\hat{x} \subseteq A$ and $\hat{a} \subseteq A^c$. So, $\hat{a} \cap \hat{x} = \emptyset$, which contradicts the fact $a \in \uparrow x$ ( $\uparrow a \subseteq \uparrow x \Rightarrow M \cap \uparrow a \subseteq M \cap \uparrow x$). Therefore $A$ is open. \hfill \Box

Let $(X, \tau_i(\leq))$, and $(X, \tau_2(\leq))$ be two UB $T_o$ A-spaces. For $i = 1, 2$, we denote $\tau_{i\alpha}$ to be the $\alpha$-topology of $\tau_i$, $M_i$ to be the set of all maximal elements with respect to $\leq_i$, $\hat{x}^i = \uparrow_i x \cap M_i$ where $\uparrow_i$ is the up of $x$ with respect to $\leq_i$.

$\hat{x}_\alpha^i = \uparrow_{i\alpha} x \cap M_{i\alpha}$.

$\leq_{i\alpha}$ to be the induced specialization order of $\tau_{i\alpha}$.
**Theorem 2.4.12.** Let $(X, \tau_1(\leq_1))$, and $(X, \tau_2(\leq_2))$ are UB $T_o$ A-spaces. Then $\hat{x}_1 = \hat{x}_2$, $\forall x \in X$ if and only if $\tau_1$, and $\tau_2$ are $\alpha$-equivalent.

**Proof.** ($\Rightarrow$) Suppose that $\hat{x}_1 = \hat{x}_2$, $\forall x \in X$. Let $x, y \in X$ such that $x \leq_{1\alpha} y$, then $y \in \{x\} \cup \hat{x}_1 = \{x\} \cup \hat{x}_2$. Therefore $x \leq_{2\alpha} y$. Similarly if $x \leq_{2\alpha} y$ then $x \leq_{1\alpha} y$. Hence the two orders $\leq_{1\alpha}$, and $\leq_{2\alpha}$ are identical, and $\tau_{1\alpha} = \tau_{2\alpha}$.

($\Leftarrow$) If $\tau_{1\alpha} = \tau_{2\alpha}$, then $\leq_{1\alpha}$, and $\leq_{2\alpha}$ are coincide, and $\forall x \in X$, $\hat{x}_1 = \hat{x}_2$. By Theorem 2.4.4 $\hat{x}_1 = \hat{x}_2$, and $\hat{x}_2 = \hat{x}_2$. Hence $\hat{x}_2 = \hat{x}_1$. $\square$
Chapter 3

g-Closed Sets on UB $T_0$ A-spaces.

Closed sets are fundamental objects in a topological space. For example, one can define the topology on a set by using either the axioms for the closed sets or the Kuratowski closure axioms. Levine [38] introduced the class of g-closed sets, a super class of closed sets. Velicko, Arya, and Nour in[48] defined gs-closed sets in which were used for characterizing s-normal spaces. Dontchev [25], and Palaniappan introduced gsp-closed sets, gpr-closed sets, and r-g-closed sets respectively.

These topics of sets has been studied extensively in recent years by many topologist. More precisely,, they studied several new properties of these sets. Many of these new properties are separation axioms which are weaker than $T_1$ and used in computer science, and digital topology. Other new properties are defined by variations of the property of submaximality. Furthermore, the study of generalized closed sets also provides new characterizations of some known classes of spaces, for example, the class of extremally disconnected spaces.
3.1 g-Closed Sets and Their Properties

In this section, we give basic concepts of several types of g-closed sets and their relations in topological space \((X, \tau)\). Then we study these concepts on UB \(T_0\) A-space and give some characterization of some properties of g-closed sets.

**Definition 3.1.1.** A subset \(A\) of a topological space \((X, \tau)\) is called

(a) a *generalized closed set* (briefly g-closed)\([38]\), if \(\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\), and \(U\) is open in \((X, \tau)\).

(b) a *generalized semi-closed set* (briefly gs-closed)\([48]\), if \(\text{scl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open in \((X, \tau)\).

(c) a *semi-generalized closed set* (briefly sg-closed)\([43]\), if \(\text{scl}(A) \subseteq U\) whenever \(A \subseteq U\), and \(U\) is semi-open in \((X, \tau)\).

(d) a *generalized \(\alpha\)-closed set* (briefly \(g\alpha\)-closed)\([16]\), if \(\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\), and \(U\) is \(\alpha\)-open in \((X, \tau)\).

(e) an \(\alpha\)-*generalized closed set* (briefly \(\alpha\)g-closed)\([17]\) if \(\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\), and \(U\) is open in \((X, \tau)\).

(f) a *semi-pre open set*\([3]\) (\(=\beta\)-open \([33]\)), if \(A \subseteq \text{cl}(\text{int}(\text{cl}(A)))\), and a semi preclosed set \([3]\) (\(=\beta\)-closed\([33]\)) if \(\text{int}(\text{cl}(\text{int}(A))) \subseteq A\). The family of all \(\beta\) open sets in \(X\) is denoted by \(SPO(X)\).

(g) a *generalized semi-preclosed set* (briefly gsp-closed)\([25]\), if \(\text{spcl}(A) \subseteq U\) whenever \(A \subseteq U\), and \(U\) is open in \((X, \tau)\). (Where \(\text{spcl}(A)\) is the smallest \(\beta\)closed set containing \(A\))

(h) a *generalized preclosed set* (briefly gp-closed)\([18]\), if \(\text{pcl}(A) \subseteq U\) whenever \(A \subseteq U\), and \(U\) is open in \((X, \tau)\).
(i) *pre-generalized closed* (briefly, pg-closed)[18], if $pcl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is pre-open set.

A subset $A$ of $X$ is *g-open* (*sg-open*) if $A^c$ is *g-closed* (*sg-closed*). Other classes of generalized open sets are defined in a similar manner.

In [27], Dontchev summarized the fundamental relationships between various types of generalized closed sets in the following diagram. He proved that in general none of the implications in that diagram is reversible.

![Figure 3.1: Relation between g-closed sets](image)

The main purpose of this section is to investigate the several types of g-closed sets, which are defined above, on UB $T_0$ A-space, and to present new relations between these sets addition to those in general. Recall that a set $A$ is $\alpha$-open if and only if $A$ is semi-open, and pre-open.

Recall that by Corollary 2.1.16, we have that if $(X, \tau(\leq))$ is a UB $T_0$ A-space and if $A \subseteq X$, then $A$ is $\alpha$-open ($\alpha$-closed) if and only if $A$ is preopen (preclosed).

**Proposition 3.1.2.** If $(X, \tau(\leq))$ is a UB $T_0$ A-space, and $A \subseteq X$ then $A$ is gp-closed set if and only if $A$ is $\alpha g$-closed set
Proof. let \((X, \tau(\leq))\) be a UB \(T_0\) A-space, and \(A \subseteq X\). Then \(A\) is gp-closed set if and only if \(pcl(A) \subseteq U\) whenever \(A \subseteq U\), and \(U\) is open in \((X, \tau)\) if and only if \(cl_{\alpha}(A) \subseteq U\) whenever \(A \subseteq U\), and \(U\) is open in \((X, \tau)\) if and only if \(A\) is \(\alpha g\)-closed set. \(\square\)

**Proposition 3.1.3.** If \((X, \tau(\leq))\) is a UB \(T_0\) A-space and \(A \subseteq X\), then \(A\) is go\(\alpha\)-closed set if and only if \(A\) is pg-closed set.

*Proof.* Similar to the proof of Proposition 3.1.2. \(\square\)

**Proposition 3.1.4.** Let \((X, \tau(\leq))\) be UB \(T_0\) A-space and \(A \subseteq X\), then all the following are equivalent:

(a) \(A\) is go\(\alpha\)-closed set.

(b) \(A\) is preclosed set.

(c) \(A\) is \((\alpha\text{-closed})\) set.

*Proof.* (a) \(\Rightarrow\) (b) is true in any topological space.

(b) \(\Rightarrow\) (c) Since \(\tau_{\alpha} = PO(X)\) in UB \(T_0\) A-space, then if \(A\) is preclosed set, then \(A^c\) is preopen, and so \(\alpha\)-open. Hence \(A\) is \(\alpha\)-closed.

(c) \(\Rightarrow\) (a) is true in any topological space. \(\square\)

**Proposition 3.1.5.** If \((X, \tau(\leq))\) is a UB \(T_0\) A-space, and \(A \subseteq X\), then \(A\) is sg-closed set if and only if \(A\) is semi-closed set.

*Proof.* In any topological space if \(A\) is semi-closed set then it is sg-closed set. So, let \(A\) be sg-closed, and assume to contrary that \(A\) is not semi-closed set. Then \(A^c\) is not semi-open set. By Proposition 2.1.13, \(\exists r \in A^c\) such that \(\hat{r} \subseteq A\). Hence by Theorem 2.1.19 \(r \in scl(A)\). Now by proposition 2.1.13 \(G := A \cup \bigcup_{x \in A} \hat{x}\) is semi open set containing \(A\), and since \(A\) is sg-closed, then \(scl(A) \subseteq G\). So, \(r \in G\). Since \(r \notin A\), we must have that \(r \in \hat{x}\) for some \(x \in A\). That is \(r\) maximal element. S, \(\hat{r} = \{r\} \subseteq A\), which contradicts the fact that \(r \notin A\). \(\square\)
Proposition 3.1.6. If \((X, \tau(\leq))\) is a UB \(T_0\) A-space and \(A \subseteq X\), then \(A\) is semi-open set if and only if \(A\) is \(\beta\) open set. That is \(SO(X) = SPO(X)\).

Proof. \((\Rightarrow)\) Let \((X, \tau(\leq))\) be a UB \(T_0\) A-space, and suppose that \(A\) is semi-open set. If \(x \in A\), then \(\hat{x} \cap A \neq \emptyset\) and \(\exists s \in \hat{x} \cap A\). Hence \(s \in \overline{A}\) since \(s\) is maximum, so \(s \in \overline{A}\), and so \(x \in \overline{A}\). Therefore \(A \subseteq \overline{A}\) and \(A\) is \(\beta\) open set.

\((\Leftarrow)\) Suppose that \(A\) is \(\beta\) open set, that is \(A \subseteq \overline{A}\). Let \(x \in A\), then \(\exists a \in \overline{A}\) such that \(x \in \downarrow a\). Equivalently \(x \leq a\) which implies that \(\hat{x} \supseteq \hat{a}\). But \(a \in \overline{A}\), so \(\hat{a} \subseteq \overline{A}\), and \(\hat{x} \cap \overline{A} \neq \emptyset\). take \(r \in \hat{x} \cap \overline{A}\), Since \(r\) is maximum, then \(r \in \hat{x} \cap A\). Hence, we have that \(\hat{x} \cap A \neq \emptyset\) and by Proposition 2.1.13 \(A\) is semi-open set.

Corollary 3.1.7. If \((X, \tau(\leq))\) is a UB \(T_0\) A-space, and \(A \subseteq X\) then \(A\) is \(\beta\) open set if and only if \(\hat{x} \cap A \neq \emptyset\), \(\forall x \in A\).

Proof. Follows directly from Proposition 2.1.13, and Proposition 3.1.6.

Corollary 3.1.8. If \((X, \tau(\leq))\) is a UB \(T_0\) A-space and \(A \subseteq X\) then \(A\) is semi-closed set if and only if \(A\) is \(\beta\) closed set.

Corollary 3.1.9. If \((X, \tau(\leq))\) is a UB \(T_0\) A-space and \(A \subseteq X\), then the following are equivalent:

(a) \(A\) is semi-closed set.

(b) \(A\) is \(\beta\) closed set.

(c) \(A\) is sg-closed set.

Proof. Comes Directly by Corollary 3.1.8, and Proposition 3.1.5.

Ganster and Reilly [21] proved that, the statement "every \(\beta\) closed set of \(X\) is sg-closed" is equivalent to the statement "every preclosed subset of \(X\) is \(g\alpha\)-closed". This fact is obvious and satisfying in UB \(T_0\) A-spaces, see Corollary 3.1.9 and Proposition 3.1.4.

The following proposition characterizes the gp-closed set in UB \(T_0\) A-space.
Proposition 3.1.10. If \((X, \tau(\leq))\) is a UB T_0 A-space and \(A \subseteq X\) then \(A\) is gp\((=\alpha g)\) closed set if and only if \(\forall x \in A \cap M\) and, \(\forall a \in \downarrow x\) we have \(A \cap \downarrow a \neq \emptyset\). In topological language, \(A\) is gp-closed if and only if for any isolated point \(x\) in \(A\), and for any \(a \in \overline{\tau}\), we have that \(A \cap \overline{\tau} \neq \emptyset\).

Proof. \((\Rightarrow)\) Suppose that \(\exists x \in A \cap M\) and \(\exists a \in \downarrow x\) such that \(\downarrow a \cap A = \emptyset\). Then \(\forall r \in A\), \(a \notin \uparrow r\), hence \(a \notin \uparrow A\) where \(\uparrow A\) is the union of \(\uparrow x\), \(x \in A\), which is open set contains \(A\). By Theorem 2.1.19, \(a \in pcl(A)\), so \(pcl(A) \notin \uparrow A\). Therefore \(A\) not gp-closed set.

\((\Leftarrow)\) Let \(x \in pcl(A)\), then by Theorem 2.1.19 either \(x \in A\) or \(x \in \downarrow y\) for some \(y \in A \cap M\). Now if \(x \in A\), then \(x \in U\) where \(U\) any open set contains \(A\). Otherwise, and by given, we have that \(A \cap \downarrow x \neq \emptyset\). Let \(s \in A \cap \uparrow x\), then \(x \in \uparrow s\) and so \(x \in \uparrow A\). Hence if \(U\) is any open contains \(A\), then \(pcl(A) \subseteq \uparrow A \subseteq U\). Therefore \(A\) is gp-closed set.

Example 3.1.11. Let \(X = \mathbb{N}\) with revers order which is UB T_0 A-space. Then \(A = \{2n : n \in \mathbb{N}\}\) is gp-closed set.

Example 3.1.12. Let \(X = \{o, b, d, c, e, h, r\}\) be set with partial order as shown in the following figure below:

![Diagram]

Then \(A = \{o, c, h, r\}\) is gp-closed set but \(B = \{o, h, c\}\) is not gp-closed set, because \(o \in B \cap M, r \in \downarrow o\) but \(A \cap \downarrow r = \emptyset\). Note that the only open contains \(A\) is \(X\), and \(\overline{A} = X\), while \(B = X\) and \(U = X - \{r\}\) is open contains \(B\) but dosen’t contain \(\overline{B}\).
Corollary 3.1.13. If \((X, \tau(\leq))\) is a BB \(T_0\) A-space, and \(A \subseteq X\), then \(A\) is gp\((=\alpha g\)–closed\))-closed set if and only if \(\forall x \in A \cap M\), we have that \(\bar{x} \subseteq A\).

Proof. (\(\Rightarrow\))Let \(A\) be a gp-closed set then by Proposition 3.1.10, \(\forall x \in A \cap M\) and \(\forall a \in \downarrow x\), we have that \(A \cap \downarrow a \neq \emptyset\). If \(y \in \bar{x}\), then \(\downarrow y \cap A = A \cap \{y\} \neq \emptyset\). Therefore \(y \in A\) and \(\bar{x} \subseteq A\).

(\(\Leftarrow\)) Let \((X, \tau(\leq))\) be a B.B \(T_0\) A-space and \(A \subseteq X\) satisfies that \(\forall x \in A \cap M\), \(\bar{x} \subseteq A\). Hence if \(a \in \downarrow x\), then any \(r \in \check{a}\) is also in \(\bar{x}\) which is containing in \(A\). Therefor \(A \cap \downarrow a \neq \emptyset\), and by Proposition 3.1.10, \(A\) is gp-closed set.

Proposition 3.1.14. If \((X, \tau(\leq))\) is a UB \(T_0\) A-space and \(A \subseteq X\), then \(A\) is gs-closed set if and only if \(A\) is gsp-closed set.

Proof. Since \(\beta\)closed set is semi closed then semi pre closure is semi-clousre, and by the definition of gsp-closed set and the gs-closed, we have that \(A\) is gs-closed if and only if \(A\) is gsp-closed.

Proposition 3.1.15. If \((X, \tau(\leq))\) is a UB \(T_0\) A-space and \(A \subseteq X\), then \(A\) is gs\((=gsp)\)-closed set if and only if \(\forall x \in X\), if \(\hat{x} \subseteq A\), \(\downarrow x \cap A \neq \emptyset\)

Proof. (\(\Rightarrow\)) Assume to contrary that \(\exists r \in X, \hat{r} \subseteq A\), and \(\downarrow r \cap A = \emptyset\). By Theorem 2.1.19, \(r \in scl(A)\) take \(G := \uparrow A = \bigcup_{x \in A} \uparrow x\), which is open set containing \(A\). Then \(scl(A) \subseteq G\) and so \(r \in G\). That is, there exists \(x \in A\) such that \(r \in \uparrow x\). But this implies that \(x \in \downarrow r \cap A\) contradicts the fact that \(\downarrow \cap A = \emptyset\).

(\(\Leftarrow\)) Let \(G\) be open set contains \(A\), and let \(m \in scl(A)\), then by Theorem 2.1.19, \(m \in A\) or \(\hat{m} \subseteq A\). If \(m \in A\), then \(m \in G\). If \(\hat{m} \subseteq A\) then by given, \(\downarrow m \cap A \neq \emptyset\). Let \(r \in \downarrow m \cap A\), then \(r \in G\) (because \(G\) contains \(A\)). Hence \(\uparrow r \subseteq G\) (\(G\) is open.) Since \(m \in \uparrow r\), we have that \(m \in G\), and so \(scl(A) \subseteq G\). Therefore \(A\) is gs-closed set.

Example 3.1.16. Let \(X = \{a, b, d, c\}\) be a set with partial order as shown in the following figure below:
The set $A = \{a, b, d\}$ is gs-closed set, because $\hat{x} \subseteq A$ and $d \in \downarrow x \cap A \forall x \in X$, so $\downarrow x \cap A \neq \emptyset$ while $B = \{a, b, c\}$ is not gs-closed set, because $\hat{d} = \{a, b\} \subseteq B$ and $\downarrow d \cap B = \{d\} \cap B = \emptyset$.

Remark 3.1.17. In a $T_0$ A-space, since for any set $A$, the up set $\uparrow A = \bigcup_{x \in A} \uparrow x$ is the smallest open neighborhood of $A$, so we use without loss of generality $\uparrow A$ instead the open set $U$ contains $A$ in the definition of generalized closed set in Definition 3.1.1.

Proposition 3.1.18. If $(X, \tau(\leq))$ is a $T_0$ A-space, and $A \subseteq X$ then $A$ is g-closed set if and only if $\forall x \in A$, and $\forall a \in \downarrow x$ we have that $\downarrow a \cap A \neq \emptyset$.

Proof. ($\Rightarrow$) Suppose that $A$ is g-closed. So $A \subseteq U$ for any $U$ open set contains $A$. But $\uparrow A = \bigcup_{x \in A} \uparrow x$ is the smallest open set containing $A$, so $\overline{A} = \downarrow A \subseteq \bigcup_{x \in A} \uparrow x$. Now, let $x \in A$ and $a \in \downarrow x \subseteq \overline{A}$. Then $a \in \uparrow A = \bigcup_{x \in A} \uparrow x$. So $\exists y \in A$ such that $a \in \uparrow y$. Equivalently, $y \leq a$. Therefore $y \in \downarrow a$ and hence $\downarrow a \cap A \neq \emptyset$.

($\Leftarrow$) Suppose that $A$ is a subset of $X$ satisfying the above condition. It suffices to prove that $A \subseteq \bigcup_{x \in A} \uparrow x = \uparrow A$. Let $r \in \overline{A}$, then $r \in \downarrow x = \uparrow A$. for some $x \in A$. So, by given, $\downarrow r \cap A \neq \emptyset$. Let $y \in A$ such that $y \in \downarrow r$, then $r \in \uparrow y \subseteq \uparrow A$. Therefore $\overline{A} \subseteq \uparrow A$. $\blacksquare$
Remark 3.1.19. The results of this section about the relations between the various types of generalized closed sets in a UB $T_0$ A-space is summarized in the following diagram, where the continuous arrows refer to the results hold in any topological space (see Figure 3.1). The dotted arrows refer to the results hold in UB $T_0$ A-spaces. Moreover, in UB $T_0$ A-spaces, since we prove that semi-closed $\Rightarrow$ sg-closed $\Rightarrow$ gs-closed $\Rightarrow$ gsp-closed, we can omit the dashed arrow from $\beta$closed to gsp-closed.

![Diagram showing relations between g-closed sets on UB space](image)

Figure 3.2: Relations between g-closed sets on UB space

3.2 $\tau^*$ on Lower Bounded Space

Dunham[50] introduced a new type of closure operator $cl^*$ which is the intersection of all g-closed sets of any space $(X, \tau)$, and he proved that $cl^*$ is a Kuratowski closure operator on $X$. He studied the topology $\tau^*$ -the topology generated by $cl^*$- , and some of it’s properties. He proved $(X, \tau^*)$ is always a $T_0$-space. Then he improved this result by establishing the stronger result where $(X, \tau^*)$ is a $T_1$-space, for any topological space $(X, \tau)$.

**Theorem 3.2.1.** Let $(X, \tau(\leq))$ be an LB $T_0$ A-space, and $A$ is a subset of $X$ then $A$ is g-closed set if and only if $\forall x \in A$, $\bar{x} \subseteq A$. 

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Proof. ($\Rightarrow$) Suppose that $A$ is g-closed, and $x \in A$. Since $X$ is LB, $\bar{x} \neq \emptyset$. If $a \in \bar{x}$, then $a \in \downarrow x$ and by Proposition 3.1.18, $\downarrow a \cap A \neq \emptyset$. But $a \in m$, so $\downarrow a = \{a\}$. Hence $a \in A$ and this proves that $\bar{x} \subseteq A$.

($\Leftarrow$) Let $x \in A$ and $a \in \downarrow x$. By given, $\bar{x} \subseteq A$, and since $a \leq x$, $\bar{a} \subseteq \bar{x} \subseteq A$. Moreover, $\bar{a} \subseteq \downarrow a$, so $\bar{a} \subseteq \downarrow a \cap A$. Therefore, $\downarrow a \cap A \neq \emptyset$ and by Proposition 3.1.18, $A$ is g-closed.

Proposition 3.2.2. If $(X, \tau(\leq))$ is a LB $T_0$ $A$-space, and $A \subseteq X$, then $A$ is g-open if and only if $\uparrow x \subseteq A$ whenever $x \in m \cap A$.

Proof. ($\Rightarrow$) Suppose to contrary that $y \in m \cap A$ such that $\uparrow y \not\subseteq A$. So, $\exists r \geq y$ such that $r \not\in A$. Then $r \in A^c$ which is g-closed, and by above Theorem, $\bar{r} \subseteq A^c$. Since $y \in m$, $y \in \bar{r} \subseteq A^c$ which contradicts that $y \in A$.

($\Leftarrow$) Let $r \in A^c$ and let $s \in \bar{r}$ then $r \in \uparrow s$. If $s \in A$, then $s \in A \cap m$, and by given, $\uparrow s \subseteq A$. Hence $r \in A$ which is impossible. So, $s \in A^c$, and hence $\bar{r} \subseteq A^c$. Therefore by Theorem 3.2.1, $A^c$ is g-closed.

Corollary 3.2.3. In a LB $T_0$ $A$-space, every open set is g-open set. This fact is true in general, but it is clearly using the characterization in Proposition 3.2.2, by the help of category of poset.

Example 3.2.4. Let $X = \{a, b, d, c, e, f, g, h, r\}$ be set with partial order as shown in the following figure below:
Then \( A = \{a, e, r, g, f\} \) is g-closed set. To see this, note that \( \check{a} = \{f, g, r\} \subseteq A \), \( \check{e} = \{r\} \subseteq A \), \( \check{r} = \{r\} \subseteq A \), \( \check{f} = \{f\} \subseteq A \), and \( \check{g} = \{g\} \subseteq A \). \( B = \{d, f, c, g, r\} \) is not g-closed set, since \( h \in \check{c} \), and \( h \notin A \). The set \( D = \{h, c, f, d, a\} \) is g-open set, since \( \{h, f\} \subseteq A \cap m \), and \( \uparrow h \subseteq A \), \( \uparrow f \subseteq A \).

**Definition 3.2.5.** For a subset \( A \) of a topological \( X \), the generalized closure operator \( cl^*(A) \) [50] is defined to be the intersection of all g-closed sets containing \( A \). That is, \( cl^*(A) \) is the smallest g-closed set containing \( A \).

In [50], it was proved that \( cl^*(A) \) is a Kuratowski closure operator on \( X \), and so a topology \( \tau^* \) on \( X \) is generated by \( cl^*(A) \).

**Definition 3.2.6.** For a topological space \((X, \tau)\), the topology \( \tau^* \) is defined by \( \tau^* = \{U : cl^*(U^c) = U^c\} \)

It is worth mentioned that if \( X \) is LB \( T_0 \) A-space the arbitrary intersection of g-closed sets is g-closed set as we will see in Theorem 3.2.9, so \( cl^*(A) \) is g-closed set for any \( A \) subset of \( X \).

**Theorem 3.2.7.** If \((X, \tau(\leq))\) is a LB \( T_0 \) A-space and \( A \subseteq X \), then \( cl^*(A) = A \cup \bigcup_{x \in A} \check{x} \).

**Proof.** From Theorem 3.2.1, the set \( A \cup \bigcup_{x \in A} \check{x} \) is g-closed containing \( A \), so \( cl^*(A) \subseteq A \cup \bigcup_{x \in A} \check{x} \). Let \( S \) be any g-closed set contains \( A \). Then \( \check{x} \subseteq S \forall x \in S \). Now, if \( S \) \( A \cup \bigcup_{x \in A} \check{x} \), and \( y \notin A \), then \( y \in \check{x} \) for some \( x \in A \subseteq S \), so \( y \in S \). Therefore \( A \cup \bigcup_{x \in A} \check{x} \) is the smallest g-closed set contains \( A \). Therefore \( cl^*(A) = A \cup \bigcup_{x \in A} \check{x} \). \( \square \)

**Example 3.2.8.** Recall Example 3.2.4. Note that \( cl^*(A) = A \), so \( A^c \in \tau^* \), while \( cl^*(B) = B \cup \{h\} \neq B \), so \( B \) is not g-closed.

**Theorem 3.2.9.** If \((X, \tau(\leq))\) is a LB \( T_0 \) A-space, then \((X, \tau^*)\) is a \( T_0 \) A-space
Proof. Clearly \((X, \tau^*)\) is \(T_0\), since any open in \((X, \tau)\) is open in \((X, \tau^*)\). Let \(\{u_\alpha : \alpha \in \Delta\}\) be a collection of g-closed sets, and let \(x \in \bigcup_{\alpha \in \Delta} u_\alpha^c\). Then by Theorem 3.2.1 \(\check{x} \subseteq u_\alpha\) for some \(\alpha \in \Delta\). Hence, \(\check{x} \subseteq \bigcup_{\alpha \in \Delta} u_\alpha\), and so \(\bigcup_{\alpha \in \Delta} u_\alpha\) is g-closed. Therefore \((X, \tau^*)\) is A-space.

By this theorem, we see that \((X, \tau(\leq))\) is a \(T_0\) A-space. We denote its Alexandroff specialization order by \(\leq^*\). The following proposition describes the partial order \(\leq^*\) on \(X\).

**Proposition 3.2.10.** Let \((X, \tau(\leq))\) be a LB \(T_0\) A-space and \(x, y\) two elements in \(X\). Then \(x \leq^* y\) if and only if \(x \in \{y\} \cup \check{y}\).

**Proof.** Let \(x, y \in X\). Then \(x \leq^* y\) if and only if \(x \in cl^*\{y\}\) if and only if \(x \in \{y\} \cup \check{y}\). □

**Remark 3.2.11.** For the notations with respect to the order \(\leq^*\) we will use the following notations

- \(M^*\) to be the set of all maximal elements.
- \(m^*\) to be the set of all minimal elements.
- \(\uparrow^* x := \{y \in X : x \leq^* y\}\).
- \(\downarrow^* x := \{y \in X : y \leq^* x\}\).
- \(\check{x}^* := \uparrow^* x \cap M^*\).
- \(\hat{x}^* := \downarrow^* x \cap m^*\).

**Theorem 3.2.12.** Let \((X, \tau(\leq))\) be a LB \(T_0\) A-space, and \((X, \leq^*)\) be the induced space of all g-open sets. Then:

(a) each element in \(X\) either maximal or minimal with respect to \(\leq^*\).
(b) \((X, \leq^*)\) is submaximal space.

(c) \((X, \leq^*)\) is g-locally finite.

(d) \(m^* = m\).

(e) \(M^* = (X \setminus m) \cup m \cap M\).

Proof. (a) Suppose to contrary that there exist \(x \leq^* y \leq^* z\). Then \(y \neq z\) and \(y \leq^* z\). By Proposition 3.2.10, \(y \in \{z\} \cup \bar{z}\). Hence \(y \in \bar{z}\) which implies that \(y\) is minimal. contradicts that \(x \leq^* y\).

(b) Direct from Theorem 1.3.20.

(c) Any chain , increasing or decreasing in \(X\) has at most two element, so \(X\) is both Artinian and Noetherian space. Hence it is g-locally finite.

(d) Comes directly from (a) and (c).

The graph of the poset \((X, \leq^*)\) consists of two rows, the row of maximal elements, and the row of minimal elements where the order is described in Proposition 3.2.10. It is worth mentioning that these two rows may have an intersection, i.e., one element can be maximal, and minimal at the same time. For a subset \(A\) of \(X\), we can use the description of the partial order \(\leq^*\) to find \(Cl^*(A)\), which is the smallest down set with respect to \(\leq^*\) contains \(A\). Similarly \(int^*(A)\) is the largest up set with respect to \(\leq^*\) inside \(A\).

**Proposition 3.2.13.** If \((X, \tau(\leq))\) is a LB \(T_0\) \(A\)-space and \(A \subseteq X\), then \(int^*(A) = \{x \in A: \uparrow^* x \subseteq A\} = (A \setminus m) \cup \{x \in m : \uparrow x \subseteq A\}\).

Proof. Direct from the description of \(\leq^*\) in Proposition 3.2.10. 

\(\square\)
Example 3.2.14. Recall the example 3.2.4 The induced order $\leq^*$ on $X$ is given in figure below:

![Figure 3.3: $(X, \tau^*)$](image)

\textbf{Theorem 3.2.15.} If $(X, \tau(\leq))$ is LB $T_0$ A-space then

(a) $m \cap M = m^* \cap M^*$.

(b) $\downarrow^* x = \{x\} \cup \bar{x}$.

(c) if $x \not\in m$, $\uparrow^* x = \{x\}$ if $x \in m$, $\uparrow^* x = \uparrow x$.

\textit{Proof.} (a) $(\Rightarrow)$let $x \in m \cap M$, let $z \leq^* x$ then $z \in \{x\} \cup \bar{x}$ since $x$ is minimal element then $\{x\} = \bar{x}$. Hence $z \in \{x\}$, $z = x$ then $x$ is minimal element with respect to $\leq^*$. Let $x \leq^* z$ then $x \in \{z\} \cup \bar{z}$ then if $x \in \bar{z}$ then $x \leq z$, then $x = z$ since $x$ is maximal element. Therefore $x$ is maximal element with respect to $\leq^*$. Thus $x \in m^* \cap M^*$.

For the reverse inclusion, If $z \in m^* \cap M^*$ then $z \in m^* = m$. If $z \notin M$, then there is $y \in X$ such that $z \leq y$, so $z \in \bar{y}$, and hence $z \leq^* y$. This implies that $z \notin M^*$ contradicting $z \in m^* \cap M^*$. Therefore, $z \in m \cap M$.
(b) Directly by definition of \( \leq^* \).

(c) If \( x \notin m \) then for each element \( y \in X \), \( x \notin y \), which implies that there is no element \( y \in X \) such that \( x \leq^* y \), so \( x \in M^* \). Therefore \( \uparrow^* x = \{x\} \). Let \( x \in m \). If \( z \in \uparrow x \) then \( x \in z \) and hence \( x \leq^* z \), therefore \( z \in \uparrow^* x \). For the other inclusion, let \( w \in \uparrow^* x \), so \( x \leq^* w \), and hence \( x \in w \). Therefore \( w \in \uparrow x \). □

### 3.3 \( \alpha \)-Open Set and \( g \)-Closed Set on UB \( T_0 \) A-space

In general, if \( (X, \tau) \) is a topological space, then the collection \( \mathcal{F} \) of all closed sets need not form a topology on \( X \). If \( X \) is A-space, then arbitrary intersection of open sets is open. Equivalently, arbitrary union of closed sets is closed. This implies that \( \mathcal{F} \) forms a topology on \( X \), and this topology is A-space. In fact, \( \tau \) is Alexandroff topology on \( X \) if and only if \( \mathcal{F} \) is Alexandroff topology on \( X \). In this case, we denote \( \mathcal{F} \) by \( \tau^d \), the dual topology of \( \tau \).

Moreover, if \( (X, \tau) \) is \( T_0 \) A-space and if \( \leq \) is its (Alexandroff ) specialization order, then \( \tau^d \) is \( T_0 \) A-space. Denote its specialization order by \( \leq^d \). For \( x \in X \), the smallest nhhood \( V(x) \) of \( x \) in \( (X, \tau) \) is \( V(x) = \uparrow x \), the smallest up set in the poset \( (X, \tau(\leq)) \) containing \( x \), and the smallest nhhood \( V^d(x) \) of \( X \) in \( (X, \tau^d) \) is \( V^d(x) = \downarrow x \), the down set in the poset \( (X, \leq) \). Therefore we have that \( x \leq y \) if and only if \( V(x) \supseteq V(y) \) if and only if \( x \in \overline{y} = \downarrow y \) if and only if \( V^d(x) \subseteq V^d(y) \) if and only if \( y \leq^d x \). That is, \( \leq^d \) is the revers order of \( \leq \). Hence, if \( (X, \tau(\leq)) \) is a UB (resp. LB) \( T_0 \) A-space, then the dual space \( (X, \tau^d(\leq^d)) \) is an LB (resp. UB) \( T_0 \) A-space. Moreover, if \( (X, \tau(\leq)) \) is a \( T_0 \) A-space, then we can construct the dual \( (X, \tau^d) \) reversing the order of poset. The set of maximal element \( M^d \) in the corresponding poset \( (X, \leq^d) \) equals the set of minimal points \( m \) in \( (X, \leq) \), and similarly \( m^d = M \). The up set of \( A \) with resp. to \( \leq^d \) is denoted by \( \uparrow^d(A) \). For \( x \in X \), \( \uparrow^d(x) \cap M^d = \hat{x}^d \) and \( \downarrow^d \cap m^d = \bar{x}^d \). It is not difficult to prove that \( \hat{x}^d = \hat{x} \) and \( \bar{x}^d = \bar{x} \).

**Example 3.3.1.** Let \( X = \{a, b, d, c, d, e, f, g, h, r\} \) be a set with partial order as shown in

\[ \begin{array}{cccccccc}
\text{a} & \text{b} & \text{c} & \text{d} & \text{e} & \text{f} & \text{g} & \text{h} & \text{r} \\
\text{b} & \text{c} & \text{d} & \text{e} & \text{f} & \text{g} & \text{h} & \text{r} & \text{b} \\
\text{c} & \text{d} & \text{e} & \text{f} & \text{g} & \text{h} & \text{r} & \text{b} & \text{c} \\
\text{d} & \text{e} & \text{f} & \text{g} & \text{h} & \text{r} & \text{b} & \text{c} & \text{d} \\
\text{e} & \text{f} & \text{g} & \text{h} & \text{r} & \text{b} & \text{c} & \text{d} & \text{e} \\
\text{f} & \text{g} & \text{h} & \text{r} & \text{b} & \text{c} & \text{d} & \text{e} & \text{f} \\
\text{g} & \text{h} & \text{r} & \text{b} & \text{c} & \text{d} & \text{e} & \text{f} & \text{g} \\
\text{h} & \text{r} & \text{b} & \text{c} & \text{d} & \text{e} & \text{f} & \text{g} & \text{h} \\
\text{r} & \text{b} & \text{c} & \text{d} & \text{e} & \text{f} & \text{g} & \text{h} & \text{r} \\
\end{array} \]
The following figure:

Then the corresponding poset of the dual space \((X, \tau^d)\) is given in the following figure:

\[ \text{Theorem 3.3.2.} \text{ Let } (X, \tau(\leq)) \text{ be a LB } T_0 \text{ A-space, and } A \text{ is a subset of } X. \text{ Then } A \text{ is } g\text{-closed in } (X, \tau) \text{ if and only if } A \text{ is } \alpha\text{--open in the dual space } (X, \tau^d(\leq^d)), \text{ and vice versa.} \]

\[ \text{Proof.} \text{ } A \text{ is } g\text{-closed set in } (X, \tau(\leq)) \text{ if and only if } \hat{x} \subseteq A \forall x \in A \text{ Theorem 3.2.1 if and only if } \hat{x}^d (= \uparrow x \cap M^d = \downarrow x \cap m = \hat{x}) \text{ is subset of } A \forall x \in A \text{ if and only if } A \text{ is } \alpha\text{-open set in } (X, \tau^d) \text{ by Proposition 2.1.11.} \]

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Proposition 3.3.3. Let \((X, \tau(\leq))\) be a UB \(T_0\) \(A\)-space, then \((\tau_\alpha)^* = \tau_\alpha\)

Proof. Let \((X, \tau(\leq))\) be a UB \(T_0\) \(A\)-space then \((X, \tau_\alpha)\) is \(T_\frac{1}{2}\) and any \(g\)-closed set in \(\tau_\alpha\) is closed. Hence any \(g\)-open is open. Therefore \((X, \tau_\alpha^*) = (X, \tau_\alpha)\). □

Proposition 3.3.4. Let \((X, \tau(\leq))\) be an UB \(T_0\) \(A\)-space then \(((\tau^d)_\alpha)^d = \tau^*\)

Proof. Let \((X, \tau(\leq))\) be an UB \(T_0\) \(A\)-space let \(U \in ((\tau^d)_\alpha)^d\) if and only if \(U\) is closed in\(((\tau^d)_\alpha)\) if and only if \(U\) is preclosed in \((X, \tau^d)\) if and only if \(U\) is \(g\)-open in \((X, \tau)\) (Theorem 3.3.2) if and only if \(U\) is open in \((X, \tau^*)\). □

Proposition 3.3.5. If \((X, \tau(\leq))\) is LB \(T_0\) \(A\) space, and \(A \subseteq X\) then \(A\) is clopen if and only if \(A\) is \(g\)-clopen

Proof. Let \(A\) be a clopen in \(\tau\), then \(A\) is clopen in \(\tau^d\) then By Theorem 2.4.11, \(A\) is preclopen, hence \(A\) is \(g\)-clopen in \(\tau\). □
Chapter 4

Generalized Separation Axioms on UB $T_0$ A-spaces.

In this chapter we will talk about separation axioms, submaximality and, extremally disconnected prosperities. Some of these results hold in UB $T_0$ A-spaces, but others hold in $T_0$ A-spaces. We will study, in section 4.1, lower separation axioms on UB $T_0$ A-spaces which was proved in Artinian $T_0$ A-spaces, to generalized these results. In section 4.2, we will deal with new separation axioms depending on several kinds of g-closed sets. In section 4.3, we present new topological concepts such as sg-submaximal, g-submaximal, $\alpha$-submaximal, and extremally disconnected. Then we study how they related to each other on UB $T_0$ A-spaces.

4.1 Lower Separation Axioms

In this section, we summarize the definitions and results of lower separation axioms studied in [14]. Some of these results hold in Artinian $T_0$ A-spaces, while others hold in $T_0$ A-spaces. We generalize the results that hold in Artinian $T_0$ A-spaces to either UB $T_0$ A-spaces or to $T_0$ A-spaces.
**Definitions 4.1.1.** A topological space \((X, \tau)\) is

1. *a semi-\(T_o\)-space* [46] if whenever \(x\) and \(y\) are distinct points in \(X\), there is a semi-open set containing one, and not the other.

2. *an \(\alpha - T_o\) space* [16] if whenever \(x\), and \(y\) are distinct points in \(X\), there is an \(\alpha\)-open set containing one, and not the other.

3. *a \(T_{1\frac{1}{2}}\) space* [9] if for every finite subset \(F\) of \(X\), and every \(y \notin F\), there exists a set \(A_y\) containing \(F\), and disjoint from \(\{y\}\) such that \(A_y\) is either open or closed.

4. *a \(T_{\frac{3}{2}}\) space* [10] if for every compact subset \(F\) of \(X\), and every \(y \notin F\), there exists a set \(A_y\) containing \(F\), and disjoint from \(\{y\}\) such that \(A_y\) is either open or closed.

5. *\(T_{1\frac{1}{2}}\)-space* [38] if every g-closed set is closed. Equivalently, if every singleton is either open or closed [51].

6. *a semi-\(T_{1\frac{1}{2}}\)-space* [43] if every sg closed set is semi-closed. Equivalently, if every singleton is either semi-open or semi-closed [31].

7. *an \(\alpha-T_{1\frac{1}{2}}\) space* if every singleton is either \(\alpha\)-open or \(\alpha\)-closed.

8. *a \(T_{\frac{3}{2}}\) space* [24] if every singleton is either regular open or closed.

9. *a semi- \(T_1\) space* [46] if whenever \(x\), and \(y\) are distinct points in \(X\), there is a semi-open set of each not containing the other. Equivalently, if every singleton is semi-closed.

10. *an \(\alpha-T_1\) space* [16] if whenever \(x\), and \(y\) are distinct points in \(X\), there is an \(\alpha\)-open set of each not containing the other. Equivalently, if every singleton is \(\alpha\)-closed [32].

11. *a \(T_1^*\) space* [35] if every nowhere dense subset of \(X\) is union of closed sets.
(12) a *feebly-\( T_1 \)* space [6] if every singleton is nowhere dense or clopen. Equivalently, if every preopen singleton \( \{x\} \) is closed, and hence clopen [24].

(13) a *semi-\( T_2 \)* space [46] if whenever \( x \) and \( y \) are distinct points in \( X \), there are disjoint semi-open sets \( U \) and \( V \) in \( X \) with \( x \in U \), and \( y \in V \).

(14) an \( \alpha - T_2 \) space [47] if whenever \( x \) and \( y \) are distinct points in \( X \), there are disjoint \( \alpha \)-open sets \( U \) and \( V \) in \( X \) with \( x \in U \) and \( y \in V \).

(15) a *semi-\( T_D \)-space* [6] if every singleton is either open or nowhere dense.

By definitions above, the space \( (X, \tau) \) is \( \alpha - T_i \) space if and only if the space \( (X, \tau_\alpha) \) is \( T_i \) space, \( i = 0, \frac{1}{2}, 1, 2 \). It is well-known that \( X \) is \( T_{3\frac{1}{2}} \) [24] if and only if \( X \) is both \( T_2 \), and semi-\( T_1 \). Cueva, and Saraf in [31] show that the space is semi-\( T_{\frac{3}{2}} \) if and only if it is semi-\( T_D \). The implication of Figure 4.1 hold in any topological space, and in general non of these implication reversed.

Suppose that \( (X, \tau(\leq)) \) be a \( T_0 \) A-space, then \( X \) is a \( T_1 \)-space if and only if it is
discrete. So these is no interest of studying separation axioms $T_i$ for $i \geq 1$. In spite of this, separation axioms in $T_0$ A-spaces have special important as we will see in the following Theorems

**Theorem 4.1.2.** [14] Let $(X, \tau(\leq))$ be a $T_\omega$ A-space. Then the following are equivalent:

1. $X$ is $T_\frac{1}{2}$.
2. $X$ is $T_\frac{1}{3}$.
3. $X$ is $T_\frac{1}{4}$.
4. Each element of $X$ is either maximal or minimal.
5. $X$ is submaximal.
6. $X$ is $T_1^*$-space.

**Theorem 4.1.3.** [14] Let $(X, \tau(\leq))$ be a $T_\omega$ A-space. Then $X$ is $T_\frac{1}{4}$-space if and only if the following two conditions are satisfying:

1. $X$ is $T_\frac{1}{2}$.
2. $\forall x \notin M, \ |\hat{x}| \geq 2$, where $\ |\hat{x}|$ is the cardinality of the set $\hat{x}$.

**Theorem 4.1.4.** [14] Let $(X, \tau(\leq))$ be an Artinian $T_\omega$ A-space. Then $X$ is $T_1$ if and only if $X$ is feebly-$T_1$-space.

In the following Theorem, we generalize the result of Theorem 4.1.4 to $T_\omega$ A-space.

**Theorem 4.1.5.** Let $(X, \tau(\leq))$ be a $T_\omega$ A-space, such that $\hat{x} \neq \emptyset$, $\forall x \in X$ Then $X$ is $T_1$ if and only if $X$ is feebly-$T_1$-space.(In this case, $X$ has the discrete topology )
Proof. ($\Rightarrow$) $T_1 \Rightarrow \text{feebly} - T_1$ is true in any space.

($\Leftarrow$) Suppose that $X$ is not $T_1$ (= not discrete topology ). Let $x \in X$ be not open, and let $\exists y \in \hat{x}$. Clearly $y \neq x$, and $\{y\}$ not closed ($\downarrow y \neq \{y\}$). Hence $\{y\}$ is not clopen. Moreover $y \in (\{y\})^\circ$, so $\{y\}$ is nowhere dense. Hence $X$ is not feebly-$T_1$-space.

We can not omit the condition ”$\hat{x} \neq \emptyset$, $\forall x \in X$” from the above theorem. In $T_o$ A-spaces, feebly-$T_1$ may not imply $T_1$. The following example gives a feebly-$T_1$- space which is not $T_1$.

**Example 4.1.6.** Let $X = \mathbb{N}$ with the usual order. If $a \in \mathbb{N}$ then $\overline{\{a\}} = \{1, 2, 3, \ldots, a\}$. So $\overline{\{a\}} = \emptyset$, and hence each singleton is nowhere dense, therefore $X$ is feebly-$T_1$- space while it is not $T_1$.

**Remark 4.1.7.** In Artinian $T_0$ A-spaces, we prove that $\forall x \in X$, $\hat{x} \neq \emptyset$, so the condition of Theorem 4.1.5 holds and we get the result of Theorem 4.1.4 in [14] as a corollary. In fact, this result can be arises to UB $T_o$ A-spaces, since $\hat{x} \neq \emptyset \forall x \in X$, to get the following corollary:

**Corollary 4.1.8.** Let $(X, \tau(\leq))$ be a $UB T_o$ A-space. Then $X$ is $T_1$ if and only if $X$ is feebly-$T_1$-space.

**Theorem 4.1.9.** [14] An Artinian $T_o$ A-space is $\alpha T_1$ if and only if $X$ is discrete

**Theorem 4.1.10.** An $UB T_o$ A-space is $\alpha - T_1$ if and only if $X$ is discrete

Proof. ($\Rightarrow$)Let $(X, \tau(\leq))$ be a $T_o$ A-space. Suppose that $X$ is $\alpha - T_1$ space. Then $(X, \tau_\alpha)$ is $T_1$ since $(X, \tau_\alpha)$ is $T_o$ A-space, we get that, $(X, \tau_\alpha)$ is discrete. Hence by Theorem 2.4.4, $X = M_\alpha = M$. Thus $(X, \tau)$ is discrete.

($\Leftarrow$) $\tau_\alpha = \tau$ so $X$ is $\alpha - T_1$.

**Theorem 4.1.11.** [14] Let $(X, \tau(\leq))$ be a an Artinian $T_o$ A-space. Then
(a) $X$ is an $\alpha - T_{\frac{1}{2}}$ space.

(b) $X$ is a semi-$T_{\frac{1}{2}}$ space.

Without essential change in the proof in [14] the result of Theorem 4.1.11 is still true in UB $T_o$ A-space.

**Theorem 4.1.12.** Let $(X, \tau(\leq))$ be a an UB $T_o$ A-space. Then

(a) $X$ is an $\alpha - T_{\frac{1}{2}}$ space.

(b) $X$ is a semi-$T_{\frac{1}{2}}$ space.

**Proof.** (a) Let $(X, \tau(\leq))$ be a UB $T_o$ A-space. Then by Corollary 2.1.17, $(X, \tau_a)$ is submaximal, and by Theorem 1.3.20, $(X, \tau_a)$ is $T_{\frac{1}{2}}$-space. Therefore $(X, \tau)$ is $\alpha - T_{\frac{1}{2}}$-space.

(b) Let $x \in X$. If $x \in M$, then $\{x\}$ is open and hence it is semi-open. If $x \notin M$, then $x \notin \hat{y} \forall y \in X$. Hence if $y \in X - \{y\}$, then $\hat{y} \cap X - \{y\} = \emptyset$. This implies that $X - \{x\}$ is semi-open and so $\{x\}$ is semi-closed.

Theorem 4.1.13. [14] Let $(X, \tau(\leq))$ be a an Artinian $T_o$ A-space. Then all the following are equivalent:

1. $X$ is semi-$T_2$-space.
2. $X$ is semi-$T_1$-space.
3. $\forall x \notin M, |\hat{x}| \geq 2$.

**Theorem 4.1.14.** Let $(X, \tau(\leq))$ be a an UB $T_o$ A-space. Then all the following are equivalent:
(1) $X$ is semi-$T_2$-space.

(2) $X$ is semi-$T_1$-space.

(3) $\forall x \notin M, |\hat{x}| \geq 2$.

Proof. Again, the proof is similar to the proof of Theorem 4.1.13 in [14].

4.2 Separation Axioms Depending on Generalized Closed Sets

In this section, we will talk about Separation axioms depending on generalized closed sets. We will study their definitions, and by using our result in chapter two and three, we will produce new theorems.

Definitions 4.2.1. A topological space $(X, \tau)$ is said to be

(1) *locally indiscrete* if every open subset is closed.

(2) *nodeg* if every nowhere dense set of $X$ is $g$-closed.

(3) a *semi − pre − $T_\frac{1}{2}$ space* [25] if every gsp-closed set in it is $\beta$-closed.

(4) *nodec* [8] if every nowhere dense set of $X$ is closed.

(5) a *$T_b$ space* [44] if every gs-closed set in it is closed.

(6) a *$T_d$ space* [44] if every gs-closed set in it is $g$-closed.

(7) an *$aT_d$ space* [45] if every $ag$ − *closed set* in it is $g$-closed.

(8) an *$aT_b$ space* [45] if every $ag$ − *closed set* in it is closed.
(9) a $T_{gs}$ [15] if every $gs$-closed subset of $X$ is $sg$-closed; or equivalently, if for each $x \in X$, \{x\} is either closed or preopen [23].

Recall that a $g$-closed set is $\alpha g$–closed set and the covers is not true in general. The following Theorem characterize when $\alpha g$–closed set is $g$-closed set in UB $T_0$ A-space.

**Proposition 4.2.2.** If $(X, \tau(\leq))$ is a $T_0$ A-space then the space, is $T_{\frac{1}{2}}$ if and only if any $\alpha g$-closed subsets are $g$-closed sets.

**Proof.** ($\Rightarrow$) Let $A$ be an $\alpha g$-closed set, $x \in A$, and $r \leq x$. Since $x$ is either maximal or minimal (X is $T_1$), if $x \in M \cap A$, and since $r \in \downarrow x$, by Proposition 3.1.10, $\exists a \in \downarrow r \cap A$, and hence $\downarrow r \cap A \neq \emptyset$. If $x \in m$, then $r = x$ and $\downarrow r = \{x\} \subseteq A$. Thus, $\downarrow r \cap A \neq \emptyset$. In both cases, we have that $\forall x \in A$ and $\forall r \in \downarrow x$, $\downarrow r \cap A \neq \emptyset$ and by Proposition 3.1.18, we get that $A$ is g-closed.

($\Leftarrow$) Suppose that $(X, \tau(\leq))$ is not $T_{\frac{1}{2}}$. Then there exist $x, y, z$ belong to $X$ such that $x < y < z$. Let $A = \{y\}$ then $(A)^o = \emptyset$ and $(\overline{A})^o = \emptyset = \emptyset \subseteq A$. Hence $A$ is $\alpha$-closed set.

This implies that $cl_\alpha(A) = A \subseteq U$ for any $U$ open set contains $A$. Thus, $A$ is $\alpha g$-closed set, the open set $G = \uparrow y$ contains $A$, and since $x \in \overline{A}$, $x \notin G$, we get that is, $\overline{A} \notin G$, so $A$ is not g-closed set.

**Corollary 4.2.3.** If $(X, \tau(\leq))$ is $T_0$ A-space then the following are equivalent:

(a) $X$ is $T_{\frac{1}{2}}$-space.

(b) $X$ is $\alpha T_d$ space.

(c) $X$ is $\alpha T_b$ space.

**Proof.** (a) $\Leftrightarrow$ (b) Comes directly from Proposition 4.2.2 and the definition of $\alpha T_d$.

(b) $\Rightarrow$ (c) Let $(X, \tau(\leq))$ be a $T_0$ A-space $X$. If $X$ is $\alpha T_d$ then by equivalent between (a) and (b), $X$ must be $T_{\frac{1}{2}}$. Now let $A$ be $\alpha g$-closed set, then by Proposition 4.2.2, $A$ is g-closed set. Hence from the definition of $T_{\frac{1}{2}}$, $A$ must be closed and $X$ is $\alpha T_b$ space.
(c) ⇒ (b) This direction is true in any topological space. \[\square\]

It is worth mentioned that $T_0$ condition is necessary in the above theorem. In A-spaces, $\alpha T_d$ spaces, are not equivalent to $T_{\frac{1}{2}}$-spaces as the following example shows:

**Example 4.2.4.** Let $X = \{a, b, c\}$, and $\tau = \{X, \emptyset, \{a, b\}\}$ then the only proper closed set is $\{c\}$. If $A \neq \{c\}$, then $\overline{A} = X$. Hence $\overline{A} = X, \forall A \neq \{c\}$ and $\overline{A} \notin A$. Thus, the collection of $\alpha$-closed sets is $\{X, \emptyset, \{c\}\}$, and so $\tau_\alpha = \tau$. Therefore $cl_\alpha(A) = cl(A)$, and any $\alpha g$-closed set is $g$-closed set. This implies that $X$ is $\alpha T_d$-space. Since $X$ is not $T_0$, then the space not $T_{\frac{1}{2}}$.

**Theorem 4.2.5.** [22],[20] For a space $(X, \tau)$ the following are equivalent:

1. $(X, \tau)$ is a $T_{gs}$ space.
2. every gp-closed subset of $(X, \tau)$ is preclosed.
3. every gp-closed subset of $(X, \tau)$ is $\beta$ closed.
4. every singleton of $(X, \tau)$ is either preopen or closed.
5. every nowhere dense subset of $(X, \tau)$ is a union of closed subsets (i.e $X$ is $T^*_1$).
6. every gsp-closed set is $\beta$ closed, i.e. $(X, \tau)$ is semi $- pre - T_{1/2}$.
7. every $\alpha g$-closed subset of $(X, \tau)$ is $g\alpha$-closed.
8. every gp-closed subset of $(X, \tau)$ is $\beta$ closed.

If $X$ is Alexandroff space, then the arbitrary union of closed set is closed, and so we get the following theorem.

**Theorem 4.2.6.** For Alexandroff space $(X, \tau)$, the following are equivalent:

a. Every nowhere dense subset of $(X, \tau)$ is a union of closed sets (i.e $X$ is $T^*_1$).
(b) Every nowhere dense subset of \((X, \tau)\) is closed set. In this case, \(X\) is nodec.

Hence the statement in (b) is equivalent to each statement in Theorem 4.2.5.

**Theorem 4.2.7.** [20] For the space \((X, \tau)\) the following are equivalent:

1. Every \(g\alpha\)-closed set is \(g\)-closed.
2. Every nowhere dense subset is locally indiscrete as a subspace.
3. Every nowhere dense subset is \(g\)-closed.
4. Every \(\alpha\)-closed set is \(g\)-closed.

**Theorem 4.2.8.** If \((X, \tau(\leq))\) is \(T_0\) A-space then the following equivalent:

1. Every nowhere dense subset is \(g\)-closed.
2. \(X\) is \(T_{1\frac{1}{2}}\) space.

**Proof.** \((\Rightarrow)\) Suppose that \(X\) is nodeg, and \(X\) is not \(T_{1\frac{1}{2}}\)-space. Then \(\exists x, y, z\) such that \(x \leq y \leq z\). Then if we take \(A = \{y\}\) is nowhere dense, since \(A^\circ = \emptyset\) but by Proposition 3.1.18, \(A\) is not \(g\)-closed, since \(y \in A\), and \(x \leq y\), but \(\downarrow x \cap A = \emptyset\).

\((\Leftarrow)\) Let \(X\) is \(T_{1\frac{1}{2}}\). Then any singleton is either maximal or minimal. Let \(A\) be a nowhere dense, then \(A \cap M = \emptyset\). Thus, \(A \subseteq m\), and \(\bar{x} = x \forall x \in A\). By 3.1.18, \(A\) is \(g\)-closed. \(\Box\)

**Theorem 4.2.9.** [20] Every \(T_{1\frac{1}{2}}\)-space is \(T_{g\delta}\).

In general, the converse of Theorem 4.2.8, is not true, but it is in true if \(X\) is \(T_o\) A-space. The following comprehensive Theorem summarize the results of \(T_o\) A-space overall the theorems and remarks that characterize the separation axioms \(T_{1\frac{1}{2}}\). Basically, we use the results of Theorems 4.1.2, 4.2.3, 4.2.6, 4.2.7, and 4.2.8.

**Theorem 4.2.10.** Let \((X, \tau(\leq))\) be a \(T_o\) A-space. Then the following are equivalent:
(1) $X$ is $T_{\frac{1}{2}}$.

(2) $X$ is $T_{\frac{1}{3}}$.

(3) $X$ is $T_{\frac{1}{4}}$.

(4) each element of $X$ is either maximal or minimal.

(5) $X$ is submaximal.

(6) $X$ is $aT_b$ space.

(7) $X$ is $T_1^*$-space.

(8) $(X, \tau)$ is a $T_{gs}$ space.

(9) every $gp$-closed subset of $(X, \tau)$ is preclosed.

(10) every singleton of $(X, \tau)$ is either preopen or closed.

(11) every $gsp$-closed set is $\beta$closed, i.e. $(X, \tau)$ is semi – pre – $T_{1/2}$.

(12) every $ag$-closed subset of $(X, \tau)$ is $ga$-closed.

(13) every $gp$-closed subset of $(X, \tau)$ is $\beta$closed.

(14) every nowhere dense subset of $(X, \tau)$ is closed subset i.e. $(X, \tau)$ is nodec.

(15) Every $ga$-closed set is $g$-closed,

(16) every nowhere dense subset is locally indiscrete as a subspace.

(17) every nowhere dense subset is $g$-closed, i.e. $(X, \tau)$ is nodeg.

(18) every $\alpha$-closed set is $g$-closed.

(19) $X$ is $\alpha T_d$ space.
Example 4.2.11. Again, $T_0$ condition is necessary in the above theorem, to see this, let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a, b\}\}$, $X$ is $T_{gs}$-space, since $\overline{\{a\}} = X^o = X \supseteq \{a\}$, $\overline{\{b\}} = X^o = X \supseteq \{b\}$. So $a, b$ are preopen, and $c$ is closed; that is, every singleton of $(X, \tau)$ is either preopen or closed. But $X$ is not $T_0$, so it is not $T_{1\frac{1}{2}}$.

In [20], the next example was given to show that there exist spaces whose nowhere dense subsets are g-closed but which are not nodec. In addition, we here use it to show that $T_0$ condition is necessary for equality between nodeg and nodec propriety.

Example 4.2.12. Let $X = \mathbb{R}$ be the set of all real numbers, let $\mathbb{R}^+ = (0, \infty)$, and $\mathbb{R}^- = (-\infty, 0)$. Define a topology on $X$ as follows: $\tau = \{X, \emptyset, \mathbb{R}^-, \mathbb{R}^+ \cup \{0\}\}$. Then the collection of closed sets is $\mathfrak{F} = \{X, \emptyset, \mathbb{R}^-, \mathbb{R}^+ \cup \{0\}\}$. For any $x \in \mathbb{R}^+$, $\overline{\{x\}} = \emptyset$, so $\overline{\{x\}}^o = \emptyset$. Thus, $\{x\}$ is non-closed nowhere dense in $X$. Hence the space is not nodec. To prove that $X$ is nodeg, note firstly that if $0 \in A$, then $\overline{A} = X$ and hence $\overline{A} \neq \emptyset$, so if $A$ is nonempty nowhere dense and $U$ contains $A$, we have that $0 \notin A$ and so $U \neq \{0\}$. If $U = X$, then Clearly $\overline{A} \subseteq U$. If $U = \mathbb{R}^+ \cup \{0\}$, then $A \subseteq \mathbb{R}^+$ and so $\overline{A} = \mathbb{R}^+ \subseteq U$. Finally If $U = \mathbb{R}^- \cup \{0\}$, then $A \subseteq \mathbb{R}^-$ and so $\overline{A} = \mathbb{R}^- \subseteq U$. In all possible cases for $A$, $\text{cl}(A) \subseteq U$, and $A$ is g-closed set. Hence $X$ nodeg.

Theorem 4.2.13. Let $(X, \tau(\leq))$ be a $T_0$ $A$-space. If $X$ is $T_d$, then $X$ is $T_{1\frac{1}{2}}$.

Proof. If $X$ is not $T_{1\frac{1}{2}}$, then $\exists x, y, z \in X$ such that $x < y < z$. Take $A = \{y\}$. $\forall r \leq y$, we have that $z \in \uparrow r$. Since $z \notin \downarrow y = \overline{\{y\}}$, we have that $\uparrow r \notin \overline{\{y\}}$. So, $\overline{\{y\}}^o = \emptyset \subseteq \{y\}$ then $\{y\}$ is semi-closed set. This implies that $\text{scl}(\{y\}) = \{y\} \subseteq U$ for any open set $U$ containing $\{y\}$. Thus, $\{y\}$ is gs-closed set. Furthermore, $U = \uparrow y$ is an open set containing $\{y\}$, and since $x \in \text{cl}(\{y\})$, $x \notin U$, we get that $\text{cl}(\{y\}) \notin U$. Thus, $\{y\}$ is not g-closed set. Therefore $X$ is not $T_d$ space.

The converse of the above Theorem is not true, as the following example shows:
Example 4.2.14. Let $X = \{1,2,3,4,5\}$, and define a partial order $\leq$ on $X$ by the following diagram:

```
  3       4       5
  |       |       |
  v       v       v
  1 - 2
```

The topology $\tau(\leq)$ is the $T_0$ $A$-space induced on $X$ by $\leq$. This topology is $T_\frac{1}{2}$ since each element is either maximal or minimal. Now consider the set $A = \{4\}$. Since $\overline{A} = \{4\} \subseteq A$, we get that $A$ is semi-closed and so $\text{scl}(A) = A \subseteq U$ for any open set $U$ containing $A$. Hence $A$ is gs-closed set in $X$. But $\uparrow A = \{4\}$ is open containing $A$ and $\overline{A} = \{1,2,4\} \nsubseteq \uparrow A$. Thus $A$ is not g-closed. Therefore $X$ is not $T_d$ space.

In the above theorem, we can not omit the $T_0$ condition. As the following example shows:

Example 4.2.15. Recall Example 4.2.11 which is not $T_\frac{1}{2}$. For any nonempty proper subset $A$ of $X$ such that $A \neq \{c\}$ we have that $\overline{A} = X \nsubseteq A$. Hence $SO(X) = \tau$ and so $\text{cl}(A) = \text{scl}(A) \forall A \subseteq X$. Thus, any gs-closed set is g-closed set and the space is $T_d$.

Lemma 4.2.16. Any $T_b$ space is $T_\frac{1}{4}$-space.

Proof. Let $(X, \tau)$ be a $T_b$ space. If $A$ is g-closed set, then $A$ is gs-closed set. Since $X$ is $T_b$ space, we get that, $A$ is closed set. Thus $X$ is $T_\frac{1}{2}$-space. \hfill \qed

Lemma 4.2.17. Any $T_b$ space is $T_d$ space.
Proof. Let \((X, \tau)\) be a \(T_b\) space. If \(A\) is a gs-closed set, then \(A\) is closed set. Hence \(A\) is g-closed set. Therefore \((X, \tau)\) is a \(T_d\) space. \(\square\)

The converse of the above theorem is not true in general, even if \(X\) is Alexandroff space, as we will see in Example 4.2.19 but if the space is \(T_o\) A-space, we have the following theorem:

**Theorem 4.2.18.** Let \((X, \tau(\leq))\) be a \(T_o\) A-space. Then if \((X, \tau(\leq))\) is \(T_d\) if and only if \((X, \tau(\leq))\) is \(T_b\).

**Proof.** \((\Rightarrow)\) By Lemma 4.2.17.

\((\Rightarrow)\) Let \(X\) be a \(T_o\) A-space. Suppose that \(A\) is gs-closed set. Then \(A\) is g-closed set. Since \(X\) is \(T_d\), by Theorem 4.2.13, \(X\) is \(T_\frac{1}{2}\). Thus, \(A\) is closed set and therefore \(X\) is \(T_b\) space. \(\square\)

**Example 4.2.19.** Recall Example 4.2.15. \(X\) is not \(T_\frac{1}{2}\), so it is not \(T_b\). But we prove that \(X\) is \(T_d\) space. This proves the covers of Lemma 4.2.17, is not true, and shows that we can not omit the \(T_0\) condition from Theorem 4.2.18.

4.3 Submaximality and Extremally Disconnected Spaces

More topological concepts will be studied on UB \(T_o\) A-space. we will present main definitions and some theorems. Using combination of theorems in this topic and other topics which studied in previous chapters, we contract a comprehensive theorem.

**Definitions 4.3.1.** A topological space \((X, \tau)\) is

1. \(g\) – submaximal [21] if every dense subset of \(X\) is \(g\)-open.
2. \(sg\) – submaxima [23] if every dense subset of \(X\) is \(sg\)-open.
3. \(\alpha\) – submaximal if every dense subset of \(X\) is \(\alpha\)-open.
Theorem 4.3.2. [23] For a space $(X, \tau)$ the following are equivalent:

1. Every preclosed subset is sg-closed.
2. $(X, \tau)$ is sg-submaximal.
3. $(X, \tau^a)$ is sg-submaximal.

Remark 4.3.3. If $X$ is a UB $T_0$ A-space, and $A$ is preclosed, then by Corollary 2.1.14, $A$ is semi-closed set. Hence by Proposition 3.1.5, $A$ is sg-closed. Therefore by Theorem 4.3.2, and we have the following result:

Theorem 4.3.4. Let $(X, \tau(\leq))$ be a UB $T_0$ A-space. Then

(a) $(X, \tau)$ is sg-submaximal, and

(b) $(X, \tau^a)$ is sg-submaximal.

Previously, we prove that if $(X, \tau(\leq))$ is a UB $T_0$ A-space then $(X, \tau_a)$ is also UB $T_0$ A-space. So, in fact, Part(b) in the above theorem is included in part(a).

Recall that a subset $A$ of a space $(X, \tau)$ is codense if $A^\circ = \emptyset$.

Theorem 4.3.5. [23] For a topological space $(X, \tau)$, the following are equivalent:

1. $X$ is sg-submaximal.
2. Every subset of $X$ is an intersection of a closed subset, and an sg-open subset of $X$.
3. Every subset of $X$ is a union of an open subset, and an sg-closed subset of $X$.
4. Every codense subset $A$ of $X$ is sg-closed.
5. $\text{cl}(A) \setminus A$ is sg-closed for every subset $A$ of $X$. 

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From Proposition 3.1.5, if $X$ is a UB $T_0$ A-space, then a set is sg-closed set if and only if it is semi-closed set. Using this, together with Theorem 4.3.4, and Theorem 4.3.5, we get directly the following theorem.

**Theorem 4.3.6.** For a UB $T_0$ A-space $X$, all the following hold:

1. $X$ is sg-submaximal.
2. Every subset of $X$ is an intersection of a closed subset, and an sg-open subset of $X$.
3. Every subset of $X$ is a union of an open subset, and an sg-closed subset of $X$.
4. Every codense subset $A$ of $X$ is sg-closed.
5. Every codense subset $A$ of $X$ is semi-closed.
6. Every subset of $X$ is an intersection of a closed subset, and an semi-open subset of $X$.
7. Every subset of $X$ is a union of an open subset, and an semi-closed subset of $X$.
8. $\text{cl}A \setminus A$ is sg-closed for every subset $A$ of $X$.
9. $\text{cl}(A) \setminus A$ is semi-closed for every subset $A$ of $X$.

**Theorem 4.3.7.** [21] For a space $(X, \tau)$ the following are equivalent:

1. Every preclosed set is $g\alpha$-closed.
2. Every $\beta$closed set is sg-closed.
3. $(X, \tau_\alpha)$ is g-submaximal.

By Proposition 3.1.4, if $(X, \tau(\leq))$ is a UB $T_0$ A-space, then a set $A$ is preclosed if and only if $A$ is $g\alpha$-closed set. This fact together with Theorem 4.3.7, we have the following theorem:
Theorem 4.3.8. If \((X, \tau(\leq))\) is a UB \(T_0\) A-space, then

(a) Every \(\beta\)closed set is sg-closed.

(b) \((X, \tau_\alpha)\) is g-submaximal.

Theorem 4.3.9. [23] For a topological space \((X, \tau)\), the following statements are equivalent:

(1) \((X, \tau)\) is extremally disconnected.

(2) \(scl(A \cup B) = scl(A) \cup scl(B)\) for all \(A, B \subseteq X\).

(3) The union of two semi-closed subsets of \(X\) is semi-closed.

(4) The union of two sg-closed subsets of \(X\) is sg-closed.

(5) Every \(\beta\)closed subset of \(X\) is preclosed.

(6) Every sg-closed subset of \(X\) is preclosed.

(7) Every semi-closed subset of \(X\) is preclosed.

(8) Every semi-closed subset of \(X\) is \(\alpha\)-closed.

(9) Every semi-closed subset of \(X\) is \(g\alpha\)-closed.

(10) Every gs-closed subset of \(X\) is \(\alpha g\)-closed.

Theorem 4.3.10. [23] For a space \((X, \tau)\), the following are equivalent:

(1) Every \(\beta\)closed set is \(g\alpha\)-closed.

(2) \((X, \tau_\alpha)\) is extremally disconnected, and g-submaximal.
Let $X$ is UB $T_0$ A-space. In Theorem 4.3.8, we see that $(X, \tau_\alpha)$ is always g-submaximal. Hence from Theorem 4.3.10, $(X, \tau_\alpha)$ if and only if every $\beta$ closed set is $g\alpha$-closed In Theorem 2.1.21 we give characterization of the extremally disconnected on $x$. Finally, Theorem 2.4.8, together with Theorem 4.3.9, we get a proof of the following comprehensive theorem that characterize the extremally disconnected on UB $T_0$ A-spaces.

**Theorem 4.3.11.** In UB $T_0$ A-spaces, the following are equivalent:

1. $(X, \tau)$ is extremally disconnected.
2. For all $x \in X$, $|\hat{x}| = 1$, i.e. $\forall x \in X$ there exists exactly one element $y \in M$ such that $x \leq y$.
3. $PO(X) = SO(X)$.
4. Every $\beta$ closed set is $g\alpha$-closed.
5. Every $\beta$ closed subset of $X$ is preclosed.
6. $scl(A \cup B) = scl(A) \cup scl(B)$ for all $A, B \subseteq X$.
7. The union of two semi-closed subsets of $X$ is semi-closed.
8. The union of two sg-closed subsets of $X$ is sg-closed.
9. Every semi-closed subset of $X$ is preclosed.
10. Every semi-closed subset of $X$ is $\alpha$-closed.
11. Every semi-closed subset of $X$ is $g\alpha$-closed.
12. Every sg-closed subset of $X$ is preclosed.
13. Every gs-closed subset of $X$ is $\alpha g$-closed.
Conclusion

During our research we were deal with some topics, which we think that they are connected with each other, and we got new theorems. We studied the proved results on Artinian $T_0$ A-space in more details and generalized them on the UB $T_0$ A-space. This thesis will open new way for other researches of what of these results are satisfying -more generally- on $T_0$ A-space, or on A-space. Giving examples for which of these are not satisfying on these spaces.
Bibliography


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