A REVIEW OF PARTIAL DIFFERENTIAL EQUATIONS OF THE FIRST ORDER IN DYNAMICAL SYSTEMS

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مراجعة للمعادلات التفاضلية الجزئية من الدرجة الأولى في الأنظمة المتحركة

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في هذا البحث تناولنا دراسة المعادلات التفاضلية الجزئية من الدرجة الأولى، وقد ناقشا بالتفصيل العلاقة بين حل المعادلات التفاضلية الجزئية من الدرجة الأولى وحل المعادلات التفاضلية الكاملة. معادلات الحركة في الأنظمة المقيمة هي معادلات تفاضلية كامنة يمكن حلها إذا خلت شروط قابلية التكامل. علاوة على ذلك قمنا بتكوين المعادلة التفاضلية الجزئية

ﮭملن-جاكوبي لكل من النظامين المنفصل والمتصل.

ABSTRACT
The first order partial differential equations are studied in this thesis. The relation between the solution of a system of first order partial differential equations and a system of total differential equations and is discussed.

The investigation of singular system leads to total differential equations which are integrable under certain integrability conditions. Furthermore; Hamilton-Jacobi partial differential equation (HJPDE) is constructed in both discrete and continuous systems.
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CHAPTER I

INTRODUCTION

Any equation that contains differential coefficient is called a differential equation. Such equations can be divide into two main types: ordinary and partial. Ordinary differential equations involve only one independent variable, and partial differential equations involve two or more independent variables. In general, any function of x, y and the derivatives of y up to any order such that

\[ F[x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots] = 0 \]  
(1.1)

defines an ordinary differential equation of y, whereas the equations

\[ F[x, y, \ldots, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \ldots, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \ldots] = 0 \]  
(1.2)

is partial differential equations for u (the dependent variable) in term of the independent variables x, y, . . .

The order of any differential equation is the order of the highest derivative that appears in the equation. These equations are linear in the sense that both u and its derivatives occur only to the first power, and the product of u and its derivatives are absent, [1], [2].

In general, the solution of partial differential equations presents a much more difficult problem than the solution of ordinary differential equations, and except for certain special types of linear partial differential equations no general method of solution is available.
Partial differential equations have many applications in applied sciences and engineering. These applications appear in [3] gravitation elastic membrane, electrostatics, fluid flow, steady state, heat conduction and many other topics in both pure and applied mathematics. Typical examples of partial differential equations of second order are the followings:

1) Laplace Equation: The general form of Laplace equation is

\[ u_{n_{1} x_{1}} + u_{n_{2} x_{2}} + \ldots + u_{n_{x} x_{x}} = 0 \]  \hspace{1cm} (1.3)

where \( u \) is a function of \( x \), \( x \in \mathbb{R}^{n} \), and \( \mathbb{R} \) is the set of real numbers.

2) Heat Equation:

\[ u_{xx} = k \ u_{t} \]  \hspace{1cm} (1.4)

where \( u \) is a real-valued function depending on ‘spatial’ variable \( x \in \mathbb{R}^{n} \) and on ‘time’ \( t \in \mathbb{R} \) where \( k \) is the temperature constant.

3) The Wave Equation:

\[ u_{tt} = c^{2} u_{xx} \]  \hspace{1cm} (1.5)

where \( u \) is a real valued function depending on spatial variables \( x \in \mathbb{R}^{n} \) and \( c \) is the velocity of light.

As an example of first order differential equations in mechanics, we consider the Hamilton-Jacobi partial differential equation, which has the form
\[
\frac{\partial S}{\partial t} + H \left( q_1, \ldots, q_n, \frac{\partial S}{\partial q_1}, \ldots, \frac{\partial S}{\partial q_n}, t \right) = 0 \tag{1.6}
\]

where \( q \)’s are the generalized co-ordinates, \( H \) is the Hamiltonian of the physical system (kinetic energy + potential energy), and \( t \) is the time. This partial differential equation of the first order in \( n+1 \) variables \( q_1, q_2, \ldots, q_n, t \) has a solution in the form:

\[
S = S(q_1, \ldots, q_n, \alpha_1, \ldots, \alpha_{n+1}, t) \tag{1.7}
\]

where, \( \alpha_1, \ldots, \alpha_{n+1} \) are \( n+1 \) independent constants, [4].

Another example of first order partial differential equation, which has many important applications in physics and field theory is the fundamental equation of a group theory, [5]. Assume that the transformation

\[
x' = f_i(x,a). \tag{1.8}
\]

form a group. The fundamental equation of group theory takes the form

\[
\frac{\partial x'}{\partial a^b} = \xi^i_{\ b}(x')A^b_i(a) \tag{1.9}
\]

where \( a \)’s are \( m \) essential parameters and \( \xi^i_{\ b}, A^b_i \) are functions of \( x' \) and \( a \) respectively. This system of equation is completely integrable and admits the solutions \( f_i(x,a) \).

In this thesis we will study the partial differential equations of the first order in dynamical systems. These systems treat in the motion of material bodies. The
example of dynamical systems, which we will study in this thesis, are discrete and continuous systems.

In chapter two we will investigate singular systems by a variational principle which leads us to the canonical equations which are total differential equations in many variables. Hence, the determination of integrability conditions of these equations is of prime importance. In section 1 we will illustrate the relations between the solutions of total differential equations and linear homogeneous partial differential equations (LHPDE). Section two concerns with complete systems and the integrability condition of the system of total differential equations. In section three, the integration of singular system will be discussed. The equations of motion lead us to total differential equations. Finally an example of singular system will be studied.

In chapter three, Hamilton-Jacobi equations of the constrained dynamical system and the integration of them will be discussed

Chapter four concerns with Hamilton-Jacobi partial differential equations of continuous systems and the integration of them is also discussed.
CHAPTER II

INTEGRABILITY CONDITION OF SINGULAR SYSTEMS

2.1 The Relations between the Solutions of Total Differential Equations and (LHPDES)

To study the relation between the solutions of the total differential equations in the form

\[ dx_i = b_{\alpha}(x_j, t) dt_{\alpha}, \quad i = 1, 2, \ldots, n; \alpha, \beta = 1, \ldots, m \quad (2.1.1) \]

where \( b_{\alpha}(t_{\beta}, x_j) = \frac{\partial x_i(t_{\beta}, u_k)}{\partial t_{\alpha}} \bigg|_{u_i = \sigma_j(t_{\beta}, u)} \), \( u_k \) are parameters

and the system of linear homogeneous partial differential equations (LHPDE) is in the form

\[ \frac{\partial F}{\partial t_{\alpha}} + b_{\alpha}(t, x_j) \frac{\partial F}{\partial x_i} = 0, \quad i, j = 1, \ldots, n; \alpha = 1, \ldots, m \quad (2.1.2) \]

(throughout this thesis, when the same two indices are repeated in a term this stands for the sum of the terms).

Let us first investigate the solutions of total differential equations (2.1.1). Let

\[ x_i = x_i(t_{\alpha}, u) \quad (2.1.3) \]

be any n-dimensional surfaces, where \( u_j \) are parameters satisfying \( x_i(t_{\alpha}, u) = u_i \).

The sufficient condition for solving equations (2.1.3) for \( u_i \) is that the determinant
\[ \frac{\partial x_i}{\partial x_j^0} \neq 0, \quad (2.1.4) \]

where \( x_j^0 = x_j(t_\alpha^0, u_j) = u_j \) is initial point.

If condition (2.1.4) is satisfied, then one can solve (2.1.3) for \( u_j \) as

\[ u_j = g_j(t_\alpha^0, x_i) \quad (2.1.5) \]

Furthermore, the relations

\[ b_i^\alpha(t_\alpha, x_j) = \frac{\partial x_i(t_\beta, u_k)}{\partial t_\alpha} \bigg|_{u_k = g_i(t_\beta, u_j)} \quad (2.1.6) \]

hold, where \( b_i^\alpha \) are continuously differentiable[6]. Thus, under these conditions, we say that \( x_i = x_i(t_\alpha, u_j) \) are solutions of the total differential equations (2.1.1). That is, equation (2.1.1) are identically satisfied if we replace the \( x_i \) by the expressions (2.1.3). If \( m = 1 \), then equations (2.1.1) are ordinary differential equations. If \( m > 1 \), then \( b_i^\alpha \) should satisfy certain integrability conditions.

Now, we will investigate the corresponding partial differential equations of the total differential equations (2.1.1). Differentiate equations (2.1.5) partially with respect to \( t_\alpha \), we have

\[ \frac{\partial u_j}{\partial t_\alpha} = \frac{\partial g_j}{\partial t_\alpha} + \frac{\partial g_j}{\partial x_i} \frac{\partial x_i}{\partial t_\alpha} = \frac{\partial g_j}{\partial t_\alpha} + \frac{\partial g_j}{\partial x_i} b_i^\alpha \quad (2.1.7) \]

Since \( u_j \) is independent of \( t_\alpha \),

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which is the same form of equations (2.1.2). Now, we want to determine the necessary and sufficient condition for the system of total differential equations (2.1.1) to be completely integrable [6]. To do this, let us consider the following theorem:

**Theorem 2.1.1**

The necessary and sufficient condition for the system of total differential equations (2.1.1) to be completely integrable is that

\[
\frac{\partial b_{ia}}{\partial t_{\alpha}} - \frac{\partial b_{i\beta}}{\partial t_{\alpha}} + \frac{\partial b_{i\alpha}}{\partial x_{j}} b_{j\beta} - \frac{\partial b_{i\beta}}{\partial x_{j}} b_{j\alpha} = 0
\]  

(2.1.9)

**Proof**

Let us consider a point in (n+m) dimensional space formed by \((t_{\alpha}, x_{i})\), and set

\[ t_{\alpha} = t_{\alpha}^{0} + \lambda_{\alpha} s \]  

(2.1.10)

where \(s\) is an arbitrary variable, and \(\lambda_{\alpha}\)'s are parameters. Thus we can express the total differential equation (2.1.1) as a set of ordinary differential equations with \(m\) parameters \(\lambda_{\alpha}\) as follows:

The system of total differential equations (2.1.1) becomes

\[
dx_{i} = b_{ia} \left(t_{\alpha}^{0} + \lambda_{\alpha} s, x_{j}\right) d \left(t_{\alpha}^{0} + \lambda_{\alpha} s\right) = b_{aa} \left(t_{\alpha}^{0} + \lambda_{\alpha} s, x_{j}\right) \lambda_{\alpha} ds
\]  

(2.1.11)
\[
\frac{dx_i}{ds} = \lambda_\alpha b_\alpha \left(t_\beta^0 + \lambda_\beta s, x_i\right)
\] (2.1.12)

These are ordinary differential equations and their solutions are functions of \(s\), arbitrary parameters \(\lambda_\alpha\), and the initial values \(u_j\), i.e.

\[
x_i = \varphi_i(s, \lambda_\alpha, u_j)
\] (2.1.13)

Hence,

\[
u_j \equiv \varphi_j(0, \lambda_\alpha, u_j)
\] (2.1.14)
or

\[
u_j \equiv \varphi_j(s, 0, u_j)
\] (2.1.15)

Thus, by using equations (2.1.10), the equations (2.1.13) take the form:

\[
x_i(t_\alpha, u_j) = \varphi_j \left(s, \frac{t_\alpha - t_\alpha^0}{s}, u_j\right)
\] (2.1.16)

Since the left hand side of equations (2.1.16) is independent of \(s\), the right hand side also should be independent of \(s\). Under the assumption that \(s \neq 0\), the above requirement means

\[
\frac{dx_j}{ds} = \frac{\partial \varphi_j}{\partial s} + \frac{\partial \varphi_j}{\partial \lambda_\alpha} \frac{\partial \lambda_\alpha}{\partial s} + \frac{\partial \varphi_j}{\partial u_j} \frac{\partial u_j}{\partial s} = 0
\] (2.1.17)
Since $\frac{\partial u_j}{\partial s} = 0$, and by using (2.1.10), we have

$$\frac{\partial \varphi_i}{\partial s} - \frac{t_\alpha - t_\alpha^0}{s^2} \frac{\partial \varphi_i}{\partial \lambda_\alpha} = 0$$  \hspace{1cm} (2.1.18)

or

$$s \frac{\partial \varphi_i}{\partial s} - \lambda_\alpha \frac{\partial \varphi_i}{\partial \lambda_\alpha} = 0$$  \hspace{1cm} (2.1.19)

Now, let us differentiate the equation (2.1.16) partially with respect to $t_\alpha$, we will obtain,

$$\frac{\partial x_i}{\partial t_\alpha} = \frac{\partial \varphi_i}{\partial s} \frac{\partial s}{\partial t_\alpha} + \frac{\partial \varphi_i}{\partial \lambda_\beta} \frac{\partial \lambda_\beta}{\partial t_\alpha} + \frac{\partial \varphi_i}{\partial u_j} \frac{\partial u_j}{\partial t_\alpha}$$  \hspace{1cm} (2.1.20)

with

$$\frac{\partial \lambda_\beta}{\partial t_\alpha} = \frac{1}{s} \delta_\alpha_\beta \frac{\partial s}{\partial t_\alpha} = 0, \text{ and } \frac{\partial u_j}{\partial t_\alpha} = 0, \text{ equation (2.1.20) becomes}

$$\frac{\partial x_i}{\partial t_\alpha} = \frac{1}{s} \frac{\partial \varphi_i}{\partial \lambda_\alpha}, \quad s \neq 0$$  \hspace{1cm} (2.1.21)

Using equations (2.1.6), we shall get

$$\frac{\partial \varphi_i}{\partial \lambda_\alpha} = s \delta_\alpha_\beta \left(t_\beta^0 + \lambda_\beta s, \varphi_j\right)$$  \hspace{1cm} (2.1.22)
Thus the solutions $\varphi_i$ should satisfy both the equations (2.1.19) and (2.1.22).

In contrast, if the functions $\varphi_i$ satisfy the system of equations (2.1.19) and (2.1.22), then the right hand side of (2.1.16) is independent of $s$, they are going to be the solutions of total differential equation (2.1.1). Since $\varphi_i$ are solutions of the ordinary differential equation (2.1.12), they also satisfy the equations

$$\frac{\partial \varphi_i}{\partial s} = \lambda_{a} b_{i,a} (t^\alpha_\beta + \lambda_\beta s, \varphi_i).$$

(2.1.23)

So the solutions $\varphi_i$ should satisfy both equations (2.1.22) and (2.1.23). In general one may set

$$\frac{\partial \varphi_i}{\partial \lambda_\alpha} = sb_{i\alpha} + \omega_{i\alpha},$$

(2.1.24)

where, $\omega_{i\alpha}$ are functions to be determined.

If $\varphi_i$ are solutions then the following condition should hold:

$$\frac{\partial}{\partial \lambda_\alpha} \left( \frac{\partial \varphi_i}{\partial s} \right) = \frac{\partial}{\partial s} \left( \frac{\partial \varphi_i}{\partial \lambda_\alpha} \right), \quad \forall \alpha,$$

(2.1.25)

Thus when differentiating the equation (2.1.23), with respect to $\lambda_\alpha$, we obtain

$$\frac{\partial}{\partial \lambda_\alpha} \left( \frac{\partial \varphi_i}{\partial s} \right) = \frac{\partial}{\partial \lambda_\alpha} \left( \lambda_{\beta} b_{i\beta} (t^\alpha_\gamma + \lambda_\gamma s, \varphi_i) \right) = b_{i\alpha} \frac{\partial \lambda_{\beta}}{\partial \lambda_\alpha}$$

$$+ \lambda_{\beta} \left( \frac{\partial b_{i\beta}}{\partial t^\alpha_\gamma} \frac{\partial t^\alpha_\gamma}{\partial \lambda_\alpha} + \frac{\partial b_{i\beta}}{\partial x_j} \frac{\partial \varphi_i}{\partial \lambda_\alpha} \right) = b_{i\alpha} + \lambda_{\beta} \left( \frac{\partial b_{i\beta}}{\partial t^\alpha_\gamma} + \frac{\partial b_{i\beta}}{\partial x_j} \frac{\partial \varphi_i}{\partial \lambda_\alpha} \right)$$

(2.1.26)
and by making use of (2.1.24), the equation (2.1.26) becomes

\[
\frac{\partial}{\partial s}\left( \frac{\partial \phi}{\partial \lambda_{\alpha}} \right) = b_{i \alpha} + \lambda_{\beta} \left( \frac{\partial b_{i \beta}}{\partial t_{\alpha}} s + \frac{\partial b_{i \beta}}{\partial x_{j}} (s b_{j a} + \omega_{j \alpha}) \right) \\
= b_{i \alpha} + s \lambda_{\beta} \left( \frac{\partial b_{i \beta}}{\partial t_{\alpha}} + \frac{\partial b_{i \beta}}{\partial x_{j}} b_{j \alpha} \right) + \lambda_{\beta} \frac{\partial b_{i \beta}}{\partial x_{j}} \omega_{j \alpha}
\]  
(2.1.27)

On the other hand, when differentiating (2.1.24), with respect to \( s \), we obtain

\[
\frac{\partial}{\partial s}\left( \frac{\partial \phi}{\partial \lambda_{\alpha}} \right) = b_{i \alpha} + s \lambda_{\beta} \left( \frac{\partial b_{i \beta}}{\partial t_{\alpha}} + \frac{\partial b_{i \beta}}{\partial x_{j}} \phi_{j} \right) + \frac{\partial \omega_{i \alpha}}{\partial s}
\]  
(2.1.28)

by using (2.1.23), we have

\[
\frac{\partial}{\partial s}\left( \frac{\partial \phi}{\partial \lambda_{\alpha}} \right) = b_{i \alpha} + s \lambda_{\beta} \left( \frac{\partial b_{i \beta}}{\partial t_{\alpha}} + \frac{\partial b_{i \beta}}{\partial x_{j}} b_{j \alpha} \right) + \frac{\partial \omega_{i \alpha}}{\partial s}
\]  
(2.1.29)

Now, by subtracting the equation (2.1.27) from (2.1.29), we obtain

\[
\frac{\partial \omega_{i \alpha}}{\partial s} + b_{i \alpha} + s \lambda_{\beta} \left( \frac{\partial b_{i \beta}}{\partial t_{\alpha}} + \frac{\partial b_{i \beta}}{\partial x_{j}} b_{j \alpha} \right) - b_{i \alpha} \\
- s \lambda_{\beta} \left( \frac{\partial b_{i \beta}}{\partial t_{\alpha}} + \frac{\partial b_{i \beta}}{\partial x_{j}} b_{j \alpha} \right) - \lambda_{\beta} \frac{\partial b_{j \beta}}{\partial x_{j}} \omega_{j \alpha} = 0
\]  
(2.1.30)
\[
\lambda_{\beta} \left( \frac{\partial b_{\beta}}{\partial t_{\alpha}} - \frac{\partial b_{\alpha}}{\partial t_{\beta}} + \frac{\partial b_{\beta}}{\partial x_{j}} b_{j}^{\alpha} - \frac{\partial b_{\alpha}}{\partial x_{j}} b_{j}^{\beta} \right) = s \lambda_{\beta} A_{i,\beta \alpha} \quad (3.1.31)
\]

If the equation (2.1.22) holds, then the function \( \omega_{\alpha} \) is identically zero. This implies that the equations (2.1.31) will lead to

\[
\lambda_{\beta} A_{i,\beta \alpha} = 0, \quad (i=1,\ldots,n; \alpha=1,\ldots,m) \quad (3.1.32)
\]

where

\[
A_{i,\alpha \beta} = \left( \frac{\partial b_{\beta}}{\partial t_{\alpha}} - \frac{\partial b_{\alpha}}{\partial t_{\beta}} + \frac{\partial b_{\beta}}{\partial x_{j}} b_{j}^{\alpha} - \frac{\partial b_{\alpha}}{\partial x_{j}} b_{j}^{\beta} \right) \quad (3.1.33)
\]

is the integrability condition in the theorem, that is the necessary condition to have a solution. By using (2.1.10), we get

\[
(t_{\beta} - t_{\beta}^{0}) A_{i,\beta \alpha} = 0 \quad (2.1.34)
\]

If the equations (2.1.34) are satisfied for a given system of \( t_{\beta}^{0} \), then a field of \( m \)-dimensional surfaces exist. Such that the total differential equations (2.1.1) are satisfied. Also when we vary \( t_{\beta}^{0} \) arbitrarily, then

\[
t_{\beta} - t_{\beta}^{0} \neq 0 \quad (2.1.35)
\]

This implies

\[
A_{i,\alpha \beta} = 0 \quad (2.1.36)
\]
In the next theorem we will establish the relation between the solutions of the total differential equations (2.1.1) and the solution of (LHPDE) (2.1.2) by using the condition (2.1.36).

**Theorem 2.1.2 [6]**

Every solution of a system of $m$ linear, homogeneous, partial differential equations of the first order

\[
\frac{\partial F}{\partial t_\alpha} + b_{\alpha i}(t_{\beta}, x_j) \frac{\partial F}{\partial x_j} = 0, \quad (\alpha=1,...,m) \quad (2.1.37)
\]

such that the $b_{\alpha i}$ satisfy the condition (2.1.9), is an integral of the total differential equations (2.1.1)

**Proof**

Let $F(t_\alpha, x_j)$ be a solution of the partial differential equation (2.1.37), which is continuously differentiable in a certain neighborhood of the points

\[
t_\alpha = t_\alpha^0, x_i = x_i^0 = x_j(t_\alpha^0, u_j) = u_j \quad (3.1.38)
\]

and let $x_j = x_j(t_{\beta}, u_j)$, be a solution of the total differential equations (2.1.1). If $F$ is an integral of the total differential equation, then $F(t_\alpha, x_j = x_j(t_{\beta}, u_j))$ should be independent of $t_\alpha$. Differentiating $F$ with respect to $t_\alpha$ we get,

\[
\frac{dF}{dt_\alpha} = \frac{\partial F}{\partial t_\alpha} + \frac{\partial F}{\partial x_j} \frac{\partial x_j(t_\alpha, u_j)}{\partial t_\alpha} \quad (2.1.39)
\]
Since \( x_i = x_i(t, u_j) \) is a solution, it satisfies

\[
\frac{\partial x_i(t, u_j)}{\partial t} \bigg|_{u_j = g_j} = b_i \tag{2.1.40}
\]

Then

\[
\frac{dF}{dt} + \frac{\partial F}{\partial t} + b_i \frac{\partial F}{\partial x_i} = 0, \quad \forall \alpha \tag{2.1.41}
\]

Hence, \( F \) is independent of \( t \). According to (2.1.5), we write

\[
F(t, x_i(t, u_j)) = \varphi(u_j) = \varphi(g_j(t, x_i)) \tag{2.1.42}
\]

Conversely, for any function \( \varphi(g_j y) \), we have

\[
\frac{d\varphi}{dt} = \frac{\partial \varphi}{\partial g_j} \frac{\partial g_j}{\partial t} + \frac{\partial \varphi}{\partial g_j} \frac{\partial x_i(t, u_j)}{\partial t} = \frac{\partial \varphi}{\partial g_j} \left( \frac{\partial g_j}{\partial t} + \frac{\partial g_j}{\partial x_i} \frac{\partial x_i}{\partial t} \right) \tag{2.1.43}
\]

By using (2.1.40), we obtain

\[
\frac{d\varphi}{dt} = 0 \tag{2.1.44}
\]

This means that every function \( \varphi(g_j) \) is a solution of (2.1.37).

Hence we conclude that, a complete manifold of solutions of LHPDE with more than one independent variable can be obtained by the functionally
independent integrals of corresponding total differential equation in analogy with
the case of one independent variables.

2.2 Complete System of LHPDES of the First Order

In this section we would like to define a complete system of partial differential
equations and its solution. Consider the system of linear partial differential
equation.

\[
a_{i\alpha}(x_j) \frac{\partial f}{\partial x_i} = 0, (\alpha = 1, \ldots, m; i, j = 1, \ldots, n) \tag{2.2.1}
\]

where \(m\) are linearly independent equations. Here, linear independence means that
the rank of the determinant \(|a_{i\alpha}|\) is \(m\).

To find the solutions, let us define the linear operators \(X_{\alpha}\), as [6]

\[
X_{\alpha}f = a_{i\alpha}(x_j) \frac{\partial f}{\partial x_i} = 0 \tag{2.2.2}
\]

These linear operators \(X_{\alpha}f\) satisfy the following relations:

\[
X_{\alpha}(f + g) = a_{i\alpha}(x_j) \frac{\partial (f + g)}{\partial x_i} = a_{i\alpha}(x_j) \frac{\partial f}{\partial x_i} + a_{i\alpha}(x_j) \frac{\partial g}{\partial x_i} = X_{\alpha}f + X_{\alpha}g \tag{2.2.3}
\]

Also

\[
X_{\alpha}fg = a_{i\alpha} \frac{\partial (fg)}{\partial x_i} = ga_{i\alpha} \frac{\partial f}{\partial x_i} + fa_{i\alpha} \frac{\partial g}{\partial x_i} = gX_{\alpha}f + fX_{\alpha}g \tag{2.2.4}
\]
Now we want to define the bracket of two operators $X_\alpha$ and $X_\beta$. Let us evaluate
\[
X_\beta X_\alpha f = X_\beta \left( a_\beta \frac{\partial f}{\partial x_i} \right) = \left( X_\beta a_\beta \right) \frac{\partial f}{\partial x_i} + a_\beta \frac{\partial f}{\partial x_i} = \left( X_\beta a_\beta \right) \frac{\partial f}{\partial x_i} + a_\beta \frac{\partial^2 f}{\partial x_i \partial x_j}
\]  
(2.2.5)

and
\[
X_\alpha X_\beta f = \left( X_\alpha a_\beta \right) \frac{\partial f}{\partial x_i} + a_\beta a_\alpha \frac{\partial^2 f}{\partial x_j \partial x_i}
\]  
(2.2.6)

Now, the symmetry between $\alpha$ and $\beta$ implies that
\[
a_\alpha a_\beta \frac{\partial^2 f}{\partial x_i \partial x_j} = a_\beta a_\alpha \frac{\partial^2 f}{\partial x_j \partial x_i}
\]  
(2.2.7)

Setting the bracket
\[
\left( X_\alpha, X_\beta \right) f = \left( X_\alpha X_\beta - X_\beta X_\alpha \right) f
\]
\[
= X_\alpha X_\beta f - X_\beta X_\alpha f = \left( X_\alpha a_\beta - X_\beta a_\alpha \right) \frac{\partial f}{\partial x_i}
\]  
(2.2.8)

One should notice that
\[
\left( X_\alpha, X_\beta \right) f = 0
\]  
(2.2.9)

represents a set of LHPDE in the form (2.2.1).
Definition 2.2.1

A system of partial differential equations

\[ X_\alpha f = b_\alpha \frac{\partial f}{\partial x_i} \]  \hspace{1cm} (2.2.10)

is called complete if the following relations hold,

\[ \left( X_\alpha, X_\beta \right) f = \gamma_{\alpha\beta} X_c f , \hspace{1cm} (\alpha, \beta, c = 1, \ldots, m) \]  \hspace{1cm} (2.2.11)

when \( \gamma_{\alpha\beta} \)'s are functions of \( x_j \)'s [5], [6].

In other words if the system is complete, the bracket of any two operators can be written as a linear combination of the operators. So every twice differential solutions of the equations (2.2.2) should satisfy \( m(m-1)/2 \) relations (2.2.9). The conclusion is that a complete system, and the linear system (2.2.10) have the same solutions. Also from the bracket relations (2.2.11), we can obtain the maximum number of linearly independent equations.

Since the rank of the matrix is \( m \), there are \( m \) linearly independent equations, which satisfy the relation (2.2.9).

If \( m = n \) the only solution of equation (2.2.10) is \( f = \text{constant} \). If \( m < n \) there is a possibility of solutions other than the trivial one. Now, if the equations (2.2.11) hold, then we adjoin this type of relations with relation (2.2.10) and we have

\( s \geq m \) linearly independent equations. If \( s > m \), we repeat this process and obtain a system of \( s' \geq s \) linearly independent equations. If \( s' > s \), we continue
this process. Finally we have either \( n \) independent equations, and the only solution in this case is \( f = \text{constant} \), or we get a number \( u \) less than \( n \) for which \( \alpha, \beta, \gamma \) in (2.2.11) take the values \( 1, \ldots, u \).

In the first case, we say that the system (2.2.10) is complete of order \( n \) and in the other case of order \( u \).

### 2.3 Integration of Singular System

The investigation of a dynamical singular system [8] by canonical method leads to equations of motion. These equations of motion are total differential equations of type (2.1.1). In this section, we would like to establish the integrability conditions of the equations of motion of a singular system.

**Definition 2.3.1**

Let \( \mathcal{L}(q_i, \dot{q}_i, t) \) be a Lagrange of coordinates \( q_i, \dot{q}_i \) and time \( t \).

The Euler-Lagrange equations of this function reads

\[
\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = 0, \quad i = 1, \ldots, n \tag{2.3.1}
\]

or, explicitly

\[
W_{ik} \dot{\dot{q}}_k = \frac{\partial \mathcal{L}}{\partial q_i} - \frac{\partial^2 \mathcal{L}}{\partial \dot{q}_i \partial q_k} - \frac{\partial^2 \mathcal{L}}{\partial q_i \partial t} \tag{2.3.2}
\]

where the matrix

\[
W_{ik} = \frac{\partial^2 \mathcal{L}}{\partial \dot{q}_i \partial \dot{q}_k} \tag{2.3.3}
\]
is called Hessian matrix [8]. If \( W = \det |W| = 0 \) we say that the Lagrangian is singular, and regular otherwise.

The set of Hamilton-Jacobi partial differential equations of a singular system is defined as

\[
\frac{\partial S}{\partial q_a} \frac{\partial S}{\partial q_b} = H_{\alpha}\left( t_\beta, q_a, p_\alpha \right), \quad a=0,1,\ldots,n \quad (2.3.4)
\]

where

\[
H'_a = H_a \left( t_\beta, q_a, p_\alpha \right) + p_\alpha, \quad (2.3.5)
\]

\( H_a \) is the Hamiltonian and \( p_\alpha \) is the generalized momenta.

Now the Poisson bracket in \((2n+2)\)-dimensional phase space of \( H'_a \) can be obtained as follows:

Differentiating (2.3.4) with respect to \( q_r \), we obtain

\[
\frac{\partial H'_a}{\partial q_a} \frac{\partial S}{\partial q_r} + \frac{\partial H'_a}{\partial p_r} \frac{\partial S}{\partial q_a} \frac{\partial q_r}{\partial q_a} = 0 \quad (2.3.6)
\]

\[
\frac{\partial H'_a}{\partial q_r} + \frac{\partial H'_a}{\partial p_r} \frac{\partial^2 S}{\partial q_s \partial q_r} = 0 \quad \forall r,s=0,1,\ldots,n \quad (2.3.7)
\]

Also
\[ \frac{\partial H'_\beta}{\partial q_r} + \frac{\partial H'_\alpha}{\partial p_s} \frac{\partial^2 S}{\partial q_r \partial q_s} = 0 \quad \forall r, s \]  

(2.3.8)

Multiplying equations (2.3.7) by \(- \frac{\partial H'_\beta}{\partial q_r}\) and equations (2.3.8) by \( \frac{\partial H'_\alpha}{\partial q_r}\) equations (2.3.7) lead to

\[ -\frac{\partial H'_\beta}{\partial q_r} \frac{\partial H'_\alpha}{\partial q_r} = \frac{\partial H'_\beta}{\partial q_r} \frac{\partial H'_\alpha}{\partial p_s} \frac{\partial^2 S}{\partial q_r \partial q_s} \]  

(2.3.9)

and equation (2.3.8) leads to

\[ \frac{\partial H'_\alpha}{\partial q_r} \frac{\partial H'_\beta}{\partial q_r} = -\frac{\partial H'_\alpha}{\partial q_r} \frac{\partial H'_\beta}{\partial p_s} \frac{\partial^2 S}{\partial q_r \partial q_s} \]  

(2.3.10)

Now, by adding the last two equations, we have the Poisson bracket

\[ [H'_\alpha, H'_\beta] = \frac{\partial H'_\alpha}{\partial q_r} \frac{\partial H'_\beta}{\partial q_r} - \frac{\partial H'_\alpha}{\partial p_s} \frac{\partial H'_\beta}{\partial q_r} \]  

\[ = \frac{\partial^2 S}{\partial q_r \partial q_s} \left( \frac{\partial H'_\alpha}{\partial q_r} \frac{\partial H'_\alpha}{\partial p_s} - \frac{\partial H'_\beta}{\partial q_r} \frac{\partial H'_\beta}{\partial p_s} \right) = 0 \]  

(2.3.11)

Now we may investigate the integrability conditions of the characteristic equations of motion [8]

\[ dq_a = \frac{\partial H'_\alpha}{\partial p_a} dt_a, \quad dp_a = -\frac{\partial H'_\alpha}{\partial q_a} dt_a \]  

(2.3.12)

\[ dz = \left( -H_a + p_a \frac{\partial H'_\alpha}{\partial p_a} \right) dt_a \]  

(2.3.13)
Now if equations (2.3.12) are integrable, then the solutions of (2.3.13) can be obtained by a quadrature. So, we want to investigate the integrability conditions of equations (2.3.12) only.

Let us define the linear operators $X_\alpha f$ as

$$X_\alpha f(t_\alpha, q_\alpha, p_\alpha) = \left[ f, H_\alpha \right] = \frac{\partial f}{\partial q_\alpha} \frac{\partial H_\alpha}{\partial p_\alpha} - \frac{\partial f}{\partial p_\alpha} \frac{\partial H_\alpha}{\partial q_\alpha} + \frac{\partial f}{\partial t_\alpha} \quad (2.3.14)$$

**Theorem 2.3.1**

The system of differential equations (2.2.2) is integrable if and only if the following conditions hold.

$$\left[ H_\alpha', H_\beta' \right] = 0, \quad \forall \alpha, \beta \quad (2.3.15)$$

**Proof**

First, let us show that if (2.3.15) is valid, then (2.2.2) is complete.

Now, forming the bracket relations (2.2.9), as

$$\left( X_\alpha X_\beta \right)f = \left( X_\alpha X_\beta - X_\beta X_\alpha \right)f$$

$$= X_\alpha X_\beta f - X_\beta X_\alpha f = X_\alpha \left[ f, H_\beta \right] - X_\beta \left[ f, H_\alpha \right] \quad (2.3.16)$$
\( (X_\alpha, X_\beta) f = \left[ [f, H'_\beta], H'_\alpha \right] - \left[ [f, H'_\alpha], H'_\beta \right] \) \quad (2.3.17)

Making use of the Jacobi relations

\[
\left[ [f, H'_\beta], H'_\alpha \right] + \left[ [H'_\alpha, f], H'_\beta \right] + \left[ [H'_\beta, H'_\alpha], f \right] = 0
\]

\quad (2.3.18)

we have

\[
\left[ [f, H'_\beta], H'_\alpha \right] = -\left[ [H'_\alpha, f], H'_\beta \right]
\]

\quad (2.3.19)

and relation (2.3.17) becomes

\[
(X_\alpha, X_\beta) f + \left[ [H'_\beta, H'_\alpha], f \right] = \left[ [f, H'_\beta], H'_\alpha \right] + \left[ [H'_\alpha, f], H'_\beta \right] + \left[ [H'_\beta, H'_\alpha], f \right] = 0
\]

\quad (2.3.20)

This implies

\[
(X_\alpha, X_\beta) f = -\left[ [H'_\beta, H'_\alpha], f \right] = [f, [H'_\beta, H'_\alpha]]
\]

\quad (2.3.21)

Identifying

\[
[H'_\beta, H'_\alpha] = A_{\beta\alpha}(t, q_a, p_a) = 0,
\]

\quad (2.3.22)

then

\[
(X_\alpha, X_\beta) f = [f, A_{\beta\alpha}] = \frac{\partial f}{\partial q_a} \frac{\partial A_{\beta\alpha}}{\partial p_a} - \frac{\partial A_{\beta\alpha}}{\partial q_a} \frac{\partial f}{\partial p_a} = 0
\]

\quad (2.3.23)
and the system is complete.

Conversely, if the system (2.2.2) is integrable, then it completes[5],[6], this implies, due to the equation (2.3.23), that

\[
\left( X_\alpha, X_\beta \right) f \left[ f, \Lambda_{\alpha \beta} \right] = \frac{\partial f}{\partial q_a} \frac{\partial \Lambda_{\alpha \beta}}{\partial p_a} - \frac{\partial \Lambda_{\alpha \beta}}{\partial q_a} \frac{\partial f}{\partial p_a} = 0 \quad (2.3.24)
\]

Then

\[
\frac{\partial \Lambda_{\alpha \beta}}{\partial p_a} = \frac{\partial \Lambda_{\alpha \beta}}{\partial q_a} = 0 \quad (2.3.25)
\]

Thus (2.3.15) is hold, and this is the end of the proof of this theorem.

Now, if the integrability conditions are satisfied, then the solutions of the total differential equations (2.3.12) and (2.3.13) are obtained as

\[
q_b = \xi_a (t_a, u_b) \quad , \quad p_a = \eta_a (t_a, u_b) \quad , \quad z = \sigma (t_a, u_b) \quad (2.3.26)
\]

where \(u_a\) are arbitrary parameters.

The function

\[
z = S(t_a, q_a) \quad (2.3.27)
\]

is obtained by solving (2.3.13)
\[ \Lambda_a (t_\beta, u_a) = -H_a (t_\beta, \xi_a, \eta_a) + \eta_a \frac{\partial \xi_a}{\partial t_a} \] (2.3.28)

then the equation (2.3.13) leads to

\[ dz = \Lambda_a dt_a \] (2.3.29)

and this equation is a total differential equation and its solution is

\[ \sigma (t_a, u_a) = s (u_a) + \int \Lambda_a dt_a \] (2.3.30)

**Example 2.3.1**

Consider a dynamical system of Hamiltonians defined as [6], [8],

\[ H_0 = 1/2 \left( \frac{p_1^2}{a_1} - \frac{p_2^2}{a_2} \right) + c \] (2.3.31)

\[ H_2 = p_3 - b \] (2.3.32)

when \( a_1, a_2, b \) are constants, and \( c(q_1) \) is a function of \( q_1 \).

The set of HJPDE (2.3.4) is

\[ \frac{\partial S}{\partial \tau} + H_0 (q_i, \tau, p_i = \frac{\partial S}{\partial q_1}, p_3 = \frac{\partial S}{\partial q_3}) = 0 \] (2.3.33)

\[ \frac{\partial S}{\partial q_2} + H_2 (q_i, \tau, p_i = \frac{\partial S}{\partial q_1}, p_3 = \frac{\partial S}{\partial q_3}) = 0 \] (2.3.34)

According to the last theorem, the system is integrable if and only if
\[ [H'_0, H'_2] = 0, \quad (2.3.35) \]

that is

\[
[H'_0, H'_2] = [p_0 + H_0, p_2 + H_2] = \frac{\partial H_3}{\partial \tau} - \frac{\partial H_0}{\partial q_2} + \frac{\partial H_2}{\partial q_3} \frac{\partial p_1}{\partial p_3} \]
\[
+ \frac{\partial H_0}{\partial q_3} \frac{\partial H_2}{\partial p_3} - \frac{\partial H_0}{\partial q_3} \frac{\partial H_2}{\partial p_1} - \frac{\partial H_0}{\partial q_3} \frac{\partial H_2}{\partial p_3} = 0 \quad (2.3.36)\]

Now, we going to show that

\[
(X_0, X_2)f = 0 \quad (2.3.37)\]

By using definition (2.3.14), we have

\[
(X_0, X_2)f = X_0 X_2 f - X_2 X_0 f = X_0[f, H'_2] - X_2[f, H'_0] \]
\[
= [[f, H'_2], H'_0] - [[f, H'_0], H'_2] \quad (2.3.38)\]

Making use of Jacobi relation

\[
[[f, H'_2], H'_0] + [[H'_0, f], H'_2] + [[H', H'_0], f] = 0 \quad (2.3.39)\]

the bracket (2.3.38) becomes

\[
(X_0, X_2)f + [[H'_2, H'_0], f] = [[f, H'_2], H'_0] - [[f, H'_0], H'_2] \]
\[
+ [[H'_2, H'_0], f] = [[f, H'_2], H'_0] + [[H'_0, f], H'_2] + [[H'_2, H'_0], f] = 0 \quad (2.3.40)\]

Then
\[(X_0, X_2) f = [f, [H', H_0']] \]  \hspace{1cm} (2.3.41)

but since \([H', H_0'] = 0\),

\[(X_0, X_2) = 0 \]  \hspace{1cm} (2.3.42)

Thus the system is integrable.

The equations of motion of this system are

\[
dp_1 = -\frac{\partial H'}{\partial q_1} d\tau - \frac{\partial H'}{\partial q_2} dq_2 \\
dp_3 = -\frac{\partial H'}{\partial q_3} d\tau - \frac{\partial H'}{\partial q_2} dq_2 \\
dq_1 = \frac{\partial H'_0}{\partial p_1} d\tau + \frac{\partial H'}{\partial p_1} dq_2 \\
dq_3 = \frac{\partial H'_0}{\partial p_3} d\tau + \frac{\partial H'}{\partial p_3} dq_2
\]

Explicitly,

\[
dp_1 = \frac{\partial p_1}{\partial \tau} d\tau + \frac{\partial p_1}{\partial q_2} dq_2 = -\frac{\partial H'_0}{\partial q_1} d\tau - \frac{\partial H'}{\partial q_1} dq_2 \]  \hspace{1cm} (2.3.43)

\[
dp_3 = \frac{\partial p_3}{\partial \tau} d\tau + \frac{\partial p_3}{\partial q_2} dq_2 = -\frac{\partial H'_0}{\partial q_3} d\tau - \frac{\partial H'}{\partial q_3} dq_2 \]  \hspace{1cm} (2.3.44)

\[
dq_1 = \frac{\partial q_1}{\partial \tau} d\tau + \frac{\partial q_1}{\partial q_2} dq_2 = \frac{\partial H'_0}{\partial p_1} d\tau + \frac{\partial H'}{\partial p_1} dq_2 \]  \hspace{1cm} (2.3.45)
\[ dq_3 = \frac{\partial q_3}{\partial \tau} d\tau + \frac{\partial q_3}{\partial q_2} dq_2 = \frac{\partial H'_1}{\partial p_1} d\tau + \frac{\partial H'_2}{\partial p_3} dq_2 \] (2.3.46)

The corresponding partial differential equations of the total differential equations (2.3.43), (2.3.44), (2.3.45) and (2.3.46) are

\[
\frac{\partial p_1}{\partial \tau} = -\frac{\partial H'_0}{\partial q_1} = -\frac{\partial c}{\partial q_1}, \quad \frac{\partial p_1}{\partial q_2} = -\frac{\partial H'_2}{\partial q_1} = 0
\] (2.3.47)

\[
\frac{\partial p_3}{\partial \tau} = -\frac{\partial H'_3}{\partial q_3} = 0, \quad \frac{\partial q_3}{\partial q_1} = \frac{\partial H'_0}{\partial q_1} = \frac{p_1}{a_1}
\] (2.3.48)

\[
\frac{\partial q_1}{\partial \tau} = \frac{\partial H'_1}{\partial p_1} = \frac{p_1}{a_1}, \quad \frac{\partial q_2}{\partial q_1} = \frac{\partial H'_2}{\partial q_1} = 0
\] (2.3.49)

\[
\frac{\partial q_3}{\partial \tau} = \frac{\partial H'_3}{\partial p_3} = -\frac{p_1}{a_2}, \quad \frac{\partial q_3}{\partial q_2} = \frac{\partial H'_3}{\partial q_2} = 1
\] (2.3.50)

The solutions of (2.3.43), (2.3.44), (2.3.45) and (2.3.46) respectively are

\[ p_1 = -\frac{\partial c}{\partial q_2}\tau + c_0, \quad p_3 = c_2 \]

\[ q_1 = \frac{p_1}{a_1}\tau + c_1, \quad q_3 = -\frac{p_3}{a_2}\tau + q_2 \]
CHAPTER III


The most important first order partial differential equation occurring in mathematical physics is the Hamilton-Jacobi partial differential equation (HJPDE).

In the first section, we will study the formulation of (HJPDE) of discrete regular constrained dynamical system and in section two, the integration of (HJPDE) will be discussed.

We say that the dynamical system is discrete if the motion of any material body remove from one point to another without passing from third point between them, i.e. there is discontinuity or jump in the motion that body.

3.1 Construction of Hamilton-Jacobi Partial Differential Equations for Discrete Constrained Dynamical Systems

The dynamical system is called regular\(^8\) if the determinant of the Hessian matrix

\[
\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}, \quad i, j = 1, \ldots, n
\]

(3.1.1)

is different from zero. Here \( L \) is the Lagrangian function \( L(q_i, \dot{q}_i, t) \) of generalized co-ordinates \( q_i \), generalized velocities \( \dot{q}_i \), and time \( t \).

Consider a regular Lagrangian \( L(q_i, \dot{q}_i, t) \), \( i = 1, \ldots, n \), and a constraint \(^{10}\) equations \( G_\alpha(t, q_i, \dot{q}_i) = 0, \quad \alpha = 1, \ldots, m, \quad m < n \). If we can determine a set of differentiable functions \( \varphi_i(t, q_j) \) and \( S(t, q_i) \) such that for \( \dot{\xi} = \varphi_i \)
a) \( L'(q_i, \varphi_i, t) = L - \frac{dS(q_i, t)}{dt} \equiv 0 \) \hspace{1cm} (3.1.2)

b) \( L' > 0 \) in the neighborhood of \( \varphi_i \) \hspace{1cm} (3.1.3)

c) \( G_{\alpha}(q_i, \varphi_i, t) = 0 \quad \alpha = 1, \ldots, m \) \hspace{1cm} (3.1.4)

then the solutions of the first order ordinary differential equations

\[ \dot{x} = \varphi_i(t, q_i) \] \hspace{1cm} (3.1.5)

are extremals[11] (extremal means that the curve making the action integral minimum) of the action integral

\[ I = \int_{t_1}^{t_2} L(q_i, \dot{x}, t) dt \] \hspace{1cm} (3.1.6)

with constraints, if the variations at the end points are zero.

If one considers \( L' \) as a function of variables \( \varphi_i \), and if \( \varphi_i \) and \( S \) satisfying (a-c), then the function \( L' \) will have a local minimum at the point

\[ \frac{dq_i}{dt} = \dot{x} = \varphi_i, \] \hspace{1cm} (3.1.7)

and its minimum value at \( \varphi_i \) is zero. This leads to the minimum of the action integral, and Hamilton’s variational problem is reduced to the determination of necessary and sufficient conditions to obtain the local minimum of the function \( L(q_i, \dot{x}, t) \) with constraint \( G_{\alpha} = 0 \). The solutions of this problem is known in
terms of Lagrange multipliers. If instead of the Lagrangian $L$, one starts with a function

$$M(t,q_i,\dot{q}_i,\ddot{q}_i) = L(t,q_i,\dot{q}_i) + \dot{\lambda}_a(t)G_a(t,q_i,\dot{q}_i)$$  \hspace{1cm} (3.1.8)

then the extremum problem with constraint is reduced to the ordinary extremum problem. Here $\dot{\lambda}_a$ are the Lagrange multipliers [12] to be determined. In this new formulation the previous Caratheodory’s equivalent Lagrangian [10] method can be rewritten as follows:

Consider a function $M(t,q_i,\dot{q}_i,\ddot{q}_i)$. If it is possible to determine $n+p$, differentiable function $\eta_{p}(t,q_a)$, $\rho,\sigma = 1,....,n+p$, and function $S(q_{\rho},t)$ such that

\begin{align*}
a) \ M'(t,q_i,\eta_{\rho}) &= M(t,q_i,\eta_{\rho}) - \frac{\partial S}{\partial q_{\rho}}\eta_{\rho} - \frac{\partial S}{\partial t} \equiv 0 \hspace{1cm} (3.1.9) \\

b) \ M'&>0 \text{ in the neighbourhood }, \hspace{1cm} (3.1.10)
\end{align*}

then the solutions of the system of first order ordinary differential equations

$$\frac{dq_{\rho}}{dt} = \eta_{\rho}(t,q_{\rho})$$ \hspace{1cm} (3.1.11)

are the extremals of the action integral (3.1.6), with constraint equations

$$G_a(t,q_i,\dot{q}_i) = 0$$ \hspace{1cm} (3.1.12)
**Proposition 3.1.1** [10]

The necessary condition for the function $M'$ to have a local minimum value zero at the point $\phi_\rho = \eta_\rho$ are:

\[
\begin{align*}
\text{a) } \frac{\partial S}{\partial t} &= M(t,q_\rho,\eta_\rho) - \frac{\partial S}{\partial q_\rho} \eta_\rho \\
&= L(t,q_\rho,\eta_\rho) + \eta_\alpha G_\alpha(t,q_\rho,\eta_\rho) - \frac{\partial S}{\partial q_\rho} \eta_\rho \\
&= -M(t,q_\rho,\eta_\rho) + \frac{\partial S}{\partial q_\rho} \eta_\rho \\
&= L(t,q_\rho,\eta_\rho) + \eta_\alpha G_\alpha(t,q_\rho,\eta_\rho) - \frac{\partial S}{\partial q_\rho} \eta_\rho \\
&= (3.1.13)
\end{align*}
\]

\[
\begin{align*}
\text{b) } \frac{\partial M}{\partial \phi_\rho} \bigg|_{\phi_\rho = \eta_\rho} &= \frac{\partial S}{\partial q_\rho}, \\
&= \rho = 1, \ldots, n+p \\
&= (3.1.14)
\end{align*}
\]

**Proof**

If $M'$ is a function of the independent variable $\phi_\rho$, then the necessary condition to have an extremum at the point $\phi_\rho = \eta_\rho$ is

\[
\frac{\partial M'}{\partial \phi_\rho} \bigg|_{\phi_\rho = \eta_\rho} = 0 \\
= (3.1.15)
\]

since

\[
M'(t,q_\rho,\eta_\rho) = M(t,q_\rho,\eta_\rho) - \frac{\partial S}{\partial q_\rho} \eta_\rho - \frac{\partial S}{\partial t}, \\
= (3.1.16)
\]

we have
\[ \frac{\partial M'}{\partial \xi} \bigg|_{\xi=n_a} = \frac{\partial M}{\partial \xi} \bigg|_{\xi=n_a} - \frac{\partial S}{\partial q_\rho} = 0 \]  

(3.1.17)

Thus

\[ \frac{\partial M}{\partial \xi} \bigg|_{\xi=n_a} = \frac{\partial S}{\partial q_\rho} \]  

(3.1.18)

Since

\[ M'(t, q_\rho, \eta_\rho) = M(t, q_\rho, \eta_\rho) - \frac{\partial S}{\partial q_\rho} \eta_\rho - \frac{\partial S}{\partial t} = 0, \]  

(3.1.19)

\[ M(t, q_\rho, \eta_\rho) - \frac{\partial S}{\partial q_\rho} \eta_\rho = \frac{\partial S}{\partial t} \]  

(3.1.20)

Using (3.1.8), (3.1.20) becomes

\[ \frac{\partial S}{\partial t} = L(t, q_\rho, \eta_\rho) + \eta_\alpha G_\alpha(t, q_\rho, \eta_\rho) - \frac{\partial S}{\partial q_\rho} \eta_\rho \]  

(3.1.21)

where \( \xi_\alpha \bigg|_{\xi=n_a} = \eta_\alpha \).

**Definition 3.1.1**

The generalized momentum [10] \( P_\rho \) corresponding to the generalized co-ordinate \( q_\rho \) is defined as

\[ P_\rho = \frac{\partial M(t, q_\rho, \xi)}{\partial \xi} \bigg|_{\xi=n_a}, \quad \rho, \sigma = 1, \ldots, n+p \]  

(3.1.22)
By using this definition we can identify the constraint functions \( G_\alpha(t, q_i, \dot{\Phi}) \) as generalized momenta corresponding to generalized velocities \( \dot{\Phi}_\alpha \).

Partial derivatives of (3.1.8) with respect to \( \dot{q}_i \) and \( \dot{\Phi}_\alpha \) are

\[
\frac{\partial M}{\partial \dot{q}_i} = \frac{\partial L}{\partial \dot{q}_i} + \dot{\Phi}_\alpha \frac{\partial G_\alpha}{\partial \dot{\Phi}_\alpha} = p_i , \quad (3.1.23)
\]

and

\[
\frac{\partial M}{\partial \dot{\Phi}_\alpha} = G_\alpha(t, \dot{\Phi}, \dot{q}_i) = p_\alpha \quad (3.1.24)
\]

Now, let us impose the condition that the determinant of the Hessian matrix [7]

\[
W_{\rho\sigma} = \frac{\partial^2 M}{\partial \dot{q}_i \partial \dot{\Phi}_\rho} , \quad \rho, \sigma = 1, \ldots, n+p \quad (3.1.25)
\]

is not zero, then \( M \) is regular function. Furthermore, we can solve the set of equations (3.1.22) for \( \dot{\Phi}_\rho \) as

\[
\dot{\Phi}_\rho = \psi_\rho (p_\rho, q_i, t) \quad (3.1.26)
\]

**Definition 3.1.2**

The Hamiltonian function \( H \) of a constrained system is defined as

\[
H(t, q_i, p_\rho) = -M(t, q_i, \psi_\rho) + p_\rho \psi_\rho . \quad (3.1.27)
\]
Now, from equation (3.1.14) and by using equation (3.1.22), we have

\[ \frac{\partial S}{\partial q_\rho} = p_\rho \]  \hspace{1cm} (3.1.28)

Also from equation (3.1.13),

\[ H(t, q_i, \frac{\partial S}{\partial q_\rho}) = -\frac{\partial S}{\partial t}. \]  \hspace{1cm} (3.1.29)

This equation is called the Hamilton-Jacobi first order partial differential equations of a constrained system[10]. Some partial derivatives of \( H \) may be computed immediately by making use of definitions (3.1.22) and (3.1.27)

\[ \frac{\partial H(t, q_i, p_\rho)}{\partial t} = -\frac{\partial M}{\partial t} - \frac{\partial M}{\partial q_\rho} \frac{\partial \psi_{\rho}}{\partial t} + p_\rho \frac{\partial \psi_{\rho}}{\partial t} \]

\[ = -\frac{\partial M}{\partial t} - p_\rho \frac{\partial \psi_{\rho}}{\partial t} + p_\rho \frac{\partial \psi_{\rho}}{\partial t} = -\frac{\partial M}{\partial t} \]  \hspace{1cm} (3.1.30)

\[ \frac{\partial H}{\partial q_i} \frac{\partial q_i}{\partial q_\alpha} = -\frac{\partial M}{\partial q_\rho} \frac{\partial q_i}{\partial q_\alpha} - \frac{\partial M}{\partial q_\rho} \frac{\partial q_i}{\partial q_\alpha} + p_\rho \frac{\partial \psi_{\rho}}{\partial q_\alpha} \frac{\partial q_i}{\partial q_\alpha} \]  \hspace{1cm} (3.1.31)

\[ \frac{\partial H}{\partial q_\alpha} = -\frac{\partial M}{\partial q_\rho} \frac{\partial \psi_{\rho}}{\partial q_\alpha} + p_\rho \frac{\partial \psi_{\rho}}{\partial q_\alpha} \]  \hspace{1cm} (3.1.32)

Furthermore, we can compute the partial derivatives of \( H \) with respect to \( p_\rho \) as
Thus, the phase space treatment has reduced the variational problem to the solution of the nonlinear first-order partial differential equation (3.1.29). Also \( S(q_\rho, t) \) satisfy HJPDE for an extremum of the function \( M \). Conversely, if \( S(q_\rho, t) \) is any solution of the HJPDE, then \( S(q_\rho, t) \) satisfies the condition (3.1.12), and the solutions of the equations

\[
\frac{\partial H}{\partial p_\rho} = -\frac{\partial M}{\partial \xi} \frac{\partial \psi_\sigma}{\partial p_\rho} + \psi_\rho + p_\sigma \frac{\partial \psi_\sigma}{\partial p_\rho} = \psi_\rho \tag{3.1.33}
\]

are extremals of the action integral.

**3.2 Integration of the HJPDE in Discrete Systems**

In this section we will determine the functions \( S(q_\rho, t) \) which satisfy the Hamilton-Jacobi partial differential equations (HJPDE),

\[
H\left(t, q_\rho, \frac{\partial S}{\partial q_\rho}\right) = -\frac{\partial S}{\partial t}, \tag{3.2.1}
\]

and this aim can be achieved by the method of characteristics[9], and this method reduces the problem from the solution of the non-linear first-order partial differential equations to the solution of the first-order ordinary differential equations which are called Hamilton’s canonical equations.

Now, if we assume that
\[ t = q_0 \quad , \quad p_0 = -H(t, q_1, p_0) \quad (3.2.2) \]

where

\[ p_p = \frac{\partial S}{\partial q_p} , \quad (3.2.3) \]

then these modifications making the HJPDE as

\[ H\left(t, q_i, \frac{\partial S}{\partial q_p}, \frac{\partial S}{\partial t}\right) = H\left(t, q_i, \frac{\partial S}{\partial q_p}\right) + \frac{\partial S}{\partial t} = 0 \quad (3.2.4) \]

or

\[ H\left(q_r, \frac{\partial S}{\partial q_r}\right) = 0 , \quad r = 0, 1, \ldots, n+p \quad (3.2.5) \]

Thus we extended the phase space from 2(n+p) dimensions to 2(n+p)+2 dimensions

**Proposition 3.2.1** [10]

If the functions \( S(q_r) \) is a solution of equation (3.2.5) and also the equations

\[ \frac{\partial H'}{\partial p_r} = \frac{dq_r}{dt} \quad (3.2.6) \]

are satisfied, then

\[ \frac{\partial H'}{\partial q_r} = -\frac{dp_r}{dt} \quad (3.2.7) \]
Proof

Assume that

\[ z(t) = S(q_r), \quad (3.2.8) \]

\[ p_r(t) = \frac{\partial S(q_r)}{\partial q_r}, \quad (3.2.9) \]

Since the function \( S(q_r) \) is a solution of the HJPDE, the derivative of (3.2.5) is given as

\[ \frac{dH'}{dq_r} = \frac{\partial H'}{\partial q_r} + \frac{\partial H'}{\partial p_r} \frac{\partial^2 S}{\partial q_r \partial q_s} = 0, \quad s = 0, 1, \ldots, n+p \quad (3.2.10) \]

Derivatives (3.2.9) with respect to \( t \), lead to

\[ \frac{dp_r}{dt} = \frac{d}{dt} \left( \frac{\partial S(q_r)}{\partial q_r} \right) = \frac{\partial^2 S}{\partial q_r \partial q_s} \frac{dq_r}{dt} = \frac{\partial^2 S}{\partial q_r \partial q_s} \frac{\partial q_r}{\partial \xi} = \eta \quad (3.2.11) \]

and

\[ \frac{dz}{dt} = \frac{\partial S}{\partial q_r} \frac{dq_r}{dt} = \frac{\partial S}{\partial q_r} \frac{\partial q_r}{\partial \xi} = p_r \eta \quad (3.2.12) \]

Making use of (3.2.6), (3.2.12) becomes

\[ \frac{dz}{dt} = \frac{\partial S}{\partial q_r} \eta = p_r \frac{\partial H'}{\partial p_r} \quad (3.2.13) \]
Adding equations (3.2.11) and (3.2.10) we obtain

\[
\frac{\partial H'}{\partial q_r} + \frac{\partial H'}{\partial p_r} \frac{\partial^2 S}{\partial q_r \partial q_s} + \frac{d p_r}{d t} - \frac{\partial^2 S}{\partial q_r \partial \xi} \xi = 0
\]  
(3.2.14)

Using (3.2.6), (3.2.14) becomes

\[
\frac{\partial H'}{\partial q_r} + \xi \frac{\partial^2 S}{\partial q_r \partial q_s} + \frac{d p_r}{d t} - \xi \frac{\partial^2 S}{\partial q_r \partial \xi} = 0, 
\]  
(3.2.15)

\[
\frac{\partial H'}{\partial q_r} = -\frac{d p_r}{d t}. 
\]  
(3.2.16)

Also since

\[
\frac{\partial H'}{\partial q_\rho} = \frac{\partial H}{\partial q_\rho}, 
\]  
(3.2.17)

\[
\frac{\partial H}{\partial q_\rho} = -\frac{d p_\rho}{d t} \quad \rho = 1, \ldots, n+p 
\]  
(3.2.18)

and

\[
\frac{\partial H'}{\partial q_0} = \frac{\partial H'}{\partial t} = \frac{\partial H}{\partial t} = -\frac{d p_0}{d t}. 
\]  
(3.2.19)

Furthermore, we can obtain the last equation by the following way:

since \( p_0 = -H(t, q_i, p_\rho) \),
By using (3.2.18), (3.2.20) becomes

\[
\frac{dp_0}{dt} = -\frac{\partial H}{\partial t}, \quad (3.2.21)
\]

In this theorem we extend the 2(n+p)-dimensional phase space in the equation (3.2.1) to 2(n+p)+2 in the equation (3.2.5) by introducing \( t \) and \( H \) as generalized co-ordinate and generalized momentum, respectively. Also, we can discuss that \( \hat{X}_a \) is explicit independence of \( H \), that is,

\[
\frac{\partial \hat{X}_a}{\partial \hat{X}_a} = 0 \quad (3.2.22)
\]

By using the equations (3.1.27) and (3.1.24), we have

\[
\frac{\partial H}{\partial \hat{X}_a} = -p_a = -G_a(t, q, \hat{X}) = 0
\]

\[
\frac{dp_a}{dt} = \frac{dG_a}{dt} = 0, \quad \alpha = 1, \ldots, m \quad (3.2.23)
\]

This implies that \( p_a = G_a(t, q, \hat{X}) \) are constants of motion.

Now the 2(n+p) first-order ordinary differential equations

\[
\frac{dp_0}{dt} = -\frac{\partial H}{\partial t}, \quad \frac{dp_a}{dt} = -\frac{\partial H}{\partial \hat{X}_a} + \frac{\partial H}{\partial p_a} \hat{X}_a \quad (3.2.20)
\]
\[ \frac{\partial H'}{\partial q_\rho} = -\xi', \quad \frac{\partial H'}{\partial p_\rho} = \xi, \quad \rho = 1, \ldots, n+p \quad (3.2.24) \]

are called Hamilton’s canonical equations [10].

Now, the general solutions of the canonical equations are given as

\[ q_\rho = \xi_\rho(t, u_\alpha), \quad p_\rho = \eta_\rho(t, u_\alpha) \quad (3.2.25) \]

where \( u_\alpha \) are arbitrarily parameters.

More explicitly

\[ q_\alpha = \xi_\alpha(t, u_\alpha), \quad q_\alpha = \lambda_\alpha = \xi_\alpha(t, u_\alpha), \quad p_\alpha = \eta_\alpha(t, u_\alpha) \]

\[ p_\alpha = \eta_\alpha(t, u_\alpha), \quad t = \xi_0(t, u_\alpha) = q_0, \quad p_0 = \eta_0(t, u_\alpha) = -H(t, \xi_\rho, \eta_\rho) \quad (3.2.26) \]

Integration of (3.2.12) gives us the Hamilton’s principal function [6], [10],

\[ S(q_\rho, t) \] in terms of arbitrary parameters \( u_\alpha \),

\[ z(t) = \sigma(t, u_\alpha) = s(u_\alpha) + \int_{\tau(u_\alpha)}^{t} \eta_r \frac{\partial H(q_r, p_r)}{\partial p_r} \bigg|_{q_r=\xi, p_r=\eta} dt \quad (3.2.27) \]

with initial values

\[ \sigma = s(u_\alpha), \quad t = \tau(u_\alpha) \quad (3.2.28) \]
CHAPTER IV

HAMILTON-JACOBI THEORY OF CONTINUOUS SYSTEMS

In this chapter we will study the construction of the Hamilton-Jacobi partial differential equation for classical field systems, or continuous systems in 5n-dimensional phase space and the integration of it by the method of characteristics. The motivation behind this chapter is to establish a valid Hamiltonian approach of classical field such that the passage from the Lagrangian to the Hamiltonian approach follows naturally and the Hamilton-Jacobi partial differential equation plays the fundamental role in this study.

We say that the dynamical system is continuous if between every two points in the system there exist a point. For example, if a particle moves on the surface, the particle will be passing at every point on the surface.

4.1 Determination of the Hamilton-Jacobi Partial Differential Equation (HJPDE) for Continuous Systems

In this section we will construct the Hamilton-Jacobi partial differential equation (HJPDE) by using the continuous indices x,y,z[4].

If Q is the configuration manifold, that means the generalized co-ordinates $q_i$ belongs to Q and if we defined the Lagrangian $L = L(q_i, \dot{q}_i, t)$ is the mapping from the tangent bundle[12] $TQ$ to $R$, i.e.

$$L : TQ \rightarrow R$$ \hspace{1cm} (4.1.1)

then we say that the curve $q_i=q_i(t)$ are extremals [4], [10], [11], [14] of the action integral

$$I = \int_{t_0}^{t_1} L(q_i, \dot{q}_i, t) \, dt \, ,$$ \hspace{1cm} (4.1.2)
if the Euler-Lagrange equations

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0 \quad \forall i$$  \hspace{1cm} (4.1.3)

are valid under the condition that there is no variation at \( t_0 \) and \( t_1 \). Also, we define the Euler-Lagrange equations under transformation

$$L \rightarrow L' = L(q_i, \phi_i, t) - \frac{dF(q_i, t)}{dt}$$  \hspace{1cm} (4.1.4)

where \( F \) is an arbitrary at least twice-differentiable function. In the continuous systems, the equivalent Lagrangian density is defined as

$$\mathcal{Z}' = \mathcal{Z} - \frac{dS_\mu(\phi_j, x_\nu)}{dx_\mu}, \mu, \nu = 1, 2, 3, 4$$  \hspace{1cm} (4.1.5)

where \( \mathcal{Z} = \mathcal{Z} \left( \phi_j, \frac{\partial \phi_j}{\partial x_\mu}, x_\nu \right) \) is the Lagrangian density defined on the tangent bundle of the manifold of field \( \phi_j( x_\nu ) \), and the arbitrary four functions \( S_\mu \) are at least twice differentiable.

**Proposition 4.1.1**

Consider a regular Lagrangian density[4], [14] \( \mathcal{Z}' \left( \phi_j, \frac{\partial \phi_j}{\partial x_\mu}, x_\nu \right) \). If it is possible to determine a set of differentiable functions \( \eta_\mu( \phi_j, x_\nu ) \) and four functions \( S_\mu( \phi_j, x_\nu ) \) such that

a) \( \mathcal{Z}'(\phi_j, \eta_\mu, x_\nu) = 0 \)  \hspace{1cm} (4.1.6)
b) \( \mathcal{Z}' > 0 \) in the neighborhood of the point with co-ordinates \( \eta_{\mu,i} \), then the solutions of the system of partial differential equations

\[
\frac{\partial \varphi_i}{\partial x_\mu} = \eta_{\mu,i} \quad (4.1.7)
\]

are extremals of the action integral

\[
I = \int \mathcal{Z} \left( \varphi_i, \frac{\partial \varphi_i}{\partial x_\mu}, x_\mu \right) d^4x \quad (4.1.8)
\]

if all the variations at the boundary are zero. The solutions of the equation (4.1.7) are extremals of the action integral (4.1.8) as well if the following equations hold:

\[
\frac{\partial S_\mu}{\partial x_\mu} = \mathcal{Z} \left( \varphi_i, \eta_{\mu,i}, x_\mu \right) - \frac{\partial S_\mu}{\partial \varphi_i} \eta_{\mu,i}, \quad (4.1.9)
\]

and

\[
\frac{\partial \mathcal{Z}}{\partial \left( \frac{\partial \varphi_i}{\partial x_\nu} \right)} \bigg|_{\varphi_i = \eta_{\mu,i}} = \frac{\partial S_\mu}{\partial \varphi_i} \quad (4.1.10)
\]

**Proof**

Let \( \mathcal{Z}' \) be a function of variables \( \varphi_i/\partial x_\mu \), then conditions a) and b) say that \( \mathcal{Z}' \) has a local minimum at the point \( \frac{\partial \varphi_i}{\partial x_\mu} = \eta_{\mu,i} \), \( \forall \mu, i \) and it is minimum value at...
that point is zero. But since equation (4.1.5) is satisfied, \( \mathcal{Z} \) is a local minimum at \( \eta_{\mu} \) and also the solution of \( \frac{\partial \phi_{\mu}}{\partial x_{\mu}} = \eta_{\mu} \) are extremals of the action integral (4.1.8) .

Now from the conditions a) and b), the solutions of equation (4.1.7) are extremals of the action integral, this implies

\[
\mathcal{Z}' = \mathcal{Z} - \frac{dS_{\mu}(\phi_{\mu}, x_{\mu})}{dx_{\mu}} = 0 \tag{4.1.11}
\]

or more explicitly

\[
\mathcal{Z}(\phi_{\mu}, \eta_{\mu}, x_{\mu}) - \frac{\partial S_{\mu}}{\partial \phi_{i}} \eta_{\mu} - \frac{\partial S_{\mu}}{\partial x_{\mu}} = 0 \tag{4.1.12}
\]

and this implies

\[
\mathcal{Z}(\phi_{\mu}, \eta_{\mu}, x_{\mu}) - \frac{\partial S_{\mu}}{\partial \phi_{i}} \eta_{\mu} = \frac{\partial S_{\mu}}{\partial x_{\mu}} \tag{4.1.13}
\]

Since \( \mathcal{Z}' \) is minimum at \( \eta_{\mu} \) and equals zero at this point,

\[
\frac{\partial \mathcal{Z}}{\partial \left( \frac{\partial \phi_{\mu}}{\partial x_{\nu}} \right) \bigg|_{\phi_{\mu} = \eta_{\mu}}} = \frac{\partial \mathcal{Z}}{\partial \left( \frac{\partial \phi_{\mu}}{\partial x_{\nu}} \right) \bigg|_{\phi_{\mu} = \eta_{\mu}}} - \frac{\partial S_{\mu}}{\partial \phi_{i}} = 0 \tag{4.1.14}
\]

and equation the (4.1.10) is hold .
Definition 4.1.1

If \( \mathcal{F}(\varphi, \frac{\partial \varphi}{\partial x^\mu}, x^\nu) \) is a regular Lagrangian density[8], the generalized momentum \( P_{\mu i} \) is defined as

\[
P_{\mu i} = -\frac{\partial \mathcal{F}}{\partial \left( \frac{\partial \varphi}{\partial x^\mu} \right)} \tag{4.1.15}
\]

since \( \mathcal{F} \) is regular, we can solve equation (4.1.15) \( \frac{\partial \varphi}{\partial x^\mu} \) in terms of \( P_{\nu i}, \varphi, \) and \( x^\nu \), i.e.

\[
\frac{\partial \varphi_i}{\partial x^\mu_i} = \xi_{\mu i}(P_{\nu i}, \varphi_j, x^\nu) \tag{4.1.16}
\]

When using equations (4.1.15) and (4.1.16), the equation (4.1.9) leads to

\[
\mathcal{F}(\varphi, \xi_{\mu i}, x^\nu) - P_{\mu i} \xi_{\mu i} = \frac{\partial S_{\mu}}{\partial x^\mu} \tag{4.1.17}
\]

Definition 4.1.2

The Hamiltonian density \( H \) of classical field systems is defined on \( T^*Q \) [13] as

\[
H(\varphi, P_{\mu i}, x^\nu) = -\mathcal{F}(\varphi, \xi_{\mu i}, x^\nu) + P_{\mu i} \xi_{\mu i} \tag{4.1.18}
\]

Using (4.1.17) and (4.1.18) the Hamilton-Jacobi equation of a continuous system will be
\[ \frac{\partial S_\mu}{\partial x_\mu} = -H \left( \varphi_i, \frac{\partial S_\mu}{\partial \varphi_i}, x_v \right) \]  

(4.1.19)

Now, let us compute the following partial derivatives of \( H \) by the aid of definitions (4.1.15) and (4.1.18):

\[ \frac{\partial H}{\partial x_v} = -\frac{\partial \Im}{\partial x_v} - \frac{\partial \Im}{\partial x_i} \frac{\partial \xi_{\mu i}}{\partial x_v} + P_\mu \frac{\partial \xi_{\mu i}}{\partial x_v} = -\frac{\partial \Im}{\partial x_v}, \]  

(4.1.20)

\[ \frac{\partial H}{\partial \varphi_i} = -\frac{\partial \Im}{\partial \varphi_i} - \frac{\partial \Im}{\partial x_i} \frac{\partial \xi_{\mu i}}{\partial \varphi_i} + P_\mu \frac{\partial \xi_{\mu i}}{\partial \varphi_i} = -\frac{\partial \Im}{\partial \varphi_i}, \]  

(4.1.21)

\[ \frac{\partial H}{\partial P_{\mu i}} = -\frac{\partial \Im}{\partial x_i} \frac{\partial \xi_{\nu j}}{\partial P_{\mu i}} + \xi_{\mu i} + P_\nu \frac{\partial \xi_{\nu j}}{\partial P_{\mu i}} \]  

(4.1.22)

From the previous notations, the phase space treatment has reduced the variational problem to the solution of the nonlinear first-order partial differential equation (4.1.19) as we discussed in the discrete case. In other words, the functions \( S_{\mu} (\varphi_i, x_v) \) should satisfy the HJPDE when \( \xi_{\mu i} \) is minimum of \( \Im' \).

Conversely, if \( S_{\mu} \) are any set of solutions of HJPDE, then

\[ \frac{\partial S_{\mu}}{\partial x_\mu} = -H \left( \varphi_i, \frac{\partial S_\mu}{\partial \varphi_i}, x_v \right). \]  

But if \( \frac{\partial S_{\mu}}{\partial x_\mu} = \Im (\varphi_i, \eta_{\mu i}, x_v) - \frac{\partial S_\mu}{\partial \varphi_i} \eta_{\mu i} \), then by previous proposition the solution of equations (4.1.7) which also equal

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\[ \frac{\partial \varphi_i}{\partial x^\mu} = \left. \frac{\partial H}{\partial P_{\mu, i}} \right|_{P_{\mu, i} = \frac{\partial S}{\partial \varphi_i}} \quad (4.1.23) \]

is an extremal of the action integral. Thus the function \( H(\varphi_i, P_{\mu, i}, x^\nu) \) completely characterizes the variational problem.

### 4.2 Integration of the HJPDE of Continuous System

In this section we will investigate the integration of the HJPDE by reducing the integration of the most general first-order partial differential equation to that of ordinary differential equation[14].

Let \( H(\varphi_i, P_{\mu, i}, x^\nu) \) be at least twice continuously differentiable function and this function is defined on a \((5n+4)\)-dimensional space. Define \( H' \) as

\[ H' = H(\varphi_i, P_{\mu, i}, x^\nu) + P_{\mu, i} \quad (4.2.1) \]

Replacing the co-ordinates in (4.2.1) by co-ordinates \( \varphi'_\rho \) and \( P'_{\mu, \rho} \) \( (\rho = 1, \ldots, n + 4) \) such that

\[
\begin{align*}
\varphi'_1 &= \varphi_1, \ldots, \varphi'_n &= \varphi_n, \varphi'_{n+1} = x_1, \ldots, \varphi'_{n+4} = x_4 \\
P'_{\mu, i} &= P_{\mu, i}, P'_{\mu, \nu} &= P_{\mu, \nu}
\end{align*}
\]

Now, we try to find a function \( S(\varphi'_\rho) \) such that when \( P_{\mu, \rho} \) is replaced by \( \frac{\partial S}{\partial \varphi'_\rho} \) in \( H' \), the HJPDE is satisfied, i.e.

\[ H' \left( \varphi'_\rho, \frac{\partial S}{\partial \varphi'_\rho} \right) = 0 \quad (4.2.2) \]
Proposition 4.2.1 [14]

If the functions \( S_m r_j(x) \) are a set of solutions of equation (4.2.2) and, besides the equations

\[
\frac{\partial \varphi'_\rho}{\partial x_\mu} = \frac{\partial H'}{\partial P'_{\mu \rho}}
\]  

(4.2.3)

are valid, then the following set of partial differential equations is satisfied:

\[
\frac{\partial P'_{\mu \rho}}{\partial x_\mu} = -\frac{\partial H'}{\partial \varphi'_\rho} \quad \forall \rho
\]  

(4.2.4)

Proof

Let

\[
z_\mu (x_\nu) = S_\mu (\varphi'_\rho(x_\nu)), \quad \mu = 1,2,3,4
\]  

(4.2.5)

\[
P'_{\mu \rho} (x_\nu) = \frac{\partial S_\mu}{\partial \varphi'_\rho}
\]  

(4.2.6)

Since the functions \( S_\mu (\varphi'_\rho(x_\nu)) \) are solutions of (4.2.2), the total variation of \( H' \) with respect to each \( \varphi'_\rho \) should be zero, i.e.

\[
\frac{d H'}{d \varphi'_\rho} \left( \varphi'_\rho, \frac{\partial S_\mu}{\partial \varphi'_\rho} \right) = \frac{\partial H'}{\partial \varphi'_\rho} + \frac{\partial H'}{\partial P'_{\mu \sigma}} \frac{\partial^2 S_\mu}{\partial \varphi'_\rho \partial \varphi'_\sigma} = 0 \quad \forall \rho, \sigma = 1,\ldots,n+4
\]  

(4.2.7)
Also since (4.2.5) is valid,

\[
\frac{\partial z_\mu}{\partial x_v} = \frac{\partial (S_\mu (\varphi'_\rho))}{\partial x_v} = \frac{\partial S_\mu}{\partial \varphi'_\rho} \frac{\partial \varphi'_\rho}{\partial x_v} \quad (4.2.8)
\]

From (4.2.3) and (4.2.6), (4.2.8) becomes

\[
\frac{\partial z_\mu}{\partial x_v} = \frac{\partial S_\mu}{\partial \varphi'_\rho} \frac{\partial \varphi'_\rho}{\partial x_v} = P^\mu_\rho \frac{\partial H'}{\partial P^\nu_\rho} = P^\mu_\nu + P^\mu_\rho \frac{\partial H}{\partial P^\nu_\rho} \quad (4.2.9)
\]

Partial derivative of (4.2.6) with respect to \( x_\mu \) gives us

\[
\frac{\partial P^\mu_\rho}{\partial x_\mu} = \frac{\partial^2 S_\mu}{\partial \varphi'_\rho \partial \varphi'_\sigma} \frac{\partial \varphi'_\sigma}{\partial x_\mu} \quad (4.2.10)
\]

Adding equations (4.2.7) and (4.2.10), and using (4.2.3), we have

\[
\frac{\partial H'}{\partial \varphi'_\rho} + \frac{\partial P^\mu_\rho}{\partial x_\mu} = \frac{\partial^2 S_\mu}{\partial \varphi'_\rho \partial \varphi'_\sigma} \left( \frac{\partial \varphi'_\sigma}{\partial x_\mu} - \frac{\partial H'}{\partial P^\mu_\sigma} \right) = 0 \quad (4.2.11)
\]

Equation (4.2.11) leads to

\[
\frac{\partial H'}{\partial \varphi'_\rho} = -\frac{\partial P^\mu_\rho}{\partial x_\mu} \quad (4.2.12)
\]

Using equations (4.2.1), one may writes another form of HJPDE as
\[ \frac{\partial H'}{\partial x_v} = \frac{\partial H}{\partial x_v} + \frac{\partial}{\partial x_v} \left( \frac{\partial S_{\mu}}{\partial x_{\mu}} \right) = 0 \quad (4.2.13) \]

or

\[ \frac{\partial H}{\partial x_v} = - \frac{\partial}{\partial x_{\mu}} \left( \frac{\partial S_{\mu}}{\partial x_v} \right) \quad (4.2.14) \]

By means of (4.2.6), we have

\[ \frac{\partial H}{\partial x_v} = - \frac{\partial P_{\mu v}}{\partial x_{\mu}} \quad (4.2.15) \]

**Proposition 4.2.2**

If the functions \( S_{\mu}(\Phi, x_v) \) are a set of solutions of the HJPDE, then the matrix

\[ P_{\mu v} = \frac{\partial S_{\mu}}{\partial x_v} = S_{\mu v} \quad (4.2.16) \]

coincides with the energy-momentum tensor \( T_{\mu v} \) [4], [14] of the physical system.

**Proof**

Elements \( P_{\mu v} \) (\( \mu, v = 1, 2, 3, 4 \)) satisfy equation (4.2.4). Hence,

\[ \frac{\partial P_{\mu v}}{\partial x_{\mu}} = \frac{\partial}{\partial x_{\mu}} \left( \frac{\partial S_{\mu}}{\partial x_v} \right) = - \frac{\partial H}{\partial x_v} \quad (4.2.17) \]
Using (4.1.18), we have

\[
\frac{\partial}{\partial x_{\mu}} \left( \frac{\partial S_\mu}{\partial x_{\nu}} \right) = -\frac{\partial H}{\partial x_{\nu}} = -\frac{\partial}{\partial x_{\nu}} \left( -\Im + P_{\mu i} \frac{\partial \Phi_i}{\partial x_{\mu}} \right)
\]

(4.2.18)

\[
\frac{\partial}{\partial x_{\mu}} \left( \frac{\partial S_\mu}{\partial x_{\nu}} \right) = \frac{\partial \Im}{\partial x_{\nu}} \left( P_{\mu i} \frac{\partial \Phi_i}{\partial x_{\nu}} \right)
\]

(4.2.19)

\[
\frac{\partial}{\partial x_{\mu}} \left( \frac{\partial S_\mu}{\partial x_{\nu}} \right) = \frac{\partial \Im}{\partial x_{\nu}} - \frac{\partial}{\partial x_{\nu}} \left( \frac{\partial S_\mu}{\partial x_{\mu}} \right)
\]

(4.2.20)

Since \( x_{\mu}, x_{\nu} \) are independent we have

\[
\frac{\partial}{\partial x_{\mu}} \left( \frac{\partial S_\mu}{\partial x_{\nu}} \right) = \frac{\partial \Im}{\partial x_{\nu}} - \frac{\partial}{\partial x_{\nu}} \left( \frac{\partial S_\mu}{\partial x_{\mu}} \right)
\]

(4.2.21)

\[
\frac{\partial}{\partial x_{\mu}} \left( \frac{\partial S_\mu}{\partial x_{\nu}} \right) = \frac{\partial \Im}{\partial x_{\nu}} \left( \frac{\partial S_\mu}{\partial \Phi_i} \frac{\partial \Phi_i}{\partial x_{\mu}} \right)
\]

(4.2.22)

\[
\frac{\partial}{\partial x_{\mu}} \left( \frac{\partial S_\mu}{\partial x_{\nu}} \right) = \frac{\partial}{\partial x_{\mu}} \left( \Im \frac{\partial x_{\nu}}{\partial x_{\mu}} + P_{\mu i} \frac{\partial \Phi_i}{\partial x_{\nu}} \right)
\]

(4.2.23)

\[
\frac{\partial}{\partial x_{\mu}} \left( \frac{\partial S_\mu}{\partial x_{\nu}} \right) = \frac{\partial}{\partial x_{\mu}} \left( \Im \frac{\partial x_{\nu}}{\partial x_{\mu}} + P_{\mu i} \frac{\partial \Phi_i}{\partial x_{\mu}} \right) = \frac{\partial T_{\mu \nu}}{\partial x_{\mu}}
\]

(4.2.24)

Also since (4.2.17) satisfied, we have

\[
\frac{\partial T_{\mu \nu}}{\partial x_{\mu}} = -\frac{\partial H}{\partial x_{\nu}}
\]

(4.2.25)
this implies that $P_{\mu v}, T_{\mu v}$ are coincides.

**Definition 4.2.1**

The first-order $5n$ partial differential equations

$$\frac{\partial \Phi_i}{\partial x_{\mu}} = \frac{\partial H}{\partial P_{\mu i}}, \quad \forall \mu, i$$

(4.2.26)

$$\frac{\partial P_{\mu i}}{\partial x_{\mu}} = -\frac{\partial H}{\partial \Phi_i}, \quad \forall i$$

(4.2.27)

are called Hamilton’s canonical equations for a classical fields[14] system and their solutions are used to determine Hamilton’s principal functions $S_i(\Phi_i, x_{\mu})$. In fact, if equations (4.2.5) and (4.2.15) are valid, then equation (4.2.8) turns out to be

$$\frac{\partial}{\partial x_{\mu}} \left( \frac{\partial z_{\mu}}{\partial x_{\nu}} \right) = -\frac{\partial H}{\partial x_{\nu}} + \frac{\partial}{\partial x_{\mu}} \left( P_{\mu} \frac{\partial \Phi_i}{\partial x_{\nu}} \right)$$

(4.2.28)

where the right-hand side is a function of $x_{\nu}$ only.

Since equations (4.2.26) and (4.2.28) are first-order partial differential equations, solutions of these equations should be given in terms of arbitrarily several parameters $u_1, ..., u_k$. If one assumes the solutions of these equations in the form

$$\phi_i = \xi_i(x_{\nu}, u), \quad i=1, ..., n$$

(4.2.29)
\[ P_{\mu,i} = \eta_{\mu,i}(x_v, u_j), \quad \mu = 1,2,3,4 \quad (4.2.30) \]
\[ z_\mu = \sigma_\mu(x_v, u_j), \quad (4.2.31) \]
\[ P_{\mu,v} = \eta_{\mu,v}(x_v, u_j) \quad (4.2.32) \]

then the functions \( \xi_j \) form the characteristic of the HJ-PDE. Also hypersurfaces
\[ z_j = S_\mu(\Phi_j, x_v), \quad \mu = 1,2,3,4 \quad (4.2.33) \]
in \((n+5)\)-dimensional space touch along the characteristic because \( \xi_j \) satisfy equation (4.2.33).

**Proposition 4.2.3** [14]

The function \( H' \) is an integral of the canonical equations (4.2.26), (4.2.27).

**Proof**

Let
\[ H'(\xi_j, \eta_{\mu,i}, \eta_{\mu,v}, x_v) = H(\xi_j, \eta_{\mu,i}, x_v) + \eta_{\mu v} \equiv K(x_v, u_j) \quad (4.2.34) \]

Taking the partial derivatives with respect to \( x_v \),
\[ \frac{\partial K}{\partial x_v} = \frac{\partial H}{\partial x_v} + \frac{\partial \eta_{\mu,v}}{\partial x_\mu} \quad (4.2.35) \]

Since \( \eta_{\mu,v} \) satisfies the equation (4.2.11),
\[ \frac{\partial \eta_{\mu,v}}{\partial x_\mu} = \frac{\partial}{\partial x_v}(\eta_{\mu,v}) = -\frac{\partial H}{\partial x_v} \quad (4.2.36) \]
Thus the equation (4.2.35) leads to

\[
\frac{\partial K}{\partial x_v} = \frac{\partial H}{\partial x_v} - \frac{\partial H}{\partial x_v} = 0
\]  

(4.2.37)

i.e. \( K \) is independent of \( x_v \). Also, the partial derivative of equation (4.2.34) with respect to \( u_j \) turns out to be

\[
\frac{\partial K}{\partial u_j} = 0
\]  

(4.2.38)

This implies

\[
H'(\xi, \eta, \eta_v, x_v) = 0
\]  

(4.2.39)

and that \( H' \) is an integral of equation (4.2.26), (4.2.27).

**Lemma 4.2.1** [6], [14]

If \( H' \) is an integral of canonical equations, then

\[
d\sigma_\mu(x_v, u_j) = \eta_{\mu v} dx_v + \eta_{\mu i} d\xi_i - \lambda_{\mu j} du_j \quad \forall \mu
\]  

(4.2.40)

where

\[
\lambda_{\mu j}(x_v, u_j) = \frac{\partial \sigma_\mu}{\partial u_j} + \eta_{\mu p} \frac{\partial \xi_p}{\partial u_j}
\]  

(4.2.41)
Proof

If equation (4.2.39) is valid, then

\[
\frac{dK}{du_j} = \frac{\partial H'}{\partial\Phi'} \frac{\partial z_{\mu}}{\partial u_j} + \frac{\partial H'}{\partial z_{\mu}} \frac{\partial\eta_{\mu,\rho}}{\partial u_j} + \frac{\partial H'}{\partial z_{\mu}} \frac{\partial\sigma_{\mu}}{\partial u_j} = 0
\]  

(4.2.42)

Now with the help of equations (4.2.3) and (4.2.4), we obtain

\[
\frac{\partial H'}{\partial\Phi'} = \frac{\partial z_{\mu}}{\partial x_{\mu}}
\]  

(4.2.43)

and

\[
\frac{\partial H'}{\partial\Phi'} = \frac{\partial H'}{\partial\Phi'} + \frac{\partial H'}{\partial z_{\mu}} \frac{\partial S_{\mu}}{\partial\Phi'}
\]  

(4.2.44)

\[
\frac{\partial H'}{\partial\Phi'} = -\frac{\partial\eta_{\mu,\rho}}{\partial x_{\mu}} + \frac{\partial H'}{\partial z_{\mu}} \eta_{\mu,\rho}
\]  

(4.2.45)

\[
\frac{\partial H'}{\partial\Phi'} = \frac{\partial H}{\partial z_{\mu}}
\]  

equation (4.2.42) leads to

\[
\frac{dK}{du_j} = -\frac{\partial\eta_{\mu,\rho}}{\partial x_{\mu}} \frac{\partial z_{\mu}}{\partial u_j} + \frac{\partial z_{\mu}}{\partial x_{\mu}} \frac{\partial\eta_{\mu,\rho}}{\partial u_j} - \frac{\partial H}{\partial z_{\mu}} \left( \frac{\partial\sigma_{\mu}}{\partial u_j} + \eta_{\mu,\rho} \frac{\partial z_{\mu}}{\partial u_j} \right) = 0
\]  

(4.2.47)

From the equation (4.2.8)
\[
\frac{\partial \sigma_{\mu}}{\partial x_{\mu}} = \eta_{\mu \rho} \frac{\partial \xi_{\rho}}{\partial x_{\mu}} \quad (4.2.48)
\]

Thus

\[
\frac{\partial^2 \sigma_{\mu}}{\partial x_{\mu} \partial u_j} = \frac{\partial}{\partial x_{\mu}} \left( \eta_{\mu \rho} \frac{\partial \xi_{\rho}}{\partial x_{\mu}} \right) = \frac{\partial \eta_{\mu \rho}}{\partial x_{\mu}} \frac{\partial \xi_{\rho}}{\partial x_{\mu}} + \frac{\partial \xi_{\rho}}{\partial x_{\mu}} \eta_{\mu \rho} \quad (4.2.49)
\]

since

\[
\frac{\partial^2 \xi_{\rho}}{\partial u_j \partial x_{\mu}} \eta_{\mu \rho} = \frac{\partial}{\partial x_{\mu}} \left( \eta_{\mu \rho} \frac{\partial \xi_{\rho}}{\partial u_j} \right) - \frac{\partial \eta_{\mu \rho}}{\partial x_{\mu}} \frac{\partial \xi_{\rho}}{\partial u_j}.
\]

Inserting the right-hand side of the equation (4.2.52) in the equation (4.2.47), one obtains

\[
\frac{\partial \lambda_{\mu, j}}{\partial x_{\mu}} - \frac{\partial H}{\partial z_{\mu}} \lambda_{\mu, j} = 0 \quad (4.2.53)
\]

since \( \sigma_{\mu} = \sigma_{\mu}(x_{\nu}, u_{j}) \).
\[ d\sigma_\mu = \frac{\partial \sigma_\mu}{\partial x_\nu} dx_\nu + \frac{\partial \sigma_\mu}{\partial u_j} du_j \] (4.2.54)

From the definition (4.2.41), and the equation (4.2.48), the equation (4.2.54) leads to

\[ d\sigma_\mu = \eta_{\mu \rho} \frac{\partial \xi_\rho}{\partial x_\nu} dx_\nu + \left( -\lambda_{\mu j} + \eta_{\mu \rho} \frac{\partial \xi_\rho}{\partial u_j} \right) du_j \] (4.2.55)

The variation of \( \xi_\rho = \xi_\rho (x_\nu, u_j) \) gives

\[ d\xi_\rho = \frac{\partial \xi_\rho}{\partial x_\nu} dx_\nu + \frac{\partial \xi_\rho}{\partial u_j} du_j \]

and the equation (4.2.55) leads to

\[ d\sigma_\mu = \eta_{\mu \rho} \left( \frac{\partial \xi_\rho}{\partial x_\nu} dx_\nu + \frac{\partial \xi_\rho}{\partial u_j} du_j \right) - \lambda_{\mu j} du_j = \eta_{\mu \rho} d\xi_\rho - \lambda_{\mu j} du_j \] (4.2.56)

or explicitly,

\[ d\sigma_\mu = \eta_{\mu \rho} d\xi_\rho + \eta_{\mu \nu} dx_\nu - \lambda_{\mu j} du_j \] (4.2.57)

let us parametrize the initial values by arbitrary function \( \tau_\nu (u_j) \) as

\[ x_\nu^0 = \tau_\nu (u_j) \] (4.2.58)

and define the initial values of \( \Phi_\rho, P_{\rho \mu} \) and \( \sigma_\mu \) as

\[ \xi_\rho (x_\nu^0, u_j) \equiv A_\rho (u_j), \] (4.2.59)

\[ \eta_{\mu \rho} (x_\nu^0, u_j) \equiv B_{\mu \rho} (u_j), \] (4.2.60)
\[ \sigma_\mu(x^0_\nu, u_j) \equiv s_\mu(u_j). \quad (4.2.61) \]

**Lemma 4.2.2** [6], [14]

The functions \( \lambda_{\mu \nu} \) are independent of \( x^0_\nu \), i.e.

\[ \lambda_{\mu \nu}(x^0_\nu, u_j) = \lambda_{\mu \nu}(x^0_\nu, u_i) \quad (4.2.62) \]

**Proof**

Consider the partial derivatives of (4.2.61) with respect to \( u_j \)

\[ -\frac{\partial \sigma_\mu}{\partial x^0_\nu} \frac{\partial \tau_\nu}{\partial u_j} - \frac{\partial \sigma_\mu}{\partial u_j} = -\frac{\partial s_\mu}{\partial u_j} \quad \forall \mu, j \quad (4.2.63) \]

Also, consider the partial derivatives of (4.2.59) with respect to \( u_j \)

\[ \frac{\partial \xi_\rho}{\partial x^0_\nu} \frac{\partial \tau_\nu}{\partial u_j} + \frac{\partial \xi_\rho}{\partial u_j} = \frac{\partial A_\rho}{\partial u_j} \quad (4.2.64) \]

Multiplying equation (4.2.64) with \( \eta_{\mu \rho} \), one obtains

\[ \eta_{\mu \rho} \frac{\partial \xi_\rho}{\partial x^0_\nu} \frac{\partial \tau_\nu}{\partial u_j} + \eta_{\mu \rho} \frac{\partial \xi_\rho}{\partial u_j} = B_{\mu \rho} \frac{\partial A_\rho}{\partial u_j} \quad (4.2.65) \]

Adding equations (4.2.63) and (4.2.65) and noticing that equation (4.2.48) is valid, one arrives at the result.
This means that \( \lambda_{\mu j} \) are independent of \( x_v \).

**Lemma 4.2.3** [14]

The initial total differential \( d\sigma_\mu(x_0^v, u_j) \) is

\[
d\sigma_\mu(x_0^v, u_j) = \frac{d[\sigma(x_v, u_j) - s_\mu(u_j)] + B_{\mu \rho} dA_\rho}{\partial u_j} = \frac{\partial s_\mu}{\partial u_j} + \eta_{\mu \rho} \frac{\partial \xi_{\rho}}{\partial u_j} = \lambda_{\mu j}
\]

(4.2.66)

**Proof**

By replacing \( \lambda_{\mu j} \) in the equation (4.2.66) by its value in the equation (4.2.56), we have

\[
d\sigma_\mu(x_0^v, u_j) = \frac{\eta_{\mu \rho} d\xi_{\rho} - \lambda_{\mu j} du_j}{\partial u_j} = \eta_{\mu \rho} d\xi_{\rho} + \frac{\partial s_\mu}{\partial u_j} du_j - B_{\mu \rho} \frac{\partial A_\rho}{\partial u_j} du_j
\]

(4.2.68)

\[
d\sigma_\mu(x_0^v, u_j) = d s_\mu + B_{\mu \rho} dA_\rho = \frac{\eta_{\mu \rho} d\xi_{\rho}}{\partial u_j} = d\sigma_\mu(x_0^v, u_j)
\]

(4.2.69)

At this step one may arrange the prescribed initial values \( A_\rho(u_j) \), \( B_{\mu \rho}(u_j) \) and \( s_\mu(u_j) \) in such a way that the left-hand side of equation (4.2.66) is zero. Hence all functions \( \lambda_{\mu \rho}(x_v, u_i) \) will be set to zero. For the sake of simplicity, assume that the functions \( s_\mu(u_j) \) and \( A_\rho(u_j) \) are given. The problem now is to calculate the functions \( B_{\mu \rho}(u_j) \) such that \( \lambda_{\mu j}(x_0^v, u_i) = 0 \quad \forall \mu, j \).

**Lemma 4.2.4** [6], [14]
For given functions \( s_\mu (u_j) \) and \( A_\rho (u_j) \) the set of equations

\[
\lambda_{\mu j} (x^\rho_0, u_j) = -\frac{\partial s_\mu}{\partial u_j} + B_{\mu \rho} \frac{\partial A_\rho}{\partial u_j} = 0 \quad \forall \mu, j \quad (4.2.70)
\]

can be solved for the functions \( B_{\mu \rho} (u_j) \) if

\[
\left| \frac{\partial \xi_j}{\partial u_j} \right| \neq 0 \quad (4.2.71)
\]

**Proof**

The explicit form of equation (4.2.66) is

\[
-\frac{\partial s_\mu}{\partial u_j} + B_{\mu \rho} \frac{\partial A_\rho}{\partial u_j} + C_{\mu \nu} \frac{\partial \tau_\nu}{\partial u_j} = 0 \quad (4.2.72)
\]

where the functions \( C_{\mu \nu} (u_j) \) are the initial values of \( P_{\mu \nu} \), i.e.

\[
P_{\mu \nu} (x^\mu_0, u_j) = C_{\mu \nu} (u_j) \quad (4.2.73)
\]

In this case, the equation (4.2.1) can be written as

\[
H (x^\mu_0, \eta_{\mu \nu}, \xi_\nu) + C_{\mu \mu} = 0 = H' \quad (4.2.74)
\]

By using the implicit function theorem, which states that one can solve the set of equations (4.2.72) for \( B_{\mu \rho} \) if the following condition holds:
\[
\left| \frac{\partial}{\partial B_{\mu i}} \left( -\frac{\partial s^u}{\partial u_j} + B_{\mu j} \frac{\partial A_i}{\partial u_j} - C_{\mu \nu} \frac{\partial \tau^\nu}{\partial u_j} \right) \right| = \left| \frac{\partial A_i}{\partial u_j} + \frac{\partial C_{\mu \nu}}{\partial B_{\mu i}} \frac{\partial \tau^\nu}{\partial u_j} \right| \neq 0 \quad (4.2.75)
\]

where \( C_{\mu \nu} \) are not independent of \( B_{\mu i} \) but

\[
\frac{\partial C_{\mu \nu}}{\partial \eta_{\mu i}} = \frac{\partial C_{\mu \nu}}{\partial H'} \frac{\partial H'}{\partial \eta_{\mu i}} \quad (4.2.76)
\]

By using (4.2.74), we have

\[
\delta_{\mu \nu} \frac{\partial C_{\mu i}}{\partial \eta_{\mu i}} = -\delta_{\mu \nu} \frac{\partial H}{\partial \eta_{\mu i}} \quad (4.2.77)
\]

\[
\frac{\partial C_{\mu \nu}}{\partial \eta_{\mu i}} = -\delta_{\mu \nu} \frac{\partial H}{\partial \eta_{\mu i}} = -\frac{\partial H}{\partial \eta_{\nu i}} \quad (4.2.78)
\]

The equation (4.2.76) becomes

\[
\frac{\partial C_{\mu \nu}}{\partial \eta_{\mu i}} = \frac{\partial C_{\mu \nu}}{\partial H'} \frac{\partial H'}{\partial \eta_{\mu i}} = -\delta_{\mu \nu} \frac{\partial H}{\partial \eta_{\mu i}} = -\frac{\partial H}{\partial \eta_{\nu i}} \quad (4.2.79)
\]

Thus the determinant (4.2.75) reads as

\[
\left| \frac{\partial A_i}{\partial u_j} - \frac{\partial H}{\partial \eta_{\nu i}} \frac{\partial \tau^\nu}{\partial u_j} \right| \neq 0 \quad (4.2.80)
\]

since \( \xi_i(\chi^0, u_j) = A_i(u_j), \chi^0 = \tau^\nu(u_j). \)
\[
\frac{\partial \xi_i}{\partial u_j} = \frac{\partial A_i}{\partial u_j} - \frac{\partial \xi_j}{\partial x_v} \frac{\partial \tau_v}{\partial u_j} \quad (4.2.81)
\]

and from equation (4.2.26), we have

\[
\frac{\partial \xi_i}{\partial x_v} = \frac{\partial H}{\partial \eta_{v,i}}. \quad (4.2.82)
\]

Then

\[
\frac{\partial \xi_i}{\partial u_j} = \frac{\partial A_i}{\partial u_j} - \frac{\partial \xi_j}{\partial x_v} \frac{\partial \tau_v}{\partial u_j} = \frac{\partial A_i}{\partial \eta_{v,i}} \frac{\partial \tau_v}{\partial u_j} \quad (4.2.83)
\]

Condition (4.2.80) takes the form

\[
\left| \frac{\partial \xi_i}{\partial u_j} \right| \neq 0 \quad i, j = 1, \ldots, n. \quad (4.2.84)
\]

As a conclusion, if we choose those characteristics for which the condition (4.2.71) is valid, then it is always possible from the last lemma to set all functions \( \lambda_{v,j}(x_v, u_j) \) to be zero such that the total differential reads as

\[
d\sigma_{\mu}(x_v, u_j) = \eta_{\mu,v} d x_v + \eta_{\mu,i} d \xi_i \quad (4.2.85)
\]

ensuring that

\[
P_{\mu v} = \frac{\partial S_\mu}{\partial x_v}, \quad P_{\mu i} = \frac{\partial S_\mu}{\partial \phi_i}. \quad (4.2.86)
\]

**Lemma 4.2.5 [14]**
Hamilton’s principal functions $S_{ij}(\varphi_i, x_v)$ are uniquely determined by the initial conditions $\xi_j(x^0_v, u_j) = A_j$ and $\sigma_j(x^0_v, u_j) = s_j$, if one determines the initial values $B_{\mu j}$ properly.

**Proof**

Assume that in the neighborhood of a point $u^0_i$, in the space of $u_i$, there exist four functions $s'_m$ as solutions of the HJPDE and satisfying the initial conditions

$$\xi_j(x^0_v, u_j) = A_j, \eta_{ij}(x^0_v, u_j) = B'_{ij} = \frac{\partial s'_m}{\partial \varphi_j}, \sigma_m(x^0_v, u_j) = s_m \quad (4.2.87)$$

since $s'_m$ are solutions of the HJPDE, $H'$ is an integral of canonical equations, i.e.

$$H'(A_i, B'_{ij}, x^0_v) = 0 \quad (4.2.88)$$

Besides

$$-\frac{\partial s'_m}{\partial u_j} + B'_{ij} \frac{\partial A_j}{\partial u_j} + C_{ij} \frac{\partial \tau_v}{\partial u_j} = 0 \quad (4.2.89)$$

By using the same way of previous lemma, we have

$$\left| \frac{\partial \xi_j}{\partial u_j} \right| \neq 0 \quad (4.2.90)$$
But since if the last condition is satisfied, then the solution of HJPDE is unique. Thus,

\[ s_{\mu} = s'_{\mu} \quad \forall \mu. \]  \hspace{1cm} (4.2.91)

At last we will take an example which establishes how we can construct the Hamiltonian function and its partial differential equations of motion.

**Example 4.2.1: Complex Scalar Fields**

The free-field Lagrangian density is given in terms of two independent world scalar fields \( \phi \) and \( \phi^* \) [7], as

\[ \Im \left( \phi \phi^*, \frac{\partial \phi}{\partial x_{\mu}}, \frac{\partial \phi^*}{\partial x_{\mu}} \right) = -c^2 \left( \frac{\partial \phi}{\partial x_{\mu}} \frac{\partial \phi^*}{\partial x_{\mu}} + \mu_0^2 \phi \phi^* \right), \]  \hspace{1cm} (4.2.92)

where \( \mu_0 \) is a constant and \( \Im \) is a world scalar.

The generalized momenta \( P_{\mu 1}, P_{\mu 2} \) corresponding to \( \phi \) and \( \phi^* \), respectively, are

\[ P_{\mu 1} = \frac{\partial \Im}{\partial \frac{\partial \phi}{\partial x_{\mu}}} = -c^2 \frac{\partial \phi^*}{\partial x_{\mu}} \]  \hspace{1cm} (4.2.93)

\[ P_{\mu 2} = \frac{\partial \Im}{\partial \frac{\partial \phi^*}{\partial x_{\mu}}} = -c^2 \frac{\partial \phi}{\partial x_{\mu}} \]  \hspace{1cm} (4.2.94)

But since the Hamiltonian density function reads as
\[ H(\varphi, \varphi^*, P_{\mu 1}, P_{\mu 2}) = -3 \left( \frac{\partial \varphi}{\partial x_\mu} \frac{\partial \varphi^*}{\partial x_\mu} \right) + P_{\mu 1} \frac{\partial \varphi}{\partial x_\mu} + P_{\mu 2} \frac{\partial \varphi^*}{\partial x_\mu} \] (4.2.95)

\[ H(\varphi, \varphi^*, P_{\mu 1}, P_{\mu 2}) = c^2 \frac{\partial \varphi}{\partial x_\mu} \frac{\partial \varphi^*}{\partial x_\mu} - c^2 \frac{\partial \varphi^*}{\partial x_\mu} \frac{\partial \varphi}{\partial x_\mu} - c^2 \frac{\partial \varphi}{\partial x_\mu} \frac{\partial \varphi^*}{\partial x_\mu} \] (4.2.96)

\[ H(\varphi, \varphi^*, P_{\mu 1}, P_{\mu 2}) = -\frac{1}{c^2} P_{\mu 1} P_{\mu 2} + \mu_0^2 c^2 \varphi \varphi^* \] (4.2.97)

Now, we need to satisfy the first set of canonical equations (4.2.26) as

\[ \frac{\partial H}{\partial p_{\mu 1}} = -\frac{1}{c^2} p_{\mu 2} = -\frac{1}{c^2} \left( -c^2 \frac{\partial \varphi}{\partial x_\mu} \right) = \frac{\partial \varphi}{\partial x_\mu} \] (4.2.98)

\[ \frac{\partial H}{\partial p_{\mu 2}} = -\frac{1}{c^2} p_{\mu 1} = -\frac{1}{c^2} \left( -c^2 \frac{\partial \varphi^*}{\partial x_\mu} \right) = \frac{\partial \varphi^*}{\partial x_\mu} \] (4.2.99)

Thus the first set of canonical equations is satisfied. the second set (4.2.27) will give the equations of motion for \( \varphi \) and \( \varphi^* \). In fact

\[ \frac{\partial p_{\mu 1}}{\partial x_\mu} = -\frac{\partial H}{\partial \varphi} = -\mu_0^2 c^2 \varphi^* \] (4.2.100)

\[ \frac{\partial}{\partial x_\mu} \left( -c^2 \frac{\partial \varphi^*}{\partial x_\mu} \right) = -\mu_0^2 c^2 \varphi^* \] (4.2.101)

\[ \frac{\partial}{\partial x_\mu} \left( \frac{\partial \varphi^*}{\partial x_\mu} \right) - \mu_0^2 \varphi^* = 0 \] (4.2.102)
\[ \frac{\partial P_{\mu \nu}}{\partial x_{\mu}} = -\frac{\partial H}{\partial \phi^\ast} = -\mu_0^2 c^2 \phi \]  \hspace{1cm} (4.2.103)

\[ \frac{\partial}{\partial x_{\mu}} \left( -c^2 \frac{\partial \phi}{\partial x_{\mu}} \right) = -\mu_0^2 c^2 \phi \]  \hspace{1cm} (4.2.104)

\[ \frac{\partial}{\partial x_{\mu}} \left( \frac{\partial \phi}{\partial x_{\mu}} \right) - \mu_0^2 \phi = 0 \]  \hspace{1cm} (4.2.105)
The importance of partial differential equations clearly appears in many physical problems. If we can solve such differential equations, we will be able find out the equations of motion of physical system.

The solution of the partial differential equation is more complicated than the solution of the ordinary differential equation. As shown in chapter two, we could simplify the problem. We proved that every solution of a system of partial differential equations of the form

$$\frac{\partial F}{\partial t_\alpha} + b_{\alpha (t_\beta, x_j)} \frac{\partial F}{\partial x_i} = 0, \quad \alpha = 1, \ldots, m$$  \hspace{1cm} (5.1)

is an integral of the total differential equations in the form

$$dx_i = b_{\alpha (t_\beta, x_j)} dt_\alpha$$  \hspace{1cm} (5.2)

We also found out the integrability conditions of the total differential equations (5.2).

A canonical formulation of singular system leads to equations of motion. These equations of motion are total differential equations of the type (5.2). The construction and integration of the equations of motion of singular system are discussed. The equations of motion of a singular system with constraints are integrable if the variation of these constraints is identically zero. This method needs more elaboration in future work.
In chapter three, Caratheodory’s equivalent Lagrangians method is used to construct the Hamilton-Jacobi formulation of a regular constrained system. This method allows us to treat the variational problem as the problem of minimization undetermined multipliers $\lambda_a$ as generalized velocities makes it possible to define the constraint functions as the generalized momenta $p_a$ conjugate to $\lambda_a$. The necessary condition to have a minimum of the function $M(t, \mathbf{q}, \dot{\mathbf{q}})$ at a specific point leads to the Hamiltonian function and the Hamilton-Jacobi partial differential equation in $2(n+p)$-dimensional phase space.

In chapter four, Hamilton-Jacobi partial differential equation of classical field systems is studied treating $x, y, z,$ and $t$ as parameters. Hamilton’s canonical equations, which are equivalent to the Euler-Lagrange equations, are treated as the first step to integrate the HJPDE, which is the fundamental equation of the Hamilton theory. Hamilton’s principal functions $S_\mu (\phi, \chi), \mu = 1, 2, 3, 4,$ have the peculiar feature that their space-time partial derivatives form the energy-momentum tensor of the physical system.

REFERENCES