On Some Types Of Ideals In Semirings

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To My Parents

To My Wife

To My Sons

To all knowledge seekers...
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List of Symbols

\[ \mathbb{N} \] Natural numbers, 3
\[ \mathbb{Z} \] Integer numbers, 3
\[ \mathbb{Z}_0^+ \] The set of positive integers with zero, 3
\[ \mathbb{Z}_n \] Set of integers modulo \( n = \{0, 1, 2, \ldots, n - 1\} \), 5
\[ n\mathbb{Z} \] Set of integers multiples \( n = \{0, \pm n, \pm 2n, \ldots\} \), 5
\[ A \times B \] Cartesian product of \( A \) and \( B \), 5
\[ \mathcal{P}(X) \] The power set of \( X \), 6
\[ a \lor b \] \( a \) join \( b \), 7
\[ a \land b \] \( a \) meet \( b \), 7
\[ \langle H \cup K \rangle \] Intersection of subgroups containing \( H \cup K \), 7
\[ a' \] Inverse of \( a \), 10
\[ Z(S) \] The set of all zeroids of \( S \), 10
\[ 1_S \] Multiplicative identity of a semiring \( S \), 11
\[ \text{lcm}(a, b) \] least common multiple of \( a, b \), 11
\[ B(n, i) \] Semiring \( B(n, i) = \{0, 1, 2, \ldots, n - 1\} \), 12

IV
$E^+(S)$ The set of all additive idempotents, \(14\)

$E^*(S)$ The set of all multiplicative idempotents, \(14\)

$\langle a \rangle$ The principal ideal generated by \(a\), \(15\)

$A + B$ The sum of ideals, \(16\)

$AB$ product of ideals \(A\) and \(B\), \(16\)

$(I : A)$ The set \(\{ r \in S \mid ra, ar \in I \text{ for all } a \in A\}\), \(16\)

$S \times T$ The direct product of the semirings \(S, T\), \(16\)

gcd\((a, b)\) The greatest common divisor between \(a\) and \(b\), \(22\)

$\bar{A}$ The \(k\)-closure of \(A\), \(23\)

$\rho$ Congruence relation, \(26\)

$x_\rho$ The congruence class of \(x\), \(27\)

$S/\rho$ The set of all congruence classes of \(S\), \(27\)

$P^+(S)$ The set of all some additively periodic elements of \(S\), \(32\)

$\hat{R}$ The smallest \(p\)-ideal containing \(R\), \(37\)

$B^2$ The set of all finite sums of products of \(B\), \(51\)

$\hat{S}a$ The \(p\)-ideal generated by \(a\), \(52\)

$\langle x_1, x_2 \rangle$ The left \(p\)-ideal generated by \(x_1, x_2\), \(63\)

V
Abstract

In this thesis we survey and introduce some concepts of semirings, additive inversive semirings, and lattices. Also ring congruences on an additive inversive semiring are characterized. Finally, we introduce ideals in semirings, and study carefully some special kinds of ideals as $k$-ideals and $p$-ideals.
Introduction

The notion of a semiring was introduced by Vandiver in 1934. Needless to say, semirings found their full place in mathematics long before years.

In 1958, Henriksen [6] defined a more restricted class of ideals in a semiring, which he called this special kind of ideals a $k$-ideal or subtractive. In 1974, Karvellas [8] studied additive inversive semirings and he proved very useful results on it, namely $x = (x')'$, $(x + y)' = y' + x'$, $(xy)' = x'y = y'x$ and $xy = x'y'$ for all $x, y \in S$ where $S$ is additive inversive semiring. In 1992, M. K. SEN[14] studied certain type of ring congruence on an additive inversive semiring with the help of full $k$-ideals, He also show that the set of full $k$-ideals of an additive inversive semiring forms a complete lattice, which is also modular. In 1999, P. Mukhopadhyay [12], provided a construction of a full $E$-inversive semiring, which are subdirect product of a semilattice and a ring. Also in 1999, P. Mukhopadhyay and S. Ghosh[11] introduced a special class of ideals called $p$-ideals, and developed the corresponding class of semifields using this kind of ideal, also they defined a new form of regularity, which is compatible with this class of ideals. In 2002, P. Mukhopadhyay and M. SEN [14] reach to more results about $p$-ideal and define a new class of regularity on a semiring called $p$-regular.
It is well known that a ring $R$ always contains the zero element $0$ which is the only additive idempotent element in $R$. In a semiring $S$ with additive idempotents, as we see that $E^+(S)$ forms an ideal of $S$ which is not necessarily a $k$-ideal. Another generalization of zero element in semirings is the concept of zeroids, $Z(S)$ of a semiring $S$. Clearly, $Z(R) = \{0\}$ for any ring $R$. We point out that $Z(S)$ is a $k$-ideal of $S$. We studied how to generalize the zero element in another way; by analyzing a particular class of additively periodic elements of $S$. The impetus behind the formation of the proposed special class of ideals in $S$ called $p$-ideal, which does not coincide with the existing class of $k$-ideals as we will see later.

The aim of this thesis is to give a survey about these topics, mainly about these classes of ideals in semirings, and study carefully some special kinds of ideals as $k$-ideals and $p$-ideals. This thesis is divided into three chapters.

In chapter one we introduce some basic definitions and theorems about groups, rings, semigroups, semirings, and lattices. Also we study some results about lattices and ideals. For more details see [1], [2], [4], [5], [7], and [9].

Chapter two is devoted for the study of $k$-ideals and full $k$-ideals in semirings and additive inversive semirings. Also we study congruences and ring congruences. For more details see [5], [13], and [17].

Chapter three is devoted for the study of $p$-ideals in semirings and $p$-regular semirings. Also we study $p$-ideals in additive inversive semirings with special conditions. For more details see [11], [14], [16], and [18].
Chapter 1

Preliminaries

In this chapter, we give basic information which will be used in the remainder of the thesis. For convenience we will use the notations $\mathbb{N}$ denotes to the set of natural numbers, that is $\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{Z}$ the set of integers, that is $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$, and $\mathbb{Z}_0^+$ is the set of all positive integers with zero, that is $\mathbb{Z}_0^+ = \{0, 1, 2, 3, \ldots\}$.

1.1 Groups and Rings

In this section we give some definitions and examples related to groups and rings, which is a background in semiring theory. For more details about this section you can see [1] and [4].

Definition 1.1.1. Let $G$ be a set, a binary operation on $G$ is a function that assigns each ordered pair of elements of $G$ an element of $G$.

Definition 1.1.2. A nonempty set $G$ together with binary operation $*$ is called a group under this operation denoted by $(G, *)$ if the following three conditions are satisfied.
1. **Associativity**, that is, \((a \ast b) \ast c = a \ast (b \ast c)\) for all \(a, b, c\) in \(G\).

2. **Identity**, that is, there exists an element \(e\) in \(G\) such that \(e \ast a = a \ast e = a\) for all \(a\) in \(G\).

3. **Inverse**, for each element \(a\) in \(G\) there is an element \(b\) in \(G\) such that \(a \ast b = b \ast a = e\).

**Example 1.1.** The set of integers \(\mathbb{Z}\) under usual addition is a group. While \(\mathbb{Z}\) under usual multiplication is not a group since the inverse dose not exist.

**Example 1.2.** The set \(\{2, 4, 6, 8\}\) under multiplication mod 10 is a group where the identity element is 6.

**Definition 1.1.3.** Let \(G\) be a group, a subset \(H\) of \(G\) is called a **subgroup** of \(G\) if \(H\) itself is a group under the operation induced by \(G\).

**Theorem 1.1.4.** Let \(I, J\) be two subgroups of a group \(G\), then \(I \cap J\) is a subgroup of \(G\).

**Definition 1.1.5.** A nonempty set \(R\) together with two binary operations, addition denoted by \(+\) and multiplication denoted by \(\cdot\), is called a **ring** if for all \(a, b, c\) in \(R\) the following conditions are satisfied:

1. **additive commutative**, that is \(a + b = b + a\).

2. **additive associative**, i.e. \((a + b) + c = a + (b + c)\).

3. **additive identity**, i.e. there exists 0 in \(R\) such that \(a + 0 = 0\)

4. **additive inverse**, i.e. there exists \(-a\) in \(R\) such that \(a + (-a) = 0\)

5. **multiplicative associative**, i.e. \((a \cdot b) \cdot c = a \cdot (b \cdot c)\).
6. distributive, i.e. \( a(b + c) = a.b + a.c \) and \((b + c)a = b.a + c.a\).

If the multiplication is commutative then the ring is called **commutative**.

**Example 1.3.** The set of integers under usual addition and multiplication is a commutative ring.

**Example 1.4.** The set \( \mathbb{Z}_n = \{0, 1, 2, \ldots, n - 1\} \) under addition and multiplication modulo \( n \) is a commutative ring.

**Definition 1.1.6.** A subset \( S \) of a ring \( R \) is a **subring** of \( R \) if \( S \) itself is a ring with the operations of \( R \).

**Definition 1.1.7.** A subring \( A \) of a ring \( R \) is called an **ideal** of \( R \) if for every \( r \in R \) and every \( a \in A \) both \( r.a \) and \( a.r \) are in \( A \).

**Example 1.5.** For any positive integer \( n \) the set \( n\mathbb{Z} = \{0, \pm n, \pm 2n, \ldots\} \) is an ideal of \( \mathbb{Z} \).

**Theorem 1.1.8.** The intersection of any two ideals of a ring is an ideal.

### 1.2 Lattices

In this section we introduce some basic notes about lattices, there is a relation between lattices and semirings as we see in future, e.g. lattices help us to give a good example for semirings which satisfies some conditions. All of the definitions and facts in this section can be found in [9].

**Definition 1.2.1.** Let \( A \) and \( B \) be two nonempty sets, we define the **cartesian product** of \( A \) and \( B \), denoted by \( A \times B \) such that:

\[
A \times B = \{(a, b) \mid a \in A, b \in B\}
\]
Definition 1.2.2. A subset $R$ of $A \times B$ is called a relation from $A$ to $B$ if $A = B$ we say $R$ is a relation on $A$, and we write $R \subseteq A \times A$. For $(a, b) \in R$ we may write $a R b$.

Definition 1.2.3. Let $R$ be a relation on a set $A$ then $R$ is called:

(i) reflexive: if $(a, a) \in R$ for all $a \in A$.

(ii) symmetric: if $(a, b) \in R$, implies that $(b, a) \in R$.

(iii) antisymmetric: if $(a, b) \in R$, and $(b, a) \in R$ implies that $a = b$.

(iv) transitive: if $(a, b) \in R$, and $(b, c) \in R$ implies that $(a, c) \in R$.

(v) equivalence: if $R$ is reflexive, symmetric, and transitive.

(iv) partial order: if $R$ is reflexive, antisymmetric, and transitive.

Definition 1.2.4. A nonempty set $P$ with a partial order relation $\leq$ is called a partial order set or a poset, denoted by $(P, \leq)$.

If $a, b$ in a poset $(P, \leq)$ such that $a \leq b$, we say $a$ and $b$ are comparable. Two elements of a poset may not be comparable.

Example 1.6. The set of natural numbers $\mathbb{N}$, under less than or equal "$\leq$" is a poset.

Example 1.7. The natural numbers $\mathbb{N}$, under divisibility is a poset.

Example 1.8. For any nonempty set $X$, the power set of $X$, denoted by $\mathcal{P}(X)$ is the set of all subsets of $X$ under contained in "$\subseteq$" is a poset, such that for any $A, B \in \mathcal{P}(X)$, $A \leq B$ means $A \subseteq B$. 

Definition 1.2.5. Let $S$ be a subset of a poset $P$, $a \in P$ is an upper bound of $S$, if $x \leq a$ for all $x \in S$. If $a$ is an upper bound, such that $a \leq b$ for all upper bounds $b$, then $a$ is called the least upper bound, we write $\text{sup} S = a$.

Definition 1.2.6. Let $S$ be a subset of a poset $P$, $a \in P$ is a lower bound of $S$, if $a \leq x$ for all $x \in S$. If $a$ is a lower bound, such that $b \leq a$ for all lower bounds $b$, then $a$ is called the greatest lower bound, write $\text{inf} S = a$.

Definition 1.2.7. A poset $L$ is said to form a lattice, if for every $a, b \in L$, $\text{sup}\{a, b\}$ and $\text{inf}\{a, b\}$ exist in $L$. We write $\text{sup}\{a, b\} = a \lor b$, read a join $b$, and $\text{inf}\{a, b\} = a \land b$, read a meet $b$.

Example 1.9. Let $X$ be a nonempty set, then the power set of $X$, $\mathcal{P}(X)$, under contained in "\subseteq" is a lattice, such that for any two sets $A, B$ in $\mathcal{P}(X)$, we have $A \cap B = A \cap B$, and $A \cup B = A \cup B$. Since $A \cap B \subseteq A$, $A \cap B \subseteq B$, and for any set $C$ such that $C \subseteq A$, $C \subseteq B$, then $C \subseteq A \cap B$, then $A \cap B = A \cap B$.

Similarly $A \cup B = A \cup B$.

Example 1.10. Let $L$ be the set of all subgroups of a group $G$, then $L$ is a poset under inclusion "\subseteq", which form a lattice, such that for any two subgroups $H, K$ in $L$,

$$H \cap K = H \cap K \quad \text{and} \quad H \cup K = \langle H \cup K \rangle.$$  

Where $\langle H \cup K \rangle$ is the intersection of all subgroups containing $H \cup K$. Remember that by theorem 1.1.4 $H \cap K$ is a subgroup. On the other hand, $\langle H \cup K \rangle$ is also a subgroup, $H \subseteq \langle H \cup K \rangle$, $K \subseteq \langle H \cup K \rangle$, and for any subgroup $T$, such that $H \subseteq T$ and $K \subseteq T$, then $H \cup K \subseteq T$, so $\langle H \cup K \rangle \subseteq T$, hence $H \cup K = \langle H \cup K \rangle$.

Theorem 1.2.8. [9] Let $L$ be a poset, then $L$ is a lattice if and only if every nonempty finite subset of $L$ has sup and inf.
Theorem 1.2.9. [9] If $L$ is a lattice, then for any $a, b, c, d \in L$ we have the following results.

1. $a \land a = a = a \lor a$.

2. $a \land b = b \land a$, and $a \lor b = b \lor a$.

3. $a \land (b \land c) = (a \land b) \land c$, and $a \lor (b \lor c) = (a \lor b) \lor c$.

4. $a \land b \leq a$, $a \land b \leq b$, $a \leq a \lor b$, and $b \leq a \lor b$.

5. $a \leq b$ if and only if $a \land b = a$ if and only if $a \lor b = b$.

6. $a \land (a \lor b) = a$, and $a \lor (a \land b) = a$.

7. $a \leq b$ and $c \leq d$ imply $a \land c \leq b \land d$, and $a \lor c \leq b \lor d$.

Theorem 1.2.10. [9] Let $L$ be a lattice, then for any $a, b, c \in L$ we have

(i) $a \land (b \lor c) \geq (a \land b) \lor (a \land c)$.

(ii) $a \lor (b \land c) \leq (a \lor b) \land (a \lor c)$.

Definition 1.2.11. A lattice $L$ is called complete if every nonempty subset of $L$ has sup and inf in $L$.

Notes:

- By theorem 1.2.8 every finite lattice is complete.

- The set of all integers $\mathbb{Z}$ with usual less than or equal “$\leq$” is a lattice, which is not complet, since the set $K = \{ x \in \mathbb{Z} \mid x \geq 0 \}$ has no upper bound.
Definition 1.2.12. A nonempty subset $S$ of a lattice $L$ is called a sublattice of $L$ if for any two elements $a, b$ in $S$, $a \land b$ and $a \lor b$ are in $S$.

Definition 1.2.13. A lattice $L$ is called a modular lattice simply modular, if for all $a, b, c \in L$ with $a \geq b$, implies

$$a \land (b \lor c) = (a \land b) \lor (a \land c) = [b \lor (a \land c)].$$

Definition 1.2.14. A lattice $L$ is called a distributive lattice if for all $a, b, c$ in $L$ we have

$$a \land (b \lor c) = (a \land b) \lor (a \land c).$$

Note: Distributive lattice is always modular.

Theorem 1.2.15. [9] A lattice $L$ is modular if and only if for all $a, b, c$ in $L$, the three relations $a \geq b$, $a \land c = b \land c$, and $a \lor c = b \lor c$ implies $a = b$.

1.3 Semigroups and Semirings

In this section we will define semigroups and semirings and study some of their properties which are important for understanding this thesis. All of the definitions and facts in this section can be found in [2],[5], [7], [13],and [14]

1.3.1 Semigroups

Definition 1.3.1. A nonempty set $S$ together with a binary operation $\ast$ is called a semigroup if $\ast$ is associative in $S$, that is, $a \ast (b \ast c) = (a \ast b) \ast c$ for all $a, b, c \in S$. If $S$ contains an identity (zero) element then the semigroup is called a monoid.
So if we drop the identity and inverse properties from the definition of a group we get a semigroup. A semigroup is called \textit{commutative} if \( a \ast b = b \ast a \) for all \( a, b \in S \).

It is clearly that every group is a semigroup which is a monoid. But the converse is not true.

**Example 1.11.** The set of natural numbers \( \mathbb{N} \) under addition is a semigroup.

**Definition 1.3.2.** A semigroup \( S \) is said to be \textit{regular} if for each element \( a \) in \( S \) there exists an element \( a' \) in \( S \) such that \( a = a \ast a' \ast a \). If the element \( a' \) is unique and satisfies \( a' = a' \ast a \ast a' \), then \( S \) is called \textit{an inverse semigroup}. \( a' \) is called the inverse of \( a \).

**Definition 1.3.3.** An element \( a \) in a semigroup \( S \) is called \textit{an idempotent} if \( a \ast a = a \). The semigroup \( S \) is called an idempotent if \( a \) is an idempotent element for each \( a \) in \( S \).

**Definition 1.3.4.** An element \( s \) in a semigroup \( S \) is called \textit{a zeroid} if \( s \ast a = a \ast s = a \) for some \( a \) in \( S \). The set of all zeroids denoted by \( Z(S) \).

### 1.3.2 Semirings

**Definition 1.3.5.** A \textit{semiring} is a nonempty set \( S \) together with two binary operations addition and multiplication denoted by \( +, \cdot \) respectively, satisfying

1. \((S, +)\) is a commutative semigroup.
2. \((S, \cdot)\) is semigroup.
3. Distributive law holds, i.e. \( a \cdot (b + c) = a \cdot b + a \cdot c \) and \((a + b) \cdot c = a \cdot c + b \cdot c\).
A semiring is called commutative if multiplication is commutative. If \((S, +)\)
is a commutative monoid, then \(S\) is called a semiring with zero, if \((S, \cdot)\) is a
monoid, then \(S\) is called a semiring with unity. The unity if exist will denoted by \(1_S\).

**Note:** In any semiring \(S\), for simplicity we will use \(xy\) instead of \(x.y\) for any elements \(x, y\) in \(S\).

**Definition 1.3.6.** The zero element 0 in a semiring \(S\) is called an **absorbing zero** if
\(a.0 = 0.a = 0\) for all \(a\) in \(S\).

**Example 1.12.** Any ring is a semiring with an absorbing zero.

**Example 1.13.** The simplest example of a semiring is the set of natural numbers \(\mathbb{N}\) under the ordinary addition and multiplication, which is commutative. Likewise, the non-negative rational numbers \(\mathbb{Q}^+\), and the non-negative real numbers \(\mathbb{R}^+\) form commutative semirings.

**Example 1.14.** The set of all \(n \times n\) matrices with non-negative entries form a semiring under ordinary addition and multiplication of matrices with zero, but not commutative.

In rings every zero is absorbing zero, but in semirings not every zero is absorbing zero, to show that see the next two examples.

**Example 1.15.** Consider the set of positive integers \(\mathbb{Z}^+\) with the operations
\[ a + b = \text{lcm}(a, b) \quad \text{and} \quad a.b = ab. \]
Then \((\mathbb{Z}^+, +, \cdot)\) is a semiring with zero element 1, but 1 is not an absorbing zero since \(1.a = a.1 = a \neq 1\) for any \(a \in \mathbb{Z}^+\) and \(a \neq 1\).
Example 1.16. In the power set \( P(X) \), define the addition and multiplication such that for any \( A, B \in P(X) \)

\[
A + B = A \cap B \quad \text{and} \quad A.B = (A \cup B) \setminus (A \cap B).
\]

Then \( (P(X),+,\cdot) \) is a semiring with zero \( X \), since \( A \cap X = A \), and the unity is \( \phi \). But for any nonempty proper subset \( A \) of \( X \) we have

\[
X.A = (A \cup X) \setminus (A \cap X) = X \setminus A \neq X.
\]

So \( X \) is not absorbing zero.

Definition 1.3.7. Let \( n, i \) be integers such that \( 2 \leq n, 0 \leq i < n \), and \( B(n, i) = \{0,1,2,\ldots,n-1\} \). We define addition and multiplication in \( B(n, i) \) by the following equations (let \( m = n - i \)):

\[
x + y = \begin{cases} 
x + y & \text{if } x + y \leq n - 1 \\
l & \text{if } x + y \geq n \\
\text{where } l \equiv (x + y) \mod m \text{ and } \\
\quad i \leq l \leq n - 1.
\end{cases}
\]

\[
x \cdot y = \begin{cases} 
xy & \text{if } xy \leq n - 1 \\
l & \text{if } xy \geq n \\
\text{where } l \equiv (xy) \mod m \text{ and } \\
\quad i \leq l \leq n - 1.
\end{cases}
\]

Proposition 1.3.8. The set \( B(n, i) \) is a commutative semiring under addition and multiplication defined in definition 1.3.7.
Example 1.17. In proposition 1.3.8, let \( n = 10 \), and \( i = 7 \) then we have

\[
B(10,7) = \{0,1,2,3,4,5,6,7,8,9\}.
\]

Where the operations are:

\[
\begin{array}{cccccccccc}
+ & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
0 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
1 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 7 \\
2 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 7 & 8 \\
3 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 7 & 8 & 9 \\
4 & 4 & 5 & 6 & 7 & 8 & 9 & 7 & 8 & 9 & 7 \\
5 & 5 & 6 & 7 & 8 & 9 & 7 & 8 & 9 & 7 & 8 \\
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5 & 0 & 5 & 7 & 9 & 8 & 7 & 9 & 8 & 8 & 9 \\
6 & 0 & 6 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 \\
7 & 0 & 7 & 8 & 9 & 7 & 8 & 9 & 7 & 8 & 9 \\
8 & 0 & 8 & 7 & 9 & 8 & 8 & 9 & 8 & 7 & 9 \\
9 & 0 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 \\
\end{array}
\]

Definition 1.3.9. A semiring \( S \) is called additive inversive if \( S \) is an inverse semigroup under addition.

Lemma 1.3.10. Let \( S \) be an additive inversive semiring, then for \( a,b \) in \( S \) we have

(i) \((a')' = a\).

(ii) \((ab)' = a'b = ab'\).

(iii) \((a+b)' = a' + b'\).

(iv) \(a b = a'b'\).
Proof. (i) Let \( a \in S \), then \( a + a' + a = a \) and \( a' + a + a' = a' \), i.e. \( a \) is inverse of \( a' \). But \( a' + (a')' + a' = a' \), and \( (a')' + a' + (a')' = (a')' \) so \( (a')' \) is inverse of \( a' \). But the inverse is unique. Hence \( (a')' = a \).

(ii) \( ab' + ab + ab' = a(b' + b + b') = ab' \) and \( ab + ab' + ab = a(b' + b + b') = ab' \), hence \( ab' \) is inverse of \( ab \). Similarly \( a'b \) is inverse of \( ab \).

(iii) \( (a + b) + a' + b' + (a + b) = (a + a' + a) + (b + b' + b) = a + b \) and \( (a' + b') + a + b + (a' + b') = (a' + a + a') + (b' + b + b') = a' + b' \).

(iv) \( a'b' + (ab)' + a'b = a'b' + a'b + a'b'[by(ii)] = a'(b' + b + b') = a'b' \) and \( (ab)' + a'b' + (ab)' = a'b + a'b' + a'b = a'(b + b' + b) = a'b = (ab)' \). So \( (a'b') \) is an inverse of \( (ab)' \). But \( ab \) is the inverse of \( (ab)' \) and the inverse is unique. Hence \( ab = a'b' \).

\[ \square \]

**Definition 1.3.11.** A semiring \( S \) is called a **halfring** if the additive cancellation law holds on \( S \), i.e. if \( a + b = a + c \) implies \( b = c \) for all \( a, b, c \in S \).

**Example 1.18.** The set of positive integers \( \mathbb{Z}^+ \) under usual addition and multiplication is a halfring.

**Definition 1.3.12.** A semiring \( S \) is called **multiplicative regular** if \( S \) is a regular semigroup under multiplication.

**Definition 1.3.13.** An element \( a \) in a semiring \( S \) is called **additive idempotent** if \( a + a = a \). The set of all additive idempotents of a semiring \( S \) is denoted by \( E^+(S) \). Similarly an element \( a \) is called **multiplicative idempotent** if \( a . a = a \). The set of all multiplicative idempotents of a semiring denoted by \( E^*(S) \). The semiring \( S \) is called an idempotent semiring if \( a \) is an additive and multiplicative idempotent element for each \( a \in S \).
Example 1.19. \( \mathbb{R} \cup \{-\infty\} \) is a commutative, additive idempotent semiring with the addition and multiplication operations defined as:

\[
a \oplus b = \max(a, b) \quad \text{and} \quad a \otimes b = a + b
\]

where + is the ordinary addition as semiring multiplication. Clearly, \(-\infty\) is the zero element, and 0 is the unity.

Example 1.20. Any distributive lattice is a commutative, idempotent semiring under join and meet.

Definition 1.3.14. A semiring \( S \) is called \textbf{E-inversive}, if for every \( a \in S \), there exists \( x \in S \) such that \( a + x \in E^+(S) \).

Definition 1.3.15. A semiring \( S \) is called \textbf{non-zeroic} if \( Z(S) \neq \emptyset \) and \( Z(S) \) is a proper subset of \( S \).

Definition 1.3.16. A subset \( I \) of a semiring \( S \) is called a \textbf{left} (resp. a \textbf{right}) \textbf{ideal} of \( S \) if

1. \( a + b \in I \) for all \( a, b \in I \).
2. for any \( a \in I \), and \( b \in S \), \( ba \in I \) (resp. \( ab \in I \)).

\( I \) is called an \textbf{ideal} if \( I \) is left and right ideal.

Lemma 1.3.17. Let \( S \) be a semiring, then for any \( a \in S \),

\[
< a > = Sa = \{ x \mid x = ra \text{ for some } r \in S \}
\]

is a left ideal of \( S \) called the principal left ideal generated by \( a \).

Lemma 1.3.18. Let \( S \) be a semiring, then \( E^+(S) \) is an ideal of \( S \).
Lemma 1.3.19. Let $S$ be a semiring, then $Z(S)$ is an ideal of $S$.

Lemma 1.3.20. If $A, B$ are two ideals of a semiring $S$, then $A \cap B$ is an ideal.

Lemma 1.3.21. Let $A, B$ be two ideals of a semiring $S$, then the sum of $A, B$ denoted by $A + B$ is an ideal of $S$ where

$$A + B = \{x = a + b \mid a \in A, \ b \in B\}.$$ 

Lemma 1.3.22. Let $A, B$ be two ideals of a semiring $S$, then the product of $A$ and $B$ denoted by $AB$ is an ideal of $S$ where

$$AB = \{a_1 b_1 + a_2 b_2 + \ldots + a_n b_n \mid a_i \in A, \ b_i \in B, \ n \in \mathbb{N}\}.$$ 

Lemma 1.3.23. [5] Let $S$ be a semiring, $I$ be a left (resp. right) ideal of $S$ and $A$ is a nonempty subset of $S$, then

$$(I : A) = \{r \in S \mid ra \in I \ (\text{resp} \ ar \in I) \ \text{for all} \ a \in A\}$$

is a left (resp. right) ideal of $S$. If $A \subseteq I$ then $(I : A) = S$.

Definition 1.3.24. An ideal $I$ of a semiring $S$ is called **full** if $E^+(S)$ contained in $I$.

Example 1.21. In any ring $R$ the set $E^+(R) = \{0\}$, and so every ideal of $R$ is a full ideal.

Example 1.22. The set of all positive integers $\mathbb{Z}^+$ under the operations max and min is an idempotent semiring, and $I_n = \{1, 2, 3, \ldots, n\}$ is an ideal of $\mathbb{Z}^+$ which is not full since $E^+(\mathbb{Z}^+) = \mathbb{Z}^+$.

Definition 1.3.25. Let $S$ and $T$ be semirings. Then the direct product of $S$ and $T$ denoted by $S \times T$ is defined such that

$$S \times T = \{(s, t) \mid s \in S, \ t \in T\}$$
Lemma 1.3.26. Let $S$ and $T$ be semirings. Then the direct product of $S$ and $T$ is a semiring under the operations defined for all $(a_1, b_1), (a_2, b_2) \in S \times T$ by:

$$(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$$

$$(a_1, b_1) \cdot (a_2, b_2) = (a_1 \cdot a_2, b_1 \cdot b_2)$$

Definition 1.3.27. A subsemiring $H$ of the direct product of two semiring $S$ and $T$ is called a subdirect product of $S$ and $T$ if the two projection mappings $\pi_1 : H \rightarrow S$ given by $\pi_1(s, t) = s$ and $\pi_2 : H \rightarrow T$ given by $\pi_2(s, t) = t$ are surjective.

Definition 1.3.28. A semiring homomorphism from a semiring $S$ to a semiring $R$ is a function $f$ from $S$ to $R$ such that for all $a, b \in S$:

(i) $f(a + b) = f(a) + f(b)$.

(ii) $f(ab) = f(a)f(b)$.

Lemma 1.3.29. Let $f$ be a semiring homomorphism from $S$ to $R$, let $I$ be an ideal of $R$, then $f^{-1}(I) = \{a \in S \mid f(a) \in I\}$ is an ideal of $S$. 
Chapter 2

$k$-Ideals in Semirings

In this chapter we will study special kind of ideals in semirings called $k$-ideal, and with more restrictions on this kind to get a full $k$-ideal. We will prove some results about these ideals. The results of this chapter can be found in [3], [5], [13] and [15].

2.1 $k$-Ideals

In this section we will study a more restricted class of ideals in a semiring, which is called $k$-ideals or subtractive, and we introduce some related results and examples.

Definition 2.1.1. An ideal $I$ of a semiring $S$ is called $k$-ideal or subtractive if for any two elements $a$ in $I$ and $x$ in $S$ such that $a + x$ in $I$, then $x$ in $I$.

Example 2.1. In any ring $R$, every ideal $I$ is $k$-ideal, since for any $a \in I$, $x \in R$ such that $a + x \in I$ then $a + x + (-a) \in I$, so $x \in I$. 

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Example 2.2. In the semiring $\mathbb{Z}^+$ under the operations $\max$ and $\min$, the set $I_n = \{1, 2, 3, ..., n\}$ is a $k$-ideal of $\mathbb{Z}^+$. Since for any element $a \in I_n$ and $x \in \mathbb{Z}^+$ such that $a + x = \max\{a, x\} \in I_n$, implies $x \in I_n$.

Example 2.3. By referring to proposition 1.3.8, the set $B(5, 2) = \{0, 1, 2, 3, 4\}$ is a commutative semiring with the operations are:

\[
\begin{array}{cccccc}
+ & 0 & 1 & 2 & 3 & 4 \\
0 & 0 & 1 & 2 & 3 & 4 \\
1 & 1 & 2 & 3 & 4 & 2 \\
2 & 2 & 3 & 4 & 2 & 3 \\
3 & 3 & 4 & 2 & 3 & 4 \\
4 & 4 & 2 & 3 & 4 & 2 \\
\end{array}
\]

\[
\begin{array}{cccccc}
\cdot & 0 & 1 & 2 & 3 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 2 & 3 & 4 \\
2 & 0 & 2 & 4 & 3 & 2 \\
3 & 0 & 3 & 3 & 3 & 3 \\
4 & 0 & 4 & 2 & 3 & 4 \\
\end{array}
\]

Then $I_1 = \{0, 3\}$ is $k$-ideal of $B(5, 2)$. But $I_2 = \{0, 2, 3, 4\}$ is an ideal which is not $k$-ideal. Indeed, $2 \in I_2$ and $2 + 1 = 3 \in I_2$, but $1 \not\in I_2$.

Lemma 2.1.2. In a semiring $S$, the set of zeroids $Z(S)$ is $k$-ideal.

Proof. By lemma 1.3.19 $Z(S)$ is an ideal. To show that $Z(S)$ is a $k$-ideal, let $s \in S$, $a \in Z(S)$ such that $a + s \in Z(S)$, so there is $x \in S$ such that $a + s + x = x$. But $a + y = y$ for some $y \in S$. Then by addition we have

\[
x + y = a + s + x + a + y = s + (a + y + a + x) = s + (a + x) = a + (y + x) = s + (x + y).
\]

Hence $s \in Z(S)$, and $Z(S)$ is $k$-ideal. \qed
Lemma 2.1.3. Let $S$ be a semiring, $I$ be a $k$-ideal of $S$ and $A$ is a nonempty subset of $S$, then

$$(I : A) = \{ r \in S \mid ra, ar \in I \text{ for all } a \in A \}$$

is a $k$-ideal of $S$.

Proof. By lemma 1.3.23 $(I : A)$ is an ideal. Next let $r \in (I : A)$, $y \in S$ such that $r + y \in (I : A)$, then $ar, ra \in I$ and $(r + y)a$, $a(r + y) \in I$ for all $a \in A$, then $ra + ya = (r + y)a \in I$ which is $k$-ideal. Hence $ya \in I$. Similarly, $ay \in I$. So $y \in (I : A)$. Hence $(I : A)$ is $k$-ideal of $S$. \[\square\]

In the following example we see that the set of all additive idempotents $E^+(S)$ is not a $k$-ideal.

Example 2.4. Let $S = \{0, a, b\}$. Define addition and multiplication on $S$ as follows:

<table>
<thead>
<tr>
<th>+</th>
<th>0</th>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>a</td>
<td>b</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>b</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>b</td>
<td>b</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>.</th>
<th>0</th>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>b</td>
<td>0</td>
<td>0</td>
<td>b</td>
</tr>
</tbody>
</table>

Then $S$ is additive inversive semiring under the operations. Moreover $E^+(S) = \{0, b\}$ is an ideal of $S$. But $a + b = b \in E^+(S)$ and $a \notin E^+(S)$, so $E^+(S)$ is not $k$-ideal.

The next example shows that the sum of two $k$-ideals need not a $k$-ideal.

Example 2.5. Consider the semiring of positive integers with zero $Z^+_0$ under usual addition and multiplication. Then $2Z^+_0$ and $3Z^+_0$ are $k$-ideals of $Z^+_0$. 

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but $2Z_0^+ + 3Z_0^+ = Z_0^+ \setminus \{1\}$ is not a $k$-ideal. Indeed, $1 + 2 = 3$, where $2, 3 \in 2Z_0^+ + 3Z_0^+$, but $1 \notin 2Z_0^+ + 3Z_0^+$.

Lemma 2.1.4. Let $S$ be a semiring. If $A$ is an ideal of $S$ such that $A = I \cup J$ where $I, J$ are two $k$-ideals, then $A = I$ or $A = J$.

Proof. Since $A = I \cup J$, then $I \subseteq A$ and $J \subseteq A$. Now suppose $A \neq I$, and $A \neq J$, then there exist $x, y \in A$ such that $x \in I$, $x \notin J$, $y \in J$, $y \notin I$, but $x + y \in A = I \cup J$, so $x + y \in I$ or $x + y \in J$, now if $x + y \in I$, then $y \in I$ as $I$ is $k$-ideal, contradiction. Also if $x + y \in J$ then $x \in J$ as $J$ is $k$-ideal, contradiction. Hence $A = I$ or $A = J$. \qed

2.2 Full $k$-Ideal

In this section, we will study more restrictions on the $k$-ideal and the semiring. We study full $k$-ideal in additive inversive semirings, so $S$ denotes an additive inversive semiring.

Definition 2.2.1. A $k$-ideal $I$ of a semiring $S$ is called \textbf{full $k$-ideal} if the set of all additive idempotents of $S$, $E^+(S)$ is contained in $I$.

Example 2.6. In any ring $R$ every ideal $I$ is a full $k$-ideal. Since $0$ is the only additive idempotent element in $R$ which belongs to any ideal $I$ of $R$. So $I$ is full $k$-ideal.

Example 2.7. In a distributive lattice $L$ with more than two elements, a proper ideal $I$ is $k$-ideal but not full $k$-ideal. Let $a \in I$, $x \in L$ such that $a \lor x \in I$, then by theorem 1.2.9, $x \leq a \lor x$. But $I$ is an ideal so $x = x \land (a \lor x) \in I$. Hence $I$ is $k$-ideal. Moreover, the set of all additive idempotents of $L$ is $L$ itself, since $a \lor a = a$ for all $a \in L$. So $I$ is not full $k$-ideal.
Example 2.8. In $\mathbb{Z} \times \mathbb{Z}^+ = \{(a, b) : a, b \text{ are integers and } b > 0\}$, define

$$(a, b) + (c, d) = (a + c, \text{lcm}(b, d))$$

and

$$(a, b) . (c, d) = (a.c, \text{gcd}(b, d)).$$

Then $\mathbb{Z} \times \mathbb{Z}^+$ is an additive inversive semiring, since for any $(a, b), (c, d), (e, f) \in \mathbb{Z} \times \mathbb{Z}^+$

Additive commutative:

$$(a, b) + (c, d) = (a + c, \text{lcm}(b, d)) = (c + a, \text{lcm}(d, b)) = (c, d) + (a, b)$$

Additive associative:

$$((a, b) + (c, d)) + (e, f) = ((a + c, \text{lcm}(b, d)) + (e, f)$$

$$= ((a + c) + e, \text{lcm}(b, d, f))$$

$$= ((a + c + e, \text{lcm}(b, d, f))$$

$$= (a, b) + ((c + e, \text{lcm}(d, f))$$

$$= (a, b) + ((c, d) + (e, f)).$$

Multiplicative associative: Similarly as additive associative

Distributivity:

$$(a, b).((c, d)) + (e, f)) = (a, b).(c + e, \text{lcm}(d, f))$$

$$= (a.(c + e), \text{gcd}(b, \text{lcm}(d, f)))$$

$$= (a.c + a.e, \text{lcm}(\text{gcd}(b, d), \text{gcd}(b, f)))$$

$$= (a.c, \text{gcd}(b, d)) + (a.e, \text{gcd}(b, f))$$

$$= (a, b).(c, d) + (a, b).(e, f)).$$
Similarly, \((c, d) + (e, f)\).\((a, b) = (c, d)\).(a, b) + (e, f).(a, b).

Additive inverse: For any \((a, b) \in \mathbb{Z} \times \mathbb{Z}^+\) there exists a unique \((-a, b) \in \mathbb{Z} \times \mathbb{Z}^+\) such that

\[
(a, b) + (-a, b) + (a, b) = (a + -a + a, \ lcm(b, b, b)) = (a, b),
\]

\[
(-a, b) + (a, b) + (-a, b) = (-a + a + -a, \ lcm(b, b, b)) = (-a, b)
\]

Moreover, the set \(A = \{(a, b) \in \mathbb{Z} \times \mathbb{Z}^+: a = 0, b \in \mathbb{Z}^+\}\) is full \(k\)-ideal of \(\mathbb{Z} \times \mathbb{Z}^+\).

Since \(E^+(\mathbb{Z} \times \mathbb{Z}^+) = \{0\} \times \mathbb{Z}^+ \subseteq A\), and for any \((0, b) \in A\), \((c, d) \in \mathbb{Z} \times \mathbb{Z}^+\) such that \((0, b) + (c, d) = (c, \ lcm(b, d)) \in A\), then \(c = 0\), so \((c, d) \in A\).

**Lemma 2.2.2.** Let \(A, B\) be two full \(k\)-ideal of a semiring \(S\), then \(A \cap B\) is full \(k\)-ideal.

**Proof.** Let \(A, B\) be two full \(k\)-ideals of \(S\), then by lemma 1.3.20 \(A \cap B\) is an ideal which is full as \(E^+(S) \subseteq A\) and \(E^+(S) \subseteq B\). Now let \(x \in S\) such that \(a + x \in A \cap B\) for some \(a \in A \cap B\), then \(a + x \in A\), \(a \in A\) and \(a + x \in B\), \(a \in B\), then \(x \in A\) and \(x \in B\) as \(A, B\) be \(k\)-ideals. Hence \(x \in A \cap B\). \(\square\)

**Lemma 2.2.3.** Every \(k\)-ideal of \(S\) is an inversive subsemiring of \(S\).

**Proof.** Clearly that every ideal of \(S\) is subsemiring of \(S\). Let \(a \in I\), then \(a \in S\), so there exist \(a' \in S\) such that

\[
a = a + a' + a = a + (a' + a) \in I.
\]

But \(I\) is a \(k\)-ideal and \(a \in I\), so \(a' + a \in I\). Again \(I\) is a \(k\)-ideal and \(a \in I\), so \(a' \in I\). Hence \(I\) is an inversive subsemiring of \(S\). \(\square\)

**Definition 2.2.4.** Let \(A\) be an ideal of an additive inversive semiring \(S\). We define the \(k\)-closure of \(A\), denoted by \(\overline{A}\) by:

\[
\overline{A} = \{a \in S: a + x \in A \text{ for some } x \in A\}
\]

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Lemma 2.2.5. Let $A$ be an ideal of $S$, then $\overline{A}$ is a $k$-ideal of $S$. Moreover $A \subseteq \overline{A}$.

Proof. Let $a, b \in \overline{A}$, then $a + x, b + y \in A$ for some $x, y \in A$. Now

$$(a + b) + (x + y) = (a + x) + (b + y) \in A.$$ 

But $x + y \in A$, so $a + b \in \overline{A}$. Next let $r \in S$, then $ra + rx = r(a + x) \in A$. But $rx \in A$, so $ra \in \overline{A}$. Similarly, $ar \in \overline{A}$. As a result $\overline{A}$ is an ideal of $S$. To show that $\overline{A}$ is $k$-ideal. Let $c, c + d \in \overline{A}$, then there exist $x$ and $y$ in $A$ such that $c + x \in A$ and $c + d + y \in A$. Now

$$d + (c + x + y) = (c + d + y) + x \in A \text{ and } c + x + y \in A.$$ 

Hence $d \in \overline{A}$ and so $\overline{A}$ is a $k$-ideal of $S$. Finally, since $a + a \in A$ for all $a \in A$, it follows that $A \subseteq \overline{A}$.

Corollary 2.2.6. Let $A$ be an ideal of $S$. Then $\overline{A} = A$ if and only if $A$ is a $k$-ideal.

Proof. Suppose $A = \overline{A}$, then by lemma 2.2.5 $\overline{A}$ is $k$-ideal, and so $A$ is $k$-ideal. Conversely, suppose $A$ is a $k$-ideal. Again by lemma 2.2.5 $A \subseteq \overline{A}$. On the other hand, let $a \in \overline{A}$ then $a + x \in A$ for some $x \in A$. But $A$ is a $k$-ideal and $x \in A$, implies $a \in A$, so $\overline{A} \subseteq A$. Therefore $A = \overline{A}$.

Lemma 2.2.7. Let $A, B$ be two ideals of a semiring $S$ such that $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$.

Proof. Let $A, B$ be two ideals of $S$ such that $A \subseteq B$, let $a \in \overline{A}$, then $a + x \in A$ for some $x \in A$, but $A \subseteq B$, so $a + x \in B$ for some $x \in B$, hence $a \in \overline{B}$, therefore $\overline{A} \subseteq \overline{B}$. 

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Lemma 2.2.8. Let $A$ be an ideal of $S$. Then $\overline{A}$ is the smallest $k$-ideal containing $A$.

Proof. Let $B$ be a $k$-ideal of $S$ such that $A \subseteq B$, let $x \in \overline{A}$. Then $x + a_1 = a_2$ for some $a_1, a_2 \in A$. Since $A \subseteq B$ and $B$ is a $k$-ideal, then $x \in B$. Therefore $\overline{A} \subseteq B$. \qed

Lemma 2.2.9. Let $A$ and $B$ be two full $k$-ideals of $S$, then $\overline{A + B}$ is a full $k$-ideal of $S$ such that

$$A \subseteq \overline{A + B} \quad \text{and} \quad B \subseteq \overline{A + B}.$$ 

Proof. By lemma 1.3.21 $A + B$ is an ideal of $S$. Then by lemma 2.2.5 $\overline{A + B}$ is a $k$-ideal and $A + B \subseteq \overline{A + B}$. Now $E^+(S) \subseteq A$ and $E^+(S) \subseteq B$. So for any $e \in E^+(S)$, $e = e + e$. Hence $E^+(S) \subseteq A + B \subseteq \overline{A + B}$. This implies that $\overline{A + B}$ is a full $k$-ideal. Finally let $a \in A$, Then

$$a = a + a' + a = a + (a' + a) \in A + B$$

as $a' + a \in E^+(S) \subseteq B$.

Hence $A \subseteq \overline{A + B}$ and similarly $B \subseteq \overline{A + B}$. \qed

Theorem 2.2.10. The set of all full $k$-ideals of $S$, denoted by $I(S)$, is a complete lattice which is also modular.

Proof. Firstly we note that $I(S)$ is a partially ordered set with respect to usual set inclusion. Let $A, B \in I(S)$. Then by lemma 2.2.2 $A \cap B \in I(S)$, and by Lemma 2.2.9, $\overline{A + B} \in I(S)$. Define

$$A \wedge B = A \cap B \quad \text{and} \quad A \vee B = \overline{A + B}.$$ 

It is clearly that $A \cap B = \inf\{A, B\}$, let $C \in I(S)$ such that $A, B \subseteq C$. Then $A + B \subseteq C$ and $\overline{A + B} \subseteq \overline{C}$. But $C = \overline{C}$. Hence $\overline{A + B} \subseteq C$, hence
\(A + B = \sup\{A, B\}\). Thus we find that \(I(S)\) is a lattice. Now by lemma 1.3.18, \(E^+(S)\) is an ideal of \(S\) which contained in in every ideal in \(I(S)\), hence \(E^+(S)\) is the smallest full \(k\)-ideal in \(I(S)\), and also \(S \in I(S)\); consequently \(I(S)\) is a complete lattice. Finally to show that \(I(S)\) is modular we will apply theorem 1.2.15, so suppose that \(A, B, C \in I(S)\) such that

\[
A \wedge B = A \wedge C \quad \text{and} \quad A \vee B = A \vee C \quad \text{and} \quad B \subseteq C.
\]

Let \(x \in C\). But by lemma 2.2.9 we have \(C \subseteq \overline{A + C} = A \vee C\), so \(x \in A \vee C = A \vee B = \overline{A + B}\). Hence there exists \(a + b \in A + B\) such that \(x + a + b = a_1 + b_1\) for some \(a_1 \in A, b_1 \in B\). Then \(x + a + a' + b = a_1 + b_1 + a'\) \hspace{1cm} (1)

But \(x \in C, a + a' \in E^+(S) \subseteq C\) since \(C\) is full ideal and \(b \in B \subseteq C\), hence \(a_1 + b_1 + a' \in C\). But \(b_1 \in B \subseteq C\) which is \(k\)-ideal, so \(a_1 + a' \in C\), also \(a_1 + a' \in A\), hence \(a_1 + a' \in C \cap A = A \cap B\). Hence \(a_1 + a' \in B\). So from (1) we find that \(x + a + a' + b = a_1 + a' + b \in B\). But \((a + a') + b \in B\) which is a \(k\)-ideal. Hence \(x \in B\), hence \(B = C\). Therefore \(I(S)\) is a modular lattice. \(\square\)

### 2.3 Ring Congruences

In this section, we will define congruence relation on a semiring, and study its relation with full \(k\)-ideal.

**Definition 2.3.1.** An equivalence relation \(\rho\) on a semiring \(S\) is called a **congruence** on \(S\) if for any \(a, b, c \in S\) such that \(a \rho b\) we have

1. \(a + c \rho b + c\).

2. \(ac \rho bc\).

3. \(ca \rho cb\).
**Definition 2.3.2.** Let $\rho$ be a congruence on a semiring $S$, then the congruence class of $x \in S$ denoted by $x\rho$ is defined such that

$$x\rho = \{ y \in S : (x, y) \in \rho \}.$$  

The set of all congruence classes of $S$ is denoted by $S/\rho$.

We can define two binary operations on $S/\rho$ such that for any $a, b \in S$

$$a\rho + b\rho = (a + b)\rho \quad \text{and} \quad a\rho \cdot b\rho = (a \cdot b)\rho$$

**Proposition 2.3.3.** Let $S$ be a semiring, then $S/\rho$ is a semiring under the operations

$$a\rho + b\rho = (a + b)\rho \quad \text{and} \quad a\rho \cdot b\rho = (a \cdot b)\rho$$

where $a, b \in S$ called *quotient semiring*.

**Proof.** Let $a\rho, b\rho, c\rho$ be elements in $S/\rho$ then

$$a\rho + b\rho = (a + b)\rho = (b + a)\rho = b\rho + a\rho.$$  

So $S/\rho$ is commutative under addition.

$$a\rho + (b\rho + c\rho) = a\rho + (c + b)\rho = (a + (b + c))\rho = (a + b + c)\rho = (a + b)\rho + c\rho.$$  

So addition is associative on $S/\rho$. Similarly multiplication is associative also.
Finally to show distributive law,

\[ a \rho (b \rho + c \rho) = a \rho ((b + c) \rho) = (a(b + c)) \rho = (ab + ac) \rho = ab \rho + ac \rho. \]

Therefore \( S/\rho \) is a semiring.

**Definition 2.3.4.** A congruence \( \rho \) on a semiring \( S \) is called a ring congruence if the quotient semiring \( S/\rho \) is a ring.

From now, let \( S \) be an additive inversive semiring. We want to characterize those ring congruences on \( S \) such that \(- (a \rho) = a' \rho \) where \( a' \) denotes the additive inverse of \( a \) in \( S \) and \(- (a \rho) \) denotes the additive inverse of \( a \rho \) in the ring \( S/\rho \)

**Theorem 2.3.5.** Let \( A \) be a full \( k \)-ideal of \( S \). Then the relation

\[ \rho_A = \{(a, b) \in S \times S : a + b' \in A\} \]

is a ring congruence on \( S \) such that \(- (a \rho_A) = a' \rho_A \)

**Proof.** Since \( a + a' \in E^+(S) \subseteq A \) for all \( a \in S \), it follows that \( \rho_A \) is reflexive. Let \( a + b' \in A \). By Lemma 2.2.3, we find that \((a + b')' \in A\). Then using lemma 1.3.10 we have

\[ b + a' = (b')' + a' = (a + b')' \in A. \]

Hence \( \rho_A \) is symmetric. Let \( a + b' \in A \) and \( b + c' \in A \). Then \( a + b + b' + c' \in A \). Also \( b + b' \in E^+(S) \subseteq A \). Since \( A \) is a \( k \)-ideal, we find that \( a + c' \in A \). Hence
\( \rho_A \) is an equivalence relation. Let \((a, b) \in \rho_A \) and \( c \in S \). Then \( a + b' \in A \).

Since

\[
(c + a) + (c + b)' = c + a + b' + c' = (a + b') + (c + c') \in A,
\]

\[
ca + (cb)' = ca + cb' = c(a + b') \in A,
\]

\[
ac + (bc)' = ac + b'c = (a + b'c) \in A,
\]

It follows that \( \rho_A \) is a congruence on \( S \). So we obtain the quotient semiring where addition and multiplication are defined by

\[
a\rho_A + b\rho_A = (a + b)\rho_A \quad \text{and} \quad (a\rho_A)(b\rho_A) = (ab)\rho_A. \]

Now let \( e \in E^+(S) \) and \( a \in S \), then \((e + a) + a' = e + (a + a') \in E^+(S) \subseteq A \), implies \((e + a)\rho_A = a'\rho_A \). But \( a + a' \in E^+(S) \subseteq A \), implies \( a\rho_A = a'\rho_A \). Then \((e + a)\rho_A = a\rho_A \). Hence \( e\rho_A + a\rho_A = a\rho_A \). Again since \( e + (a + a') \in A \), implies \((e, (a + a')) = (e, a + a') \in \rho_A \). Hence

\[
a\rho_A + a'\rho_A = (a + a')\rho_A = e\rho_A. \]

Hence \( e\rho_A \) is the zero element and \( a'\rho_A \) is the additive inverse of \( a\rho_A \) in the ring \( S/\rho_A \). \( \square \)

**Theorem 2.3.6.** Let \( \rho \) be a congruence on \( S \) such that \( S/\rho \) is a ring and \(-\rho = a' \rho \). Then there exists a full \( k \)-ideal \( A \) of \( S \) such that \( \rho_A = \rho \)

**Proof.** Consider the set

\[
A = \{a \in S : (a, e) \in \rho \text{ for some } e \in E^+(S)\}.
\]

Since \( \rho \) is reflexive, then \((e, e) \in \rho \) for all \( e \in E^+(S) \), it follows that \( E^+(S) \subseteq A \), then \( A \) is nonempty, since \( E^+(S) \) is nonempty. Let \( a, b \in A \). Then there exist
$e, f \in E^+(S)$ such that $(a, e) \in \rho$ and $(b, f) \in \rho$. Then $(a + b, e + f) \in \rho$ as $\rho$ is a congruence. But $e + f \in E^+(S)$. Hence $a + b \in A$. Again for any $r \in S, (ra, re) \in \rho$ and $(ar, er) \in \rho$. But $re, er \in E^+(S)$. Hence $A$ is an ideal of $S$.

Let $a + b \in A$ and $b \in A$. Then there exist $e, f \in E^+(S)$ such that $(a + b, f) \in \rho$ and $(b, e) \in \rho$. Hence $f \rho = (a + b) \rho = a \rho + b \rho = a \rho + e \rho$. But $f \rho$ and $e \rho$ are additive idempotents in the ring $S/\rho$ as $f, e \in E^+(S)$. Hence $e \rho = f \rho$ is the zero element of $S/\rho$. As a result, $a \rho$ is the zero element of $S/\rho$. Then $a \rho = e \rho$.

This implies $a \in A$. Therefore $A$ is a full $k$-ideal of $S$. Finally consider now the congruences $\rho_A$ and $\rho$. Let $(a, b) \in \rho$. Then $(a + b', b + b') \in \rho$ as $\rho$ is a congruence. But $b + b' \in E^+(S)$. Hence $a + b' \in A$ and $(a, b) \in \rho_A$. Conversely suppose that $(a, b) \in \rho_A$ Then $a + b' \in A$. Hence $(a + b', e) \in \rho$ for some $e \in E^+(S)$. As a result, $e \rho = a \rho + b' \rho = a \rho - b \rho$ holds in the ring $S/\rho$. But $e \rho$ is the zero element of $S/\rho$. Consequently $a \rho = b \rho$. This show that $(a, b) \in \rho$ and hence $\rho_A = \rho$.  

\[ \square \]
Chapter 3

*p*-Ideals in *p*-Regular Semirings

In this chapter, we introduce *p*-ideals in semirings. A new form of regularity, which is compatible with *p*-ideals is defined. Our aim is to explore the possibilities of establishing an ideal theory in semirings, going alongside the existing literature of semiring theory. All definitions and facts in this chapter can be found in [11], [14], [16], and [18].

3.1 *p*-Ideals

In this section, we introduce new class of ideals called *p*-ideals in semiring, and study some results on it.

In what follows a general semiring will always contain an absorbing zero, with the exception for the results on additive inversive semirings, where we do not consider the presence of such an absorbing zero (unless otherwise stated). The reason behind such a consideration is that, in an additive inversive semiring with absorbing zero, this new class of ideal boils down to a full ideal and we intend to distinguish between these two classes of ideals as far as possible.
Definition 3.1.1. Let $S$ be a semiring, we define the set $P^+(S)$ which consists of some additively periodic elements of $S$ by:

$$P^+(S) = \{ x \in S \mid nx = (n+1)x \text{ for some } n \in \mathbb{N} \}.$$ 

Clearly that for any semiring $S$, $E^+(S) \subseteq P^+(S)$. Also we note that for any ring $R$, $P^+(R) = \{0\}$.

Lemma 3.1.2. In any semiring $S$, $P^+(S)$ is an ideal of $S$.

Proof. Let $x, y \in P^+(S)$ then $nx = (n+1)x$, $my = (m+1)y$ for some $m, n \in \mathbb{N}$. Now,

$$\begin{align*}
(n + m + 1)(x + y) &= nx + ny + mx + my + x + y \\
&= (n + 1)x + (m + 1)y + mx + ny \\
&= nx + my + mx + ny \\
&= n(x + y) + m(x + y) \\
&= (n + m)(x + y).
\end{align*}$$

Therefore, $x + y \in P^+(S)$. Let $s \in S$, then

$$n(sx) = s(nx) = s((n + 1)x) = (n + 1)sx.$$ 

Hence $sx \in P^+(S)$. Similarly we can show that $xs \in P^+(S)$. Therefore $P^+(S)$ is an ideal of $S$. 

Proposition 3.1.3. In a semiring $S$, let $a \in P^+(S)$ such that for some $b \in S$ and some $n \in \mathbb{N}$, $a + nb = (n + 1)b$ holds. Then $b \in P^+(S)$. 

Proof. Let \( a \in P^+(S) \), then \( mA = (m+1)a \) for some \( m \in \mathbb{N} \). (1)

Since \( a + nb = (n+1)b \) for some \( b \in S \) and some \( n \in \mathbb{N} \), then

\[ ma + mnb = m(n+1)b. \]

Implies

\[ ma + mnb + nb = m(n+1)b + nb \]

Implies by (1)

\[ (m+1)a + mnb + nb = m(n+1)b + nb. \] (2)

Thus

\[ (mn + m + n)b = mnb + mb + nb \]
\[ = m(n+1)b + nb \]
\[ = (m+1)a + mnb + nb \] [by (2)]
\[ = (m+1)a + (m+1)nb \]
\[ = (m+1)(a + nb) \]
\[ = (m+1)(n+1)b \]
\[ = (mn + m + n + 1)b \]

Hence \( b \in P^+(S) \). \(\square\)

The above lemma and proposition motivates the following definition:

**Definition 3.1.4.** An ideal \( I \) of a semiring \( S \) is called a **p-ideal** if for any \( x \in S \), such that

\[ nx + a = (n+1)x \] for some \( a \in I \), \( n \in \mathbb{N} \) implies \( x \in I \).
In particular, if $S$ is an additive inversive semiring, then for each $e \in S$ there exist $e'$ such that $e + e' + e = e$. Thus

\[
ne + a = (n + 1)e \\
ne + e' + a = (n + 1)e + e' \\
(n-1)e + a = ne \\
(n-1)e + e' + a = ne + e' \\
(n-2)e + a = (n-1)e \\
\vdots \\
\vdots \\
e + 2e + a = 3e + e' \\
e + a = 2e
\]

Hence the definition boils down to:

For any $x \in S$, such that $a + x = 2x$, for some $a \in I$ then $x \in I$.

Note that if $S$ is a semiring with absorbing zero, then $P^+(S) \subseteq I$ for all $p$-ideals $I$, since $nx = nx + 0 = (n + 1)x$ for any $x \in P^+(S)$.

**Lemma 3.1.5.** *In any halfring $S$, every ideal is a $p$-ideal.*

**Proof.** Let $I$ be an ideal of a halfring $S$, let $x \in S$, $n \in \mathbb{N}$, such that $nx + a = (n+1)x$, for some $a \in I$. Then $nx + a = (n+1)x = nx + x$, implies $a = x \in I$ Hence $I$ is $p$-ideal. \qed

**Example 3.1.** *Not all $p$-ideals are $k$-ideals. For example, The set $\mathbb{Z}^+_0$ of all positive integers with zero is a halfring under the usual addition and multiplication. By lemma 3.1.5 the ideal $I = 3\mathbb{Z}^+_0\setminus\{3\}$ is $p$-ideal. But $I$ is not a $k$-ideal, since $3 \in S$, $6 \in I$, such that $3 + 6 = 9 \in I$ and $3 \notin I$.**
Example 3.2. Not all $k$-ideals are $p$-ideals in general. For example, in the semiring $Z^+$ under the operations max, min. By example 2.2 the set $I_n = \{1, 2, 3, \ldots, n\}$ is a $k$-ideal. But $I_n$ is not a $p$-ideal, since in $I_5$, $3 \cdot 6 + 5 = 6 = 4 \cdot 6$, $5 \in I_5$ and $6 \notin I_5$.

Proposition 3.1.6. In an additive inversive semiring $S$, $E^+(S)$ is a $p$-ideal. In fact, any full ideal of $S$ is a $p$-ideal.

Proof. Let $x \in S$, such that $e + x = 2x$ for some $e \in E^+(S)$. But $S$ is inversive, implies that $e + (x + x') = 2x + x' = x$, so $x \in E^+(S)$, since $x + x' \in E^+(S)$ which is an ideal by lemma 1.3.18. Therefore $E^+(S)$ is a $p$-ideal. And so any full ideal $I$ is a $p$-ideal since $E^+(S) \subseteq I$.

Example 3.3. In example 2.8, the set

$$A = \{(a, b) \in Z \times Z^+ : a = 0, \ b \in Z^+\}$$

is a full $k$-ideal in the additive inversive semiring $Z \times Z^+$ under the operations

$$(a, b) + (c, d) = (a + c, \text{lcm}(b, d)) \text{ and } (a, b).(c, d) = (a.c, \text{gcd}(b, d)).$$

Therefore, by proposition 3.1.6, the set $A$ is a $p$-ideal.

Proposition 3.1.7. In an $E$-inversive semiring $S$, every full $k$-ideal is a $p$-ideal.

Proof. Let $I$ be a full $k$-ideal of $S$, i.e., $E^+(S) \subseteq I$. Let $x \in S$, such that $a + nx = (n + 1)x$ for some $a \in I$, $n \in \mathbb{N}$. But $S$ is $E$-inversive, so there exists $y \in S$ such that $nx + y = e \in E^+(S)$. So $a + (nx + y) = (nx + y) + x$ implies $a + e = x + e$. Whence $x + e \in I$ as $a + e \in I$, as $I$ is full. But $I$ is also a $k$-ideal of $S$ and $e \in I$, so $x \in I$. Consequently, $I$ is a $p$-ideal of $S$.  

\[35\]
**Proposition 3.1.8.** In a non-zeroic semiring $S$, the zeroids $Z(S)$ is a $p$-ideal of $S$.

*Proof.* Since $S$ a non-zeroic, so $Z(S) \neq \phi$. Let $x \in S$ such that $a + nx = (n + 1)x$ for some $a \in Z(S)$ and $n \in \mathbb{N}$. Since $a \in Z(S)$ so there exists $y \in S$ such that $a + y = y$. Hence $a + nx + y = (n + 1)x + y$, so that $y + nx = nx + x + y = (y + nx) + x$ which implies that $x \in Z(S)$. Hence $Z(S)$ is a $p$-ideal. \qed

**Proposition 3.1.9.** In an additive inversive semiring $S$, an ideal $I$ is a $p$-ideal if and only if $I = I + E^+(S)$.

*Proof.* Suppose that $S$ is inversive, $I$ is $p$-ideal of $S$, let $t \in I + E^+(S)$, then $t = a + x$ where $a \in I$, $x \in E^+(S)$, so

$$2t = 2a + 2x = 2a + x = a + (a + x) = a + t$$

Then $t \in I$ as $I$ is $p$-ideal. Hence $I + E^+(S) \subseteq I$. On the other hand, let $t \in I$ then $t = t + (t' + t) \in I + E^+(S)$ as $t + t' \in E^+(S)$. hence $I \subseteq I + E^+(S)$. Therefore $I = I + E^+(S)$.

Conversely, suppose $I = I + E^+(S)$, and let $x \in S$ such that $a + x = 2x$ for some $a \in I$. Then $a + (x + x') = 2x + x' = x$, so $x \in I$ as $a \in I$ and $x + x' \in E^+(S)$. Therefore $I$ is $p$-ideal. \qed

**Corollary 3.1.10.** In an additive inversive semiring $S$, the sum of any two $p$-ideals is also a $p$-ideal.

*Proof.* Suppose that $I, J$ are two $p$-ideals in the additive inversive semiring $S$.

Then by lemma 1.3.21 $I + J$ is an ideal. Now to show that $I + J$ is $p$-ideal, let $x \in S$ such that $a + x = 2x$ for some $a \in I + J$, then $a = i + j$ where $i \in I$ and $j \in J$.

Then $2a = 2i + 2j = 2i + x = x$ which implies that $x \in I + J$. Therefore $I + J$ is $p$-ideal. \qed
and $j \in J$. Then $i + j + x + x' = 2x + x' = x$, i.e. $i + [j + (x + x')] = x$. But $J$ is $p$-ideal in the additive inversive semiring $S$. So by proposition 3.1.9, $j + (x + x') = j_1 \in J$ so $i + j_1 = x \in I + J$. Therefore $I + J$ is $p$-ideal.

**Proposition 3.1.11.** For any two $p$-ideals $I$, $J$ of a semiring $S$, $I \cap J$ is a $p$-ideal.

**Proof.** Suppose $I, J$ are two $p$-ideals of $S$, then by lemma 1.3.20 $I \cap J$ is an ideal. To show $I \cap J$ is $p$-ideal, let $x \in S$ such that $a + nx = (n + 1)x$ for some $a \in I \cap J$ and $n \in \mathbb{N}$. Then as $a \in I$, $I$ is $p$-ideal so $x \in I$, again as $a \in J$, $J$ is $p$-ideal so $x \in J$. Then $x \in I \cap J$. Therefore $I \cap J$ is $p$-ideal.

**Proposition 3.1.12.** Let $f$ be a semiring homomorphism from $S$ to $R$, let $I$ be a $p$-ideal of $R$. Then $f^{-1}(I) = \{a \in S \mid f(a) \in I\}$ is a $p$-ideal of $S$.

**Proof.** By lemma 1.3.29 $f^{-1}(I)$ is an ideal of $S$. Now let $x \in S$ such that $a + nx = (n + 1)x$ for some $a \in f^{-1}(I)$, $n \in \mathbb{N}$. Then $f(a + nx) = f((n + 1)x)$, i.e. $f(a) + nf(x) = (n + 1)f(x)$. But $I$ is a $p$-ideal and $f(a) \in I$, so $f(x) \in I$. Hence $x \in f^{-1}(I)$. Therefore $f^{-1}(I)$ is a $p$-ideal.

Next, our target is to search for the smallest $p$-ideal containing a given ideal. So we introduce the following:

**Definition 3.1.13.** For any subsemiring $R$ of a semiring $S$, we define the set

$$\hat{R} = \{x \in S \mid a + nx = (n + 1)x, \text{ for some } n \in \mathbb{N}, a \in R\}.$$ 

In the case of an additive inversive semiring, the definition of $\hat{R}$ becomes as

$$\hat{R} = \{x \in S \mid a + x = 2x \text{ for some } a \in R\}.$$ 

**Lemma 3.1.14.** For any ideal $I$ of a semiring $S$, $I \subseteq \hat{I}$
Proof. Suppose $I$ is an ideal of a semiring $S$, let $x \in I$ then $x + nx = (n+1)x$ for any $n \in \mathbb{N}$. Hence $x \in \hat{I}$. 

Lemma 3.1.15. For any ideal $I$ of a semiring $S$, $\hat{I}$ is a $p$-ideal of $S$.

Proof. Firstly, we will show that $\hat{I}$ is an ideal of $S$. Let $x, y \in \hat{I}$, then $a + nx = (n+1)x$ and $b + my = (m+1)y$ for some $n, m \in \mathbb{N}$ and $a, b \in I$. Let $k \in \mathbb{N}$ such that $k > \max\{n, m\}$. So

\[
k(x + y) + (a + b) = kx + ky + a + b
\]

\[
= (k + n - n)x + (k + n - n)y + a + b
\]

\[
= nx + (k - n)x + my + (k - m)y + a + b
\]

\[
= (k - n)x + (nx + a) + (my + b) + (k - m)y
\]

\[
= (k - n)x + (n + 1)x + (m + 1)y + (k - m)y
\]

\[
= (k - n + n + 1)x + (k - m + m + 1)y
\]

\[
= (k + 1)(x + y)
\]

so $x + y \in \hat{I}$ as $a + b \in I$. Let $r \in S$ then

\[
(n + 1)rx = r(n + 1)x = r(a + nx) = ra + nrx., \quad ra \in I.
\]

So $rx \in \hat{I}$ as $ra \in I$. similarly $xr \in \hat{I}$. Hence $\hat{I}$ is an ideal.

Finally, to show $\hat{I}$ is $p$-ideal let $u \in S$ such that

\[
uu + z = (n + 1)u \quad \text{for some} \quad z \in \hat{I} \text{ and } n \in \mathbb{N}.......(1)
\]

then since $z \in \hat{I}$ we have

\[
mz + a = (m + 1)z \quad \text{for some} \quad a \in I \text{ and } m \in \mathbb{N}.......(2)
\]
Now choose $p \in \mathbb{N}$ such that $p > m + mn + n$, then

\[ pu + a = (p - m(n + 1))u + m(n + 1)u + a \]
\[ = (p - mn - m)u + m(nu + z) + a \]
\[ = (p - mn - m)u + mnu + mz + a \]
\[ = (p - mn - m + mn)u + (mz + a) \]
\[ = (p - m)u + (m + 1)z \]
\[ = (p - m)u - n(m + 1)u + n(m + 1)u + (m + 1)z \]
\[ = (p - m - n(m + 1))u + (m + 1)(nu + z) \]
\[ = (p - m - n(m + 1))u + (m + 1)(n + 1)u \]
\[ = (p - m - mn - n + mn + m + n + 1)u \]
\[ = (p + 1)u \]

Hence, $u \in \hat{I}$. Therefore $\hat{I}$ is $p$-ideal. \hfill \Box

**Lemma 3.1.16.** For any two ideals $I, J$ of a semiring $S$ such that $I \subseteq J$ then $\hat{I} \subseteq \hat{J}$

**Proof.** Suppose that $I \subseteq J$, let $x \in \hat{I}$ then $a + nx = (n + 1)x$ for some $a \in I$, $n \in \mathbb{N}$. But $I \subseteq J$, then $x \in \hat{J}$. Therefore $\hat{I} \subseteq \hat{J}$. \hfill \Box

**Lemma 3.1.17.** For any ideal $I$ of a semiring $S$ we have $\hat{\hat{I}} = \hat{I}$

**Proof.** Let $I$ be an ideal of $S$, then by lemma 3.1.14 $I \subseteq \hat{I}$, so by lemma 3.1.16 we have $\hat{I} \subseteq \hat{\hat{I}}$. Now let $x \in \hat{I}$, then $a + nx = (n + 1)x$ for some $n \in \mathbb{N}$ and some $a \in \hat{I}$. But by lemma 3.1.15 $\hat{I}$ is a $p$-ideal, so $x \in \hat{\hat{I}}$, hence $\hat{I} \subseteq \hat{\hat{I}}$. Therefore $\hat{\hat{I}} = \hat{I}$ \hfill \Box

**Lemma 3.1.18.** For any ideal $I$ of a semiring $S$, $\hat{I}$ is the smallest $p$-ideal of $S$ containing $I$. 
Proof. Let \( J \) be a \( p \)-ideal of \( S \) such that \( I \subseteq J \), we will show that \( \hat{I} \subseteq J \). To show that let \( x \in \hat{I} \), then \( a + nx = (n+1)x \) for some \( n \in \mathbb{N} \) and some \( a \in I \).

But \( I \subseteq J \), so \( a \in J \) which is \( p \)-ideal, hence \( x \in J \). Hence \( \hat{I} \subseteq J \). \( \square \)

According to above lemma and proposition 3.1.9 we arrive to the following corollary.

**Corollary 3.1.19.** In an additive inversive semiring \( S \), \( \hat{I} = I + E^+(S) \) for any ideal \( I \) of \( S \).

**Proof.** Let \( x \in \hat{I} \), then \( a+x = 2x \) for some \( a \in I \), then \( a+(x+x') = 2x+x' = x \), then \( x \in I + E^+(S) \). So \( \hat{I} \subseteq I + E^+(S) \). On the other hand, let \( x \in I + E^+(S) \), then \( x = a + i \) for some \( a \in I \), \( i \in E^+(S) \), then \( 2x = 2a + 2i = 2a + i = a + (a + i) = a + x \), hence \( x \in \hat{I} \), so

\[
I + E^+(S) \subseteq \hat{I}.
\]

Therefore \( \hat{I} = I + E^+(S) \). \( \square \)

**Example 3.4.** Let \( S = B(10, 7) \) in example 1.17, Consider \( T = S \times \{1, 2\} \), where we define the operations on \( T \) such that for any \( (a, i), (b, j) \in T \),

\[
(a, i) + (b, j) = (a + b, \max(i, j)),
\]

\[
(a, i) \cdot (b, j) = (ab, \max(i, j)).
\]

Then it is easy to show that \( T \) a semiring with \( P^+(T) = \{(0, i), (3, i), (6, i), (9, i)\} \).

Indeed, \( 1(0, i) = 2(0, i); 3(3, i) = 4(3, i); 2(6, i) = 3(6, i); 1(9, i) = 2(9, i) \).

Now consider the ideal

\[
P = \{(0, i), (3, i), (6, i), (9, i), (1, 2), (2, 2), (4, 2), (5, 2), (7, 2), (8, 2) \mid i = 1, 2\}.
\]

we will show that \( P \) is a proper \( p \)-ideal of \( T \) by show that \( \hat{P} \subseteq P \).
Let \((a, i) \in \hat{P}\) such that \((a, i) \not\in P\), than \(i = 1\) and there exist \((b, j) \in P\) and \(n \in \mathbb{N}\) such that

\[(b, j) + n(a, 1) = (n + 1)(a, 1)\]

If \(b = 0\) then \((a, i) \in P^+(T) \subset P\). So \(b \neq 0\). Then

\[(b + na, \max(j, 1)) = ((n + 1)a, 1),\]

\[b + na = (n + 1)a \quad \text{and} \quad \max(j, 1) = 1,
\]

\[b + na = (n + 1)a \quad \text{and} \quad j = 1.\]

Hence \((b, j) = (b, 1) \in P\), so \(b\) is one of 3, 6, 9 only. But \(b + na = (n + 1)a\), so we have two cases.

Case(I): When \(b + na\), \((n + 1)a \leq 6\), \(b + na = (n + 1)a\), implies \(b = a\), whence \((a, i) = (b, j) \in P\), but \((a, i) \not\in P\). Contradiction.

Case(II): When \(b + na\) and \((n + 1)a\) are grater than 6, then \((b + na) - (n + 1)a = 3k\) for some \(k \in \mathbb{Z}_0^+\), implies \(b - a = 3k\), so \(b = a + 3k\), implies \(b = a + 3k\), but we choose \(b\) from 3,6,9. hence \((a, i) \in P^+(T) \subseteq P\). Contradiction. Therefore \(P\) is a \(p\)-ideal of \(T\).

**Lemma 3.1.20.** In any semiring \(S\) with absorbing zero \(0\), we have \(\hat{\{0\}} = P^+(S)\) and \(P^+(S) \subseteq \hat{I}\) for any ideal \(I\)

**Proof.** Suppose \(S\) is a semiring with absorbing zero, then

\[
\hat{\{0\}} = \{x \in S \mid 0 + nx = (n + 1)x, \text{ for some } n \in \mathbb{N}\}
\]

\[
= \{x \in S \mid nx = (n + 1)x, \text{ for some } n \in \mathbb{N}\}
\]

\[
= P^+(S).
\]

Next, Let \(I\) be an ideal of \(S\), and let \(x \in P^+(S)\) then \(nx = 0 + nx = (n + 1)x\) for some \(n \in \mathbb{N}\). But \(0 \in I\), so \(x \in \hat{I}\). Therefore \(P^+(S) \subseteq \hat{I}\) }
Theorem 3.1.21. Let $S$ be a semiring without absorbing zero be a subdirect product of a distributive lattice $D$ and a ring $R$. Then $I$ is a full ideal of $S$ if and only if $I$ is a $p$-ideal of $S$.

Proof. Let $S$ be a subdirect product of a distributive lattice $D$ and a ring $R$. Let $I$ be a full ideal of $S$. Let $x \in S$ such that $nx + a = (n + 1)x$ for some $a \in I$, $n \in \mathbb{N}$. Let $x = (\alpha, r)$, $a = (\beta, s)$ for some $\alpha, \beta \in D$ and $r, s \in R$. So we have

$$n(\alpha, r) + (\beta, s) = (n + 1)(\alpha, r).$$

Implies $nr + s = (n + 1)r$ which implies that $r = s$ and $\alpha + \beta = \alpha$ as $R$ is a ring, and $D$ is a distributive lattice. But $I$ is a full ideal so we have $(\alpha, 0) \in I$ as $S$ is a subdirect product. Hence,

$$(\beta, s) + (\alpha, 0) = (\alpha + \beta, s) = (\alpha, r) = x.$$

Whence $x \in I$. Consequently, $I$ is a $p$-ideal.

Conversely, let $I$ be a $p$-ideal of the semiring $S$. As $I \neq \emptyset$, there exists some $a \in I$, so that, $a = (\delta, r)$ for some $\delta \in D$ and $r \in R$. Now, idempotents of $S$ are of the form $(\alpha, 0)$ for each $\alpha \in D$. But $I$ is an ideal of $S$, so $(\delta, r)(\alpha, 0) = (\delta\alpha, 0) \in I$. We see that,

$$(\delta\alpha, 0) + (\alpha, 0) = (\delta\alpha + \alpha, 0) = (\alpha, 0) = 2(\alpha, 0)$$

which implies $(\alpha, 0) \in I$ for all $\alpha \in D$. Hence $I$ is a full ideal of $S$. \hfill \Box

Definition 3.1.22. A $p$-ideal $I$ of a semiring $S$ is called a maximal $p$-ideal of $S$ if there does not exist any $p$-ideal $J$ of $S$ satisfying $I \subsetneq J \subseteq S$.

Proposition 3.1.23. Let $I$ be a $p$-ideal of a semiring $S$. Then the $k$-closure of $I$ given by $\bar{I} = \{a \in S | a + i_1 = i_2, \text{ for some } i_1, i_2 \in I\}$ is a $p$-ideal of $S$. 42
**Proof.** Let \( x \in S \) such that \( nx + a = (n + 1)x \) for some \( a \in \bar{I}, n \in \mathbb{N} \), then \( a + i_1 = i_2 \) for some \( i_1, i_2 \in I \), implies \( nx + a + i_1 = (n + 1)x + i_1 \), implies \( nx + i_2 = (n + 1)x + i_1 \), by adding \( ni_1 \) to both sides we have \( nx + i_2 + ni_1 = (n + 1)x + i_1 + ni_1 \), implies \( n(x + i_1) + i_2 = (n + 1)(x + i_1) \) implies \( x + i_1 \in I \) as \( I \) is a \( p \)-ideal. Hence \( x + i = i_3 \) for some \( i_3 \in I \) as \( I \) is an ideal, therefore \( x \in \bar{I} \), hence \( \bar{I} \) is a \( p \)-ideal. \( \square \)

By recalling that \( I \subseteq \bar{I} \), from the above proposition we arrive at a conclusion in the next corollary.

**Corollary 3.1.24.** In a semiring \( S \) a maximal \( p \)-ideal of \( S \) must always be either a \( k \)-ideal or its \( k \)-closure must be \( S \).

### 3.2 \( p \)-Regular Semirings

In this section \( p \)-regularity of a semiring \( S \) is defined in keeping with the form of a \( p \)-ideal in \( S \) and we obtain several characterizations of \( p \)-regularity of \( S \) in connection with \( p \)-ideals in it.

**Definition 3.2.1.** In a semiring \( S \) an element \( a \) is called \textbf{\( p \)-regular} if there exists some \( b \in S \) such that,

\[
na + aba = (n + 1)a \quad \text{for some } n \in \mathbb{N}.
\]  

(3.1)

If \( S \) is an additive inversive semiring, this relation reduces to,

\[
a + aba = 2a.
\]

(3.2)

The semiring \( S \) is called \( p \)-regular if each \( a \in S \) is \( p \)-regular.

**Lemma 3.2.2.** Every multiplicative regular semiring is \( p \)-regular.
Proof. Let $S$ be a multiplicative regular semiring. Then for each $a \in S$ there exists $b \in S$ such that $aba = a$. Then for any $a \in S, n \in \mathbb{N}$, we have $na + aba = na + a = (n + 1)a$. Hence $S$ is $p$-regular.

The converse of above lemma need not be true in general. To show that let us see the next example. Also the example satisfies equation 3.1 but not equation 3.2.

Example 3.5. By referring to example 1.17, in $B(10, 7)$ we have:

1. $0 + 0 \cdot 0 \cdot 0 = 0 + 0 = 0 = 2.0$;
2. $1 + 1 \cdot 1 = 1 + 1 = 2 = 2.1$;
3. $2 + 2 \cdot 2 = 6 + 4 \cdot 2 = 6 + 8 = 8 = 4.2$;
4. $3 + 3 \cdot 3 = 9 + 3 \cdot 3 = 9 + 9 = 9 = 4.3$;
5. $4 + 4 \cdot 4 = 4 + 4 \cdot 4 = 4 + 7 = 8 = 2.4$;
6. $5 + 5 \cdot 5 = 5 + 7 \cdot 5 = 5 + 8 = 7 = 2.5$;
7. $6 + 6 \cdot 6 = 6 + 6 \cdot 6 = 6 + 9 = 9 = 2.6$;
8. $7 + 7 \cdot 7 = 7 + 7 \cdot 7 = 7 + 7 = 8 = 2.7$;
9. $8 + 8 \cdot 8 = 8 + 7 \cdot 8 = 8 + 8 = 7 = 2.8$;
10. $9 + 9 \cdot 9 = 9 + 9 \cdot 9 = 9 + 9 = 9 = 2.9$.

Hence $S$ is $p$-regular. But $S$ is not multiplicative regular since $2 \cdot x \cdot 2 = 4 \cdot x \geq 4 > 2$, so $2 \cdot x \cdot 2 \neq 2$ for all $x \in B(10, 7)$. Also equation 3.2 not hold. Indeed $2 + 2 \cdot x \cdot 2 = 2 + 4 \cdot x \geq 6 > 4 = 2(2)$. So $2 + 2 \cdot x \cdot 2 \neq 2.2$ for all $x \in B(10, 7)$.

Hence for each $a \in B(10, 7)$, there exists some $b \in B(10, 7)$ such that $na + a \cdot b \cdot a = (n + 1)a$ for some $n \in \mathbb{N}$. But $a \cdot b \cdot a = a$ does not hold for all $a \in B(10, 7)$, and $a + a \cdot b \cdot a = 2.2$ does not hold for all $a \in B(10, 7)$.

Lemma 3.2.3. In any semiring $S$ with absorbing zero, $P^+(S)$ is $p$-regular.
Proof. Let \( a \in P^+(S) \), then \( na = (n + 1)a \) for some \( n \in \mathbb{N} \). But \( 0 \in P^+(S) \) as \( m0 = (m + 1)0 \) for any \( m \in \mathbb{N} \). Hence for any \( a \in P^+(S) \) there exists \( b = 0 \in P^+(S) \) such that \( na + aba = (n + 1)a \). Therefore \( P^+(S) \) is \( p \)-regular.

\[ \square \]

**Lemma 3.2.4.** *In a semiring \( S \), If \( Z(S) \subseteq E^*(S) \) then \( Z(S) \) is \( p \)-regular.*

Proof. Suppose \( Z(S) \subseteq E^*(S) \), let \( b \in Z(S) \), then \( bbb = b \), so for any natural number \( n \) we have, \( nb + bbb = nb + b = (n + 1)b \). Therefore \( Z(S) \) is \( p \)-regular.

\[ \square \]

The converse of the above lemma not necessarily true in general as we see in the following examples.

**Example 3.6.** *Let \( S \) be an additive idempotent semiring with absorbing zero which is not multiplicative idempotent. Recall that \( S = Z(S) \) since \( a + a = a \) for all \( a \in S \) as \( S \) is additive idempotent. Now for any \( b \in S \) we have \( nb + b0b = b = (n + 1)b \) for any \( n \in \mathbb{N} \). Therefore \( Z(S) \) is \( p \)-regular. On the other hand \( Z(S) \notin E^*(S) \) as \( S \) is not multiplicative idempotent by assumption. For example, consider the set \( S = \{0, a, b\} \). Define addition and multiplication as:

\[
\begin{array}{ccc}
+ & 0 & a & b \\
0 & 0 & a & b \\
a & a & a & b \\
b & b & b & b \\
\end{array}
\quad
\begin{array}{ccc}
. & 0 & a & b \\
0 & 0 & 0 & 0 \\
a & 0 & 0 & 0 \\
b & 0 & 0 & b \\
\end{array}
\]

Then \( S \) is additive idempotent semiring, and \( S \) is not multiplicative idempotent. Hence \( S \) is \( p \)-regular.*

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Example 3.7. Let $S$ be any additive idempotent semiring with zero multiplication i.e., $ab = 0$ for any $a, b \in S$, then $S$ is $p$-regular. Since

$$na + a + a = na + 0 = a = (n + 1)a$$

for any $a \in S$, $n \in \mathbb{N}$ as $a \in E^+(S)$. On the other hand $S = Z(S) \nsubseteq E^*(S)$ as $aa = 0$ for any $a \in S$.

The next example show that the semiring $S$ is not $p$-regular, but the corresponding zeroids $Z(S)$ is $p$-regular

Example 3.8. Let $S$ be a subdirect product of a distributive lattice $D$ and a non-regular ring $R$. Then $S$ is a semiring which is not $p$-regular, since $R$ is non-regular i.e., there exists $x \in R$ such that $x \neq xy$ for all $y \in R$, so we have for any $a, b \in D$,

$$n(a, x) + (a, x)(b, y)(a, x) = (a, nx) + (a, b, xyx) = (a + ab, nx + xyx)$$

$$= (a, nx + xyx) \neq (n + 1)(a, x).$$

Otherwise, if $(a, nx + xyx) = (n + 1)(a, x)$, then $nx + xyx = (n + 1)x$, implies $xyx = x$ contradiction. But $Z(S) = D \times \{0\}$ is $p$-regular since for any $(a, 0) \in Z(S)$ and for any $b \in D$ we have

$$n(a, 0) + (a, 0)(b, 0)(a, 0) = (a + ab, 0) = (a, 0) = (n + 1)(a, 0).$$

Finally, $Z(S) \subseteq E^*(S)$, since $(x, 0)(x, 0) = (x, 0)$ for all $(x, 0) \in Z(S)$.

Proposition 3.2.5. In a $p$-regular semiring $S$, every ideal of $S$ is $p$-regular.

Proof. Let $I$ be an ideal of $p$-regular semiring $S$, let $a \in I$, then there exist $b \in S$ such that $na + a + a = (n + 1)a$ for some $n \in \mathbb{N}$ \hfill (*)
since $I$ is an ideal then $b a b \in I$, Now

$$n(n + 1)a + ababa = n(na + aba) + ababa \quad \text{[by (*)]}$$

$$= n^2a + naba + ababa$$

$$= n^2a + (na + aba)baba$$

$$= n^2a + (n + 1)aba \quad \text{[by (*)]}$$

$$= n(na + aba) + aba$$

$$= n(n + 1)a + aba \quad \text{[by (*)]}$$

$$= n^2a + na + aba$$

$$= n^2a + (n + 1)a \quad \text{[by (*)]}$$

$$= (n^2 + n + 1)a.$$ 

Hence $I$ is $p$-regular.

**Proposition 3.2.6.** In a $p$-regular semiring $S$, if for some $a, b \in S$ we have $na + aba = (n + 1)a$, for some $n \in \mathbb{N}$ then,

$$ma + a(bab)a = (m + 1)a \text{ for some } m \in \mathbb{N} \quad (3.3)$$

and,

$$m'(bab) + (bab)a(bab) = (m' + 1)bab \text{ for some } m' \in \mathbb{N}. \quad (3.4)$$

**Proof.** Suppose for some $a, b \in S$ we have $na + aba = (n + 1)a$ for some $n \in \mathbb{N}$. Take $m = n(n + 1)$, then by the proof of proposition 3.2.5 we have

$$ma + a(bab)a = n(n + 1)a + a(bab)a$$

$$= (n^2 + n + 1)a = (m + 1)a.$$ 

To prove the second part multiply the equation 3.3 by $b$ from right and left to get $m(bab) + (bab)a(bab) = (m + 1)bab$. 

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Definition 3.2.7. In a semiring $S$ an element $e$ is called $p$-idempotent if

$$ne + e^2 = (n + 1)e$$

for some $n \in \mathbb{N}$.

In case of an additive inversive semiring $S$ this definition reduces to

$$e + e^2 = 2e.$$ 

It is clear that every multiplicative idempotent element in a semiring $S$ is $p$-idempotent, but the converse need not be true. To show that see the next example:

Example 3.9. In example 1.17, we have $4$ is $p$-idempotent but not multiplicative idempotent. Indeed, $1.4 + 4^2 = 4 + 7 = 8 = 2.4$. But $4^2 = 4 \cdot 4 = 7 \neq 4$.

Proposition 3.2.8. In a semiring $S$, $p$-idempotents are $p$-regular.

Proof. Let $e \in S$ be a $p$-idempotent, then $ne + e^2 = (n + 1)e$ for some $n \in \mathbb{N}$. Then by multiplying both sides by $e$, implies

$$ne^2 + e^3 = (n + 1)e^2.$$ 

Then by adding $n^2 e$ to both sides, implies

$$n^2 e + ne^2 + e^3 = n^2 e + e^2 + ne^2,$$

$$n(ne + e^2) + e^3 = n(ne + e^2) + e^2,$$

$$n(n+1)e + e^3 = n(n+1)e + e^2,$$

$$(n^2 + n)e + e^3 = n^2 e + (ne + e^2),$$

$$= n^2 e + (n + 1)e,$$

$$(n^2 + n)e + eee = (n^2 + n + 1)e$$

Whence $e$ is $p$-regular. \qed

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Theorem 3.2.9. A semiring $S$ with $1_S$ is p-regular if and only if for every right p-ideal $A$ and left p-ideal $B$ we have $A \cap B = \hat{AB}$.

Proof. Let $S$ be a p-regular semiring. Suppose $A, B$ are right and left p-ideals respectively, then $AB \subseteq A$ and $AB \subseteq B$. But $A, B$ are p-ideals, so $A = \hat{A}$ and $B = \hat{B}$, implies

$$\hat{AB} \subseteq \hat{A} = A \text{ and } \hat{AB} \subseteq \hat{B} = B, \text{ hence } \hat{AB} \subseteq A \cap B.$$ 

On the other hand let $a \in A \cap B$, then $na + aba = (n + 1)a$ for some $b \in S$, $n \in \mathbb{N}$. But $aba \in AB$, so $a \in \hat{AB}$, hence $A \cap B \subseteq \hat{AB}$, therefore $A \cap B = \hat{AB}$.

Conversely, suppose $A \cap B = \hat{AB}$, then for any $a \in S$ and $a \neq 0$, then $a = a.1 \in aS \subseteq \hat{aS}$, then we have

$$a \in \hat{aS} \cap \hat{Sa} = \hat{aSSa}$$

Implies that

$$qa + \sum_{i=1}^{k} x_i y_i = (q + 1)a \quad (1)$$

for some $q, k \in \mathbb{N}$, where $x_i \in \hat{aS}$, $y_i \in \hat{Sa}$ for all $i = 1, 2, 3, \ldots, k$. Implies that

$$m_i x_i + ar_1 = (m_i + 1)x_i \quad (2)$$

$$p_i y_i + r_2 a = (p_i + 1)y_i \quad (3)$$

for some $m_i, p_i \in N$, $i = 1, 2, \ldots, k$; and some $r_1, r_2 \in S$. We point out that, for simplicity of calculation, without loss of generality, we may consider $n = \max(q, m_i, p_i)$ for all $i = 1, 2, \ldots, k$ in all of the three equations (1), (2), (3) instead of $q, m_i, p_i$ respectively. Now multiplying (1) by $n$ we have,

$$n^2a + n \sum_{i=1}^{k} x_i y_i = n(n + 1)a \quad (4)$$

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Again, multiplying (2) by \( y_i \) we have,

\[
nx_i y_i + ar_1 y_i = (n + 1)x_i y_i \text{ for all } i = 1, 2, \ldots, k
\]

By taking the sum implies that,

\[
n \sum_{i=1}^{k} x_i y_i + ar_1 \sum_{i=1}^{k} y_i = (n + 1) \sum_{i=1}^{k} x_i y_i
\]

By adding \( n^2a \) to both sides we have,

\[
n^2a + n \sum_{i=1}^{k} x_i y_i + ar_1 \sum_{i=1}^{k} y_i = (n + 1) \sum_{i=1}^{k} x_i y_i + n^2a
\]

so,

\[
n(n + 1)a + ar_1 \sum_{i=1}^{k} y_i = n \sum_{i=1}^{k} x_i y_i + \sum_{i=1}^{k} x_i y_i + n^2a \quad \text{[by (4)]}
\]

By adding \( na \) to both sides we have,

\[
(n^2 + n)a + ar_1 \sum_{i=1}^{k} y_i + na = n(n + 1)a + na + \sum_{i=1}^{k} x_i y_i \quad \text{[by (4)]}
\]

So,

\[
(n^2 + 2n)a + ar_1 \sum_{i=1}^{k} y_i = (n^2 + n)a + (n + 1)a \quad \text{[by (4)]}
\]

So,

\[
(n^2 + 2n)a + ar_1 \sum_{i=1}^{k} y_i = (n^2 + 2n + 1)a \quad \text{(5)}
\]

Again by multiply (3) by \( ar_1 \) we have,

\[
nar_1 y_i + ar_1 r_2 a = (n + 1)ar_1 y_i \text{ for all } i = 1, 2, \ldots, k
\]

i.e.,

\[
\sum_{i=1}^{k} ar_1 r_2 a + nar_1 \sum_{i=1}^{k} y_i + n(n^2 + 2n)a = (n + 1)ar_1 \sum_{i=1}^{k} y_i + n(n^2 + 2n)a
\]

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\[a(kr_1r_2)a + n(n^2 + 2n + 1)a = nar_1 \sum_{i=1}^{k} y_i + n(n^2 + 2n)a + ar_1 \sum_{i=1}^{k} y_i \]  \[\text{[by (5)]}\]

so by using (5) we get,

\[a(kr_1r_2)a + (n^3 + 2n^2 + n)a + (n^2 + 2n)a = n(n^2 + 2n + 1)a + ar_1 \sum_{i=1}^{k} y_i + (n^2 + 2n)a\]

i.e.,

\[a(kr_1r_2)a + (n^3 + 3n^2 + 3n)a = (n^3 + 3n^2 + 3n + 1)a\]

i.e.,

\[pa + ara = (p + 1)a \quad \text{where } p = n^3 + 3n^2 + 3n \in \mathbb{N}\]  \[\text{and } r = kr_1r_2\]

but \(a\) was arbitrary. Therefore \(S\) is \(p\)-regular. \(\square\)

**Definition 3.2.10.** A \(p\)-ideal \(B\) of a semiring \(S\) is called idempotent if \(B = \widehat{B}^2\). where \(B^2\) is the set of all finite sums of products of \(B\), i.e.,

\[B^2 = \{a_1b_1 + a_2b_2 + \ldots + a_nb_n \mid a_i, b_i \in B, n \in \mathbb{N}\}\]

**Theorem 3.2.11.** A commutative semiring \(S\) with \(1_S\) is \(p\)-regular if and only if every \(p\)-ideal of \(S\) is idempotent.

**Proof.** Let \(S\) be a commutative \(p\)-regular semiring, then by theorem 3.2.9 take \(B = A\), we get

\[A \cap A = \widehat{A}A. \quad \text{That is, } A = \widehat{A}^2.\]

Therefore \(A\) is an idempotent.

Conversely, let \(A, B\) are two \(p\)-ideals of \(S\), then by proposition 3.1.11 \(A \cap B\) is \(p\)-ideal, so by the given condition we have

\[A \cap B = (\widehat{A \cap B})^2. \quad (\ast)\]
Now, let \( a \in (\widehat{A \cap B})^2 \) so there exist \( b \in (A \cap B)^2 \) such that \( b + na = (n + 1)a \) for some \( n \in \mathbb{N} \), let \( b = b_1b_2 \) where \( b_1, b_2 \in A \cap B \), so we have

\[
b_1b_2 + na = (n + 1)a
\]

So \( a \in \widehat{AB} \) as \( b_1b_2 \in AB \). Hence

\[
(\widehat{A \cap B})^2 \subseteq \widehat{AB}
\]

Hence by (*) we have

\[
A \cap B \subseteq \widehat{AB}.
\]

On the other hand we have \( AB \subseteq A \) and \( AB \subseteq B \) which implies that

\[
\widehat{AB} \subseteq \widehat{A \cap B} = A \cap B
\]

as \( A \cap B \) is \( p \)-ideal. So \( \widehat{AB} \subseteq \widehat{A \cap B} \), hence

\[
\widehat{AB} = \widehat{A \cap B}
\]

Therefore, by theorem 3.2.9 we have \( S \) is \( p \)-regular.

**Theorem 3.2.12.** In a commutative semiring \( S \) with \( 1_S \), \( S \) is \( p \)-regular if and only if for any \( a, b \in S \),

\[
\widehat{S}ab = \widehat{Sa} \cap \widehat{Sb}
\]

Where \( \widehat{Sa} \) is the \( p \)-ideal generated by \( a \).

**Proof.** Let \( S \) be a \( p \)-regular semiring, by Theorem 3.2.9 we have,

\[
\widehat{Sa} \cap \widehat{Sb} = \widehat{SaSb}.
\]

(*)

Now, we recall that

\[
Sab = Sa1_Sb \subseteq SaSb \subseteq \widehat{SaSb}
\]
so that,

\[ \hat{S}ab \subseteq \hat{S}a \hat{S}b. \]

On the other hand, let

\[ y \in \hat{S}a \hat{S}b \]

then through some calculations essentially similar to that of Theorem 3.2.9 we can show that \( y \in \hat{S}ab \). So we have

\[ \hat{S}a \hat{S}b \subseteq \hat{S}ab \]

Hence

\[ \hat{S}a \hat{S}b = \hat{S}ab \]

Therefore, by (*) we have

\[ \hat{S}a \cap \hat{S}b = \hat{S}ab. \]

Conversely, Let the given condition is hold, then by taking \( b = a \) we have

\[ \hat{S}a^2 = \hat{S}a \cap \hat{S}a = \hat{S}a \]

whence we have \( a \in \hat{S}a^2 \) so that, for some \( s \in \hat{S} \), \( sa^2 + na = (n + 1)a, \ n \in \mathbb{N} \). But \( S \) is commutative, so \( na + asa = (n + 1)a \) whence \( S \) is \( p \)-regular.

\[ \square \]

**Theorem 3.2.13.** If a semiring \( S \) with \( 1_S \) be \( p \)-regular, then we have for \( a \in S, \hat{S}a = \hat{S}e \) where \( e \) is a \( p \)-idempotent of \( S \) i.e., every principal left \( p \)-ideal is generated by a \( p \)-idempotent.

**Proof.** Let \( S \) be a \( p \)-regular semiring with \( 1_S \). Then for any \( a \in S \) there exist \( b \in S \) such that

\[ na + aba = (n + 1)a \quad \text{for some } n \in \mathbb{N}. \quad (1) \]

Then by multiplying both sides by \( b \) from left, so \( nba + bab = (n + 1)ba \) for some \( n \in \mathbb{N} \), so \( ba \) is a \( p \)-idempotent. Take \( e = ba \), so we have that
\( ne + e^2 = (n + 1)e \) for some \( n \in \mathbb{N} \). Let \( p \in \widehat{Se} \), then there exist \( r_1 \in S \) such that

\[
\begin{align*}
  r_1 e + mp &= (m + 1)p \quad \text{for some } m \in \mathbb{N} \\
  r_1 b a + mp &= (m + 1)p, \\
  r_2 a + mp &= (m + 1)p \quad \text{for some } r_2 = r_1 b \in S
\end{align*}
\]

Implies \( p \in \widehat{Sa} \). Hence \( \widehat{Se} \subseteq \widehat{Sa} \). On the other hand. Let \( t \in \widehat{Sa} \), then

\[
ra + kt = (k + 1)t \quad \text{for some } r \in S, k \in \mathbb{N} \quad (2)
\]

Again, from (1) we have

\[
\begin{align*}
  na + ab a &= (n + 1)a, \\
  na + ae &= (n + 1)a, \\
  n r a + r ae &= (n + 1)ra. \quad (3)
\end{align*}
\]

From (2) we have

\[
\begin{align*}
  n r a + nk t &= n(k + 1)t, \\
  n r a + nk t + r a e &= n(k + 1)t + r a e, \\
  (n + 1) r a + nk t &= r a e + (nk + n) t \quad \text{[by (3)]}, \\
  (n + 1) r a + (n + 1) k t + nk t &= r a e + (nk + n) t + (n + 1) k t, \\
  (n + 1) (r a + kt) + nk t &= (r a)e + (nk + n) t + (n + 1) k t, \\
  (n + 1)(k + 1)t + nk t &= (r a)e + (nk + n + nk + k)t \quad \text{[by (2)]}, \\
  (nk + n + k + 1)t + nk t &= (r a)e + (2nk + n + k)t, \\
  (2nk + n + k + 1)t &= (r a)e + (2nk + n + k)t.
\end{align*}
\]

Hence \( t \in \widehat{Se} \). So \( \widehat{Sa} \subseteq \widehat{Se} \). Therefore \( \widehat{Sa} = \widehat{Se} \). \( \square \)
3.3 Inversive semirings with special conditions

In this section we shall consider an additive inversive semiring $S$ enjoying certain conditions. We consider the class of *additive inversive semirings* $S$ with $1_S$ for which $E^+(S)$ is a sublattice of $S$, and we will show that the condition in Theorem 3.2.13 becomes necessary and sufficient for this class of semirings to be $p$-regular. We will denote such a semiring by $R$.

Any ring with unity and distributive lattice are examples of this type of semirings. The next example is also satisfies the conditions of the semiring $R$.

**Example 3.10.** Consider the set $R = \{0, 1, 2\}$. Define addition and multiplication on $S$ as follows:

\[
\begin{array}{c|ccc}
+ & 0 & 1 & 2 \\
\hline
0 & 0 & 1 & 2 \\
1 & 1 & 0 & 2 \\
2 & 2 & 2 & 2 \\
\end{array}
\quad
\begin{array}{c|ccc}
\cdot & 0 & 1 & 2 \\
\hline
0 & 0 & 0 & 0 \\
1 & 0 & 1 & 2 \\
2 & 0 & 2 & 2 \\
\end{array}
\]

Then $R$ is an additive inversive semiring with $1_S = 1$. Indeed, $0 + 0 + 0 = 0; 1 + 1 + 1 = 1; \text{And } 2 + 2 + 2 = 2$. Moreover $E^+(R) = \{0, 2\}$ is a sublattice of $S$. Since $0 + 0 = 0 = 0.0$ and $2 + 2 = 2 = 2.2$.

The next proposition is very useful, which will be used many times in this section.

**Proposition 3.3.1.** For every $a \in R$ and for every $e \in E^+(R)$, $a + ae = a$.

*Proof.* Let $a \in R, e \in E^+(R)$, then $ae \in E^+(R)$. But $E^+(R)$ is sublattice, so
\[ ae + ae = ae, \text{ then} \]
\[ a + ae = (a + a' + a) + (ae + ae) = (a + a' + a) + ae + (ae + (ae)') + ae \]
\[ = (a + a' + a) + ae + (ae + (ae)') = (a + a' + a) + (ae + (ae)') \]
\[ = (a + a' + a) + (ae + a' e) = a + (a + a') + (a + a')e \]
\[ = a + a + a' = a. \]

\[ \square \]

**Theorem 3.3.2.** In \( R \), if for every \( a \in R \), there exists some \( p \)-idempotent \( e \in R \) satisfying \( \widehat{Ra} = \widehat{Re} \), then \( R \) is \( p \)-regular.

**Proof.** Suppose \( R \) be an additive inversive semiring with \( 1_S \) such that \( E^+(R) \) is a sublattice, let \( \widehat{Ra} = \widehat{Re} \), then \( 1.a + a = 2a \) so \( a \in \widehat{Ra} \), implies \( a \in \widehat{Re} \), so
\[ re + a = 2a \text{ for some } r \in R, \quad (1) \]
implies
\[ re^2 + ae = 2ae, \quad (2) \]
then by adding (1) and (2) we have,
\[ re^2 + ae + re + a = 2ae + 2a \]
implies
\[ re^2 + ae + re + a + a' = 2ae + 2a + a' \]
implies
\[ r(e^2 + e) + ae + a + a' = 2ae + a \]
But $e$ is $p$-idempotent, so $e^2 + e = 2e$, implies that

\[
2ae + a = 2re + ae + a + a' + a + a' \\
= 2(re + a) + 2a' + ae \\
= 4a + 2a' + ae \\
= 2a + ae
\]

(3)

Again from assumption $e \in \widehat{Ra}$, so we have

\[
ba + e = 2e \quad \text{for some } b \in R
\]

implies

\[
aba + ae = 2ae
\]

implies

\[
a + aba + ae = 2ae + a = 2a + ae \quad \text{[by (3)]}
\]

implies

\[
a + aba + ae + ae' = 2a + ae + ae'
\]

implies

\[
a + aba + a(e + e') = 2a + a(e + e')
\]

But $e + e' \in E^+(R)$, so by proposition 3.3.1 we have $a + aba = 2a$. Hence $R$ is a $p$-regular. \hfill \square

By combining theorem 3.2.13 and theorem 3.3.2 we can arrive to the following theorem.

**Theorem 3.3.3.** $R$ is $p$-regular if and only if every principal left $p$-ideal of $R$ is generated by a $p$-idempotent of $R$.  

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We now state a lemma which tells us a useful property of $p$-idempotents in $R$, so we will use it in proving the next few results.

**Lemma 3.3.4.** An element $e \in R$ is a $p$-idempotent if and only if $e \in E^\bullet(R)$.

**Proof.** Let $e$ be an element in the additive inversive semiring $R$, then we have

\[
e^2 + (e^2)' = e^2 + (e^2)' + e^2 + (e^2)' = e^2 + (ee)' + (ee)' + (ee)' \quad \text{[by lemma 1.3.10]}
\]

\[
e^2 + ee' + e'e' + e'e = (ee')^2
\]

\[
= e + e' \quad \text{[as, $e + e' \in E^+(R)$ which is a sublattice of $R$]}
\]

Now, $e$ is $p$-idempotent if and only if $e + e^2 = 2e$ if and only if $e + e^2 + e' = 2e + e'$ if and only if $e^2 + (e + e') = e$ if and only if $e^2 + e^2 + (e^2)' = e$ if and only if $e^2 = e$ if and only if $e \in E^\bullet(R)$.

**Lemma 3.3.5.** If $e$ is a $p$-idempotent in $R$ then $(1_R + e')$ is $p$-idempotent.

**Proof.** Since $e$ is $p$-idempotent, then by lemma 3.3.4 $e^2 = e$. Now

\[
(1_R + e')(1_R + e') = 1_R + e' + e' + e'e'
\]

\[
= 1_R + e' + e' + e^2 \quad \text{[by lemma 1.3.10 part(iv)]}
\]

\[
= 1_R + e' + e' + e = 1_R + e'.
\]

Hence by lemma 3.3.4 $(1_R + e')$ is $p$-idempotent.

**Lemma 3.3.6.** In $R$, if $e$ is $p$-idempotent and $b \in R$ then

\[
\widehat{Re} + \widehat{Rb} = \widehat{Re} + \widehat{Rc}
\]

where $c = b(1_R + e')$.  

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Proof. Let \( a, d \) are any two elements in \( R \), then \( ae + db \in Re + Rb \), but

\[
ae + db = ae + d(b + b(e + e')) \quad \text{[as } E^+(R) \text{ is a sublattice of } R]\]
\[
= (a + db)e + db + db'e' = (a + db)e + db(1_R + e')
\]
\[
= (a + db)e + dc \in Re + Rc
\]

which implies that

\[
Re + Rb \subseteq Re + Rc \quad \ldots \ldots (1)
\]

Now let \( x \in \widehat{Re} + \widehat{Rb} \), then \( x = y + z \) for some \( y \in \widehat{Re} \) and \( z \in \widehat{Rb} \), implies that \( y + r_1 e = 2y \) and \( z + r_2 b = 2z \) for some \( r_1, r_2 \in R \), implies \( y + z + r_1 e + r_2 b = 2(y + z) \), implies \( x + r_1 e + r_2 b = 2x \), but by (1) we have \( r_1 e + r_2 b = r_3 e + r_4 c \) for some \( r_3, r_4 \in R \), so \( x + r_3 e + r_4 c = 2x \), so \( x \in \widehat{Re} + \widehat{Rc} \). Hence

\[
\widehat{Re} + \widehat{Rb} \subseteq \widehat{Re} + \widehat{Rc}.
\]

But

\[
Re + Rc \subseteq \widehat{Re} + \widehat{Rc}
\]

Hence by applying the corollary 3.1.10 to the left \( p \)-ideal in \( R \) we have

\[
\widehat{Re} + \widehat{Rc} \subseteq \widehat{Re} + \widehat{Rc} = \widehat{Re} + \widehat{Rc}.
\]

Hence we have,

\[
\widehat{Re} + \widehat{Rb} \subseteq \widehat{Re} + \widehat{Rc}.
\]

On the other hand, Let \( a, d \in R \), then \( ae + dc \in Re + Rc \), but

\[
ae + dc = ae + d(b + b(e + e')) = ae + db + db'e'
\]
\[
= ae + db1_{1_R} + db = ae + db1_{1_R} e + db
\]
\[
= (a + db1_{1_R})e + db \in Re + Rb
\]
So, \( Re + Rc \subseteq Re + Rb \), hence similar as above we can get that

\[
\hat{Re} + \hat{Rc} \subseteq \hat{Re} + \hat{Rb}.
\]

Therefore, we get that

\[
\hat{Re} + \hat{Rb} = \hat{Re} + \hat{Rc}.
\]

\[\square\]

**Theorem 3.3.7.** If \( R \) is \( p \)-regular, then the sum of any two principal left \( p \)-ideals of \( R \) is a principal left \( p \)-ideal of \( R \).

*Proof.* Let \( \hat{Ra} \), and \( \hat{Rb} \) be any two principal left \( p \)-ideals of \( R \), then by theorem 3.2.13 and lemma 3.3.4 there exist \( p \)-idempotent \( e \in R \) such that

\[
\hat{Ra} = \hat{Re} \text{ and } e^2 = e.
\]

But by lemma 3.3.6 we have that

\[
\hat{Re} + \hat{Rb} = \hat{Re} + \hat{Rc}, \text{ where } c = b(1_R + e'). \tag{1}
\]

Also, by theorem 3.2.13 and lemma 3.3.4 there exist \( p \)-idempotent \( f \in R \) such that

\[
\hat{Rc} = \hat{Rf} \text{ and } f^2 = f.
\]

Then \( f = 1_R f \in \hat{Rc} \), implies \( f + r_1 c = 2f \) for some \( r_1 \in R \), implies \( fe + r_1 c e = 2fe \), implies \( fe + r_1 b(1_R + e')e = 2fe \), implies \( fe + r_1 b(e^2 + e'e) = 2fe \), implies \( fe + r_1 b(e + e')e = 2fe \). But \( e + e' \in E^+(R) \) which is \( p \)-ideal. Hence

\[
fe \in E^+(R) = E^+(R).
\]

So \( fee' + fee' = fee' \). But by 1.3.10 we have \( ee' = (ee)' = (e^2)' = e' \), implies \( fe' + fe' = fe' \) implies \( fe' f + fe' f = fe' f \). Hence \( fe' f \in E^+(R) \).
Now, take \( g = (1_R + e')f \), then \( g \in Rf \), implies \( Rg \subseteq Rf \), hence \( \widehat{Rg} \subseteq \widehat{Rf} \). Also, since \( f + r_1 c = 2f \), implies \( f^2 + r_1 c f = 2f^2 \), implies \( f + r_1 b(1_R + e')f = 2f \), implies \( f + r_1 bg = 2f \), implies \( rf + rr_1 bg = 2rf \) for any \( r \in R \). Implies \( rf \in \widehat{Rg} \) for any \( r \in R \). Implies \( Rf \subseteq \widehat{Rg} \), implies \( \widehat{Rf} \subseteq \widehat{Rg} \). Hence

\[
\widehat{Rf} = \widehat{Rg}
\]  

(2)

Hence we have

\[
\widehat{Ra} + \widehat{Rb} = \widehat{Re} + \widehat{Rb} = \widehat{Re} + \widehat{Rc} \quad \text{[by (1)]}
\]

\[
= \widehat{Re} + \widehat{Rf} = \widehat{Re} + \widehat{Rg} \quad \text{[by (2)]}
\]

(3)

Moreover,

\[
g^2 = (1_R + e')f(1_R + e')f
\]

\[
= f^2 + e'ff + fef' + e'f'e'
\]

\[
= f + e'f + fef' + e'f'e'
\]

\[
= f + f^2(e'f) + e'(f + f^2(e'f))
\]

\[
= [f + f(fe')f] + [e'(f + f(fe')f)]
\]

\[
= f + e'f \quad \text{[as } fe'f \in E^+(R) \text{ which is sublattice]}
\]

\[
= (1_R + e')f = g
\]

Finally, we claim that

\[
\widehat{Re} + \widehat{Rg} = \widehat{R(e + g)}.
\]

to show that let \( x \in R(e + g) \), so \( x = r(e + g) = re + rg \in Re + Rg \), so \( R(e + g) \subseteq Re + Rg \), implies

\[
R(e + g) \subseteq \widehat{Re + Rg}.
\]
But by corollary 3.1.10 we have

\[
\widehat{Re + Rg} \subseteq \widehat{Re + Rg} = \widehat{Re + Rg}.
\]

Hence

\[
\widehat{R(e + g)} \subseteq \widehat{Re} + \widehat{Rg}.
\]

On the other hand,

\[
e g = e(1_R + e')f = (e + ee')f = (e^2 + ee')f = (e + e')ef \in E^+(R) \quad \text{as} \quad (e + e') \in E^+(R).
\]

Also,

\[
ge = (1_R + e')fe \in E^+(R) \quad \text{as} \quad fe \in E^+(R).
\]

Then

\[
e = e + e(eg) = e^2 + eg = e(e + g) \in R(e + g),
\]

\[
g = g + g(ge) = g^2 + ge = g(e + g) \in R(e + g).
\]

Thus

\[
Re \subseteq R(e + g) \quad \text{and} \quad Rg \subseteq R(e + g),
\]

\[
\widehat{Re} \subseteq \widehat{R(e + g)} \quad \text{and} \quad \widehat{Rg} \subseteq \widehat{R(e + g)},
\]

\[
\widehat{Re + Rg} \subseteq \widehat{R(e + g)},
\]

Hence,

\[
\widehat{Re} + \widehat{Rg} = \widehat{R(e + g)}.
\]

Therefore, by (3) we have

\[
\widehat{Ra} + \widehat{Rb} = \widehat{R(e + g)}.
\]
Proposition 3.3.8. In $R$, for any $x_1, x_2 \in R$ we have $\langle x_1, x_2 \rangle \subseteq R_{x_1} + R_{x_2}$, where $\langle x_1, x_2 \rangle$ is the left $p$-ideal generated by $x_1, x_2$.

Proof. Since $\langle x_1, x_2 \rangle = \{ r_1x_1 + r_2x_2 | r_1r_2 \in R \} \subseteq R_{x_1} + R_{x_2}$, then

$\langle x_1, x_2 \rangle \subseteq R_{x_1} + R_{x_2} \subseteq R_{x_1} + R_{x_2} = \widehat{Rx_1} + \widehat{Rx_2} \ [\text{by corollary 3.1.10}].$

On the other hand, let $x \in \widehat{Rx_1} + \widehat{Rx_2}$, then $x = y + z$ for some $y \in \widehat{Rx_1}$ and $z \in \widehat{Rx_2}$, so $y + r_3x_1 = 2y$ and $z + r_4x_2 = 2z$ for some $x_3, x_4 \in R$, implies $y + z + r_3x_1 + r_4x_2 = 2(y + z)$, implies $x + r_3x_1 + r_4x_2 = 2x$, so that $x \in \langle x_1, x_2 \rangle$, hence

$\widehat{Rx_1} + \widehat{Rx_2} \subseteq \langle x_1, x_2 \rangle.$

therefore we get that,

$\langle x_1, x_2 \rangle = \widehat{Rx_1} + \widehat{Rx_2}$

We can generalize the above proposition for any finite number of elements of $R$ to get that

$\langle x_1, x_2, \ldots, x_n \rangle = \widehat{Rx_1} + \widehat{Rx_2} + \ldots + \widehat{Rx_n}$

and so we can also generalize theorem 3.3.7 in the following corollary

Corollary 3.3.9. If $R$ is $p$-regular then the left $p$-ideal generated by a finitely number of elements of $R$ is a principal left $p$-ideal.

Proposition 3.3.10. If $e$ is a $p$-idempotent in $R$ then

$\widehat{Re} \cap R(1_R + e') \subseteq E^{+}(R).$
Proof. Let \( a \in \hat{Re} \cap R(1 + e') \), then \( a \in R(1 + e') \), so \( a + t_1(1 + e') = 2a \) for some \( t_1 \in R \), implies \( ae + t_1(1 + e')e = 2ae \), implies \( ae + t_1(e + e'e) = 2ae \).

But by lemma 1.3.10 we have \( e'e = (ee)' = e' \) as \( e \) is \( p \)-idempotent, implies \( ae + t_1(e + e') = 2ae \). But \( e + e' \in E^+(R) \) which is \( p \)-ideal by proposition 3.1.6. Hence

\[
ae \in E^+(R) \quad \text{......(1)}
\]

Also, \( a \in \hat{Re} \), so

\[
a + t_2e = 2a \quad \text{......(2)}
\]

for some \( t_2 \in R \), implies \( ae + t_2ee = 2ae \), implies

\[
ae + t_2e = 2ae \quad \text{......(3)}
\]

implies \( a + ae + t_2e = a + 2ae \). By (1) implies that \( a + ae + t_2e = a + ae \). But \( a + t_2e = 2a \). Hence

\[
2a + ae = a + ae \quad \text{......(4)}.
\]

Again by (3) we have \( ae + ae' + t_2e = 2ae + ae' = 2ae + (ae)' = ae \), implies \( a + a(e + e') + t_2e = a + ae \), but \( e + e' \in E^+(R) \) which is sublattice, so \( a + t_2e = a + ae \), implies \( a + t_2e + ae' = a + ae + ae' = a + a(e + e') \), by (2) implies \( 2a + ae' = a \). But since \( ae \in E^+(R) \), so \( ae + ae + ae = ae \) implies \( ae = (ae) = ae' \). Then

\[
2a + ae = a \quad \text{......(5)},
\]

\[
a + ae = a \quad \text{......(6)}.
\]

Thus, we have, \( 2a = a + a = a + a + ae \)  [by (6)]

\[
= 2a + ae = a \quad \text{[by (5)]}
\]

Hence \( a \in E^+(R) \). Therefore, the result is hold.
Proposition 3.3.11. In $R$, if $e$ is $p$-idempotent then we have

$$\widehat{Re} + R(1_R + e') = R.$$  

Proof. Since $1_R + [e + (1_R + e')] = 1_R + [1_R + (e + e')1_R] = 1_R + 1_R = 2 1_R$ as $e + e' \in E^+(R)$ which is a sublattice of $R$, hence

$$1_R \in Re + R(1_R + e').$$

But $Re \subseteq \widehat{Re}$ and $R(1_R + e') \subseteq R(\widehat{1_R + e'})$, so

$$Re + R(1_R + e') \subseteq \widehat{Re} + R(\widehat{1_R + e'}).$$

By applying corollary 3.1.10 to the left $p$-ideal we have

$$\widehat{Re} + R(\widehat{1_R + e'}) \subseteq \widehat{Re} + R(\widehat{1_R + e'}) = \widehat{Re} + R(\widehat{1_R + e'}).$$

Hence $1_R \in \widehat{Re} + R(\widehat{1_R + e'})$. Therefore

$$\widehat{Re} + R(\widehat{1_R + e'}) = R.$$  

\qed

By using proposition 3.3.10 and proposition 3.3.11 we arrive immediately at the following:

Theorem 3.3.12. If $R$ is $p$-regular and contains absorbing zero, then the set of all principal left $p$-ideals of $R$ form a lattice

Proof. Let $L$ be the set of all principal left $p$-ideals of $R$, then $L$ is a poset with respect to set inclusion as partial ordering. Now for any two principal left $p$-ideals of $R \widehat{Ra}$, $\widehat{Rb}$ define

$$\widehat{Ra} \vee \widehat{Rb} = \widehat{Ra + Rb}, \quad \text{and} \quad \widehat{Ra} \wedge \widehat{Rb} = \widehat{Ra \cap Rb}.$$
Then by theorem 3.3.7 $\hat{Ra} + \hat{Rb} \in L$, also by theorem 3.2.12 $\hat{Ra} \cap \hat{Rb} = \hat{Rab} \in L$. To show that $\sup\{\hat{Ra}, \hat{Rb}\} = \hat{Ra} + \hat{Rb}$, clearly that $\hat{Ra} = \hat{Ra} + 0 \subseteq \hat{Ra} + \hat{Rb}$, and $\hat{Rb} \subseteq \hat{Ra} + \hat{Rb}$, let $\hat{Rc} \in L$ such that $\hat{Ra} \subseteq \hat{Rc}$ and $\hat{Rb} \subseteq \hat{Rc}$, then $\hat{Ra} + \hat{Rb} \subseteq \hat{Rc}$. Hence

\[
\sup\{\hat{Ra}, \hat{Rb}\} = \hat{Ra} + \hat{Rb}.
\]

To show that $\inf\{\hat{Ra}, \hat{Rb}\} = \hat{Ra} \cap \hat{Rb}$. Clear $\hat{Ra} \cap \hat{Rb} \subseteq \hat{Ra}$ and $\hat{Ra} \cap \hat{Rb} \subseteq \hat{Rb}$, let $\hat{Rx} \in L$ such that $\hat{Rx} \subseteq \hat{Ra}$ and $\hat{Rx} \subseteq \hat{Rb}$, then $\hat{Rx} \subseteq \hat{Ra} \cap \hat{Rb}$. Hence

\[
\inf\{\hat{Ra}, \hat{Rb}\} = \hat{Ra} \cap \hat{Rb}.
\]

Therefore, $L$ is a lattice.
Conclusion

In this research we introduced some algebraic structures in semirings, and studied carefully some special kinds of ideals in it as $k$-ideal and $p$-ideal, also we illustrated this research with many examples. But there are a lot of special kinds of ideals in semirings, we hope to study many of it in future, also there is some research problems related with the special kinds of ideals, we hope that we can find solutions for them. We hope in the future there will be other wide researches on other algebraic structures that has important applications in the practical life as cryptography, coding, and communication.
Bibliography


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