SPECTRAL THEORY IN EFFECT ALGEBRAS

M.Sc. Thesis

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If one supposes a quantum logic $L$ to be a $\sigma$-effect algebra, then the observables on $L$ are identified with the $L$-valued measures defined on the Borel subsets of the real line. In this structure (and without the aid of Hibert space formalism) we will show that

1. the spectrum of an observable can be completely characterised by studying the observable $\left(A - \lambda\right)^{-1}$, and
2. corresponding to every observable $A$ there is a spectral resolution uniquely determined by $A$ and uniquely determining $A$.

Also, we study the existence of spectral measures corresponding to elements of a $\sigma$-MV-algebra, and we apply such a result to obtain a similar result concerning $\sigma$-complete lattice effect algebras.
INTRODUCTION

In the last few years the notion of effect algebras has received much attention within the studies on the mathematical foundations of quantum mechanics [1,6,7,11]. Effect algebras appear to be the natural outcome in the search of a mathematical structure that captures the fundamental aspects of the elementary two-valued physical quantities, or effects, pertaining to a physical system. The notion of an effect algebra is sufficiently general to encompass the traditional order structures accompanying classical systems (Boolean algebras) and quantum systems (orthomodular posets), but it is sufficiently structured to carry a meaningful interplay with the physically relevant notions of states and of observables [1,6,7,11].

Until quite recently the observables in non-relativistic quantum mechanics have been identified with the set of self-adjoint operators on a separable, infinite dimensional, complex Hilbert space. Likewise, on the same Hilbert space, the states have been identified with the trace operators of trace class 1 [4]. However, with the advent of Mackey’s book on the mathematical foundations of quantum mechanics [20], both observables and states have assumed a more abstract character having no overt connection with Hilbert space. This had led some investigators to consider the problem of deciding which quantum mechanical results are essentially consequences of Hilbert space formalism and which can be obtained without involving Hilbert space [12,25,29]. In this thesis we will show that most of the desirable theorems involving spectra can be obtained without the use of Hilbert space.

Thus, in this thesis, we shall study the relationship between the notions of observable and the so-called spectral resolutions on an effect algebras (EA), originally introduced by Foulis and Bennett [11]. In 1967, D. Catlin [4] studied this relationship on an orthomodular poset (OMP). Our work will rely heavily on Catlin’s paper and we will try to extend the results in [4] to the more general setting of an effect algebra, a generalization of an OMP.
The thesis is organized in four chapters. In Chapter 1, we recall some elementary definitions and facts pertaining to lattices, OMP theory and spectral resolutions [3, 4], and we establish some basic results which will be used in the subsequent chapters.

In Chapter 2, we present some preliminary notions and results on effect algebras, and we illustrate these notions with appropriate examples. Moreover, we shall be interested in the relation between effect algebras, orthoalgebras and OMPs. More precisely, we will justify, how is each effect algebra is a generalization of an orthoalgebra and of an OMP. Furthermore, we shed some light on the compatibility concept, which is useful for Chapter 4. In the last section of this chapter, we will present the definition of a $\sigma$-effect algebra and some results concerning it, which will be used in Chapter 3.

In Chapter 3, we will give the definition of the spectrum of an observable on an effect algebra, and we derive the most important properties of this spectrum. Depending on these properties we shall obtain a classification of spectra, which is similar to the well-known classification of spectra in operator theory. In the remaining part of this chapter, we will study the relationship between observables and spectral resolutions, and hence we conclude the chapter with general results on effect algebras.

Finally, Chapter 4 is concerned with the notion of spectral measure introduced by S. Pulmannova [22, 23]. Firstly, S. Pulmannova showed the existence of spectral measures for elements of $\sigma$-MV algebras. Hence we can use the last result and the results in Section 2.2, to show the existence of spectral measures on $\sigma$-complete lattice effect algebras, which is an analogue of a spectral theorem for self-adjoint Hilbert space operators [22].
Chapter 1

Preliminary Results

1.1 Basic Properties of Posets

Definition 1.1 [3] A relation $R$ on a nonempty set $X$ is a subset of $X \times X$. If $(x, y) \in R$ we write $xRy$. A relation $R$ on $X$ is called

(i) reflexive iff $xRx \ \forall x \in X$;
(ii) symmetric iff $xRy \Rightarrow yRx$;
(iii) anti-symmetric iff $xRy$ and $yRx \Rightarrow x = y$;
(iv) transitive iff $xRy$ and $yRz \Rightarrow xRz$.

A relation $R$ on a set $X$ is partial order provided $R$ is reflexive, anti-symmetric and transitive. In this case $(X, R)$ is called a partially ordered set or simply a poset.

Definition 1.2 [3] A subset $B$ of a poset $(P, \leq)$ is a chain if every two elements $a, b$ in $B$ are comparable; that is, either $a \leq b$ or $b \leq a$.

Let $(P, \leq)$ be a poset. The smallest element of $P$ (if it exists) is the element $a_0$ such that $a_0 \leq a \ \forall a \in P$, and the largest element of $P$ (if it exists) is the element $a_1$ such that $a \leq a_1 \ \forall a \in P$. The smallest and the largest elements are unique, when they exist, by anti-symmetry of $\leq$. They may not exist; e.g., $\mathbb{R}$ with the usual order $\leq$ has no smallest or largest elements. The smallest and the largest elements of $P$ will be denoted by 0 and 1, respectively. A poset with 0 and 1 is called a bounded poset. An element $u \in P$ is called the supremum (or join) of a subset $M$ of $P$, and write $\text{sup} M$ (or $\sqcup M$), if $u$ is an
upper bound for \( M \) (i.e., \( x \leq u \ \forall \ x \in M \)), and whenever \( v \) is an upper bound for \( M \), \( u \leq v \). An element \( e \in P \) is called the infimum (or meet) of a subset \( M \) of \( P \) and we write \( \inf M \) (or \( \bigwedge M \)), if \( e \) is a lower bound for \( M \) (i.e., \( e \leq x \ \forall \ x \in M \)), and whenever \( l \) is a lower bound for \( M \), \( l \leq e \). By antisymmetry of \( \leq \), these elements are unique whenever they exist.

**Zorn’s Lemma 1.3** If \( P \) is a partially ordered set such that every chain in \( P \) has an upper bound in \( P \), then \( P \) has a maximal element.

### 1.2 Lattices

**Definition 1.4** [3] A lattice is a partially ordered set \( (L, \leq) \) such that for each pair of elements \( x, y \in L \), the supremum and infimum of the set \( \{x, y\} \) exist in \( L \). A lattice \( (L, \leq) \) is called a \( \sigma \)-lattice iff every countable subset of \( L \) has a join and meet in \( L \).

**Theorem 1.5** For all elements \( x, y, z \) of a lattice \( (L, \leq) \), we have the following:

(a) (Idempotency) \( x \wedge x = x \) and \( x \vee x = x \).

(b) (Commutativity) \( x \wedge y = y \wedge x \) and \( x \vee y = y \vee x \).

(c) (Associativity) \( x \wedge (y \wedge z) = (x \wedge y) \wedge z \) and \( x \vee (y \vee z) = (x \vee y) \vee z \).

(d) (Absorption) \( x \wedge (x \vee y) = x \) and \( x \vee (x \wedge y) = x \).

**Proof.** Parts (a) and (b) follow directly from the definition of sup and inf.

(c) Notice that \( x \wedge (y \wedge z) \leq y \wedge z \leq y \) and \( x \wedge (y \wedge z) \leq x \), so by definition of inf, \( x \wedge (y \wedge z) \leq x \wedge y \). We also have \( x \wedge (y \wedge z) \leq y \wedge z \leq z \). This implies that \( x \wedge (y \wedge z) \leq (x \wedge y) \wedge z \), and likewise we get \( (x \wedge y) \wedge z \leq x \wedge (y \wedge z) \). Therefore by anti-symmetry of \( \leq \), \( x \wedge (y \wedge z) = (x \wedge y) \wedge z \).

(d) Clearly \( x \leq x \) and \( x \leq x \vee y \). Hence \( x \leq x \wedge (x \vee y) \). Also \( x \wedge (x \vee y) \leq x \). Therefore by anti-symmetry of \( \leq \), \( x \wedge (x \vee y) = x \). Similarly \( x \vee (x \wedge y) = x \). \( \square \)

**Definition 1.6** A lattice \( L \) is said to be **distributive** iff it satisfies the following two **distributive laws** for all \( x, y, z \in L \):

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Theorem 1.7 For any lattice $L$, (i) and (ii) in above definition are equivalent.

Proof. (i) $\Rightarrow$ (ii) : Suppose that (i) holds. Then from parts (c), (d) of Theorem 1.5, we have

\[
(x \lor y) \land (x \lor z) = [(x \lor y) \land x] \lor [(x \lor y) \land z] = [x \lor ((x \lor y) \land z)] = x \lor [(x \land z) \lor (y \land z)] = [x \lor (x \land z)] \lor (y \land z) = x \lor (y \land z)
\]

Therefore (ii) holds.

(ii) $\Rightarrow$ (i) : The proof of this part is done similarly. □

1.3 Orthomodular Posets

Definition 1.8 [3] Let $(P, \leq)$ be a bounded poset. By an orthocomplementation on $P$ we mean a function $': P \to P$ such that

(a) $x \leq y \Rightarrow y' \leq x' \ \forall \ x, y \in P$;
(b) $x = x''$;
(c) $x \land x', x \lor x'$ exists, and $x \lor x' = 1; x \land x' = 0$.

The element $x'$ is called the orthocomplement of $x$, and if such function exists, then $(P, \leq)$ is called an orthocomplemented poset or simply orthoposet. Let $x, y \in P$, where $P$ is an orthoposet. Then $x$ and $y$ are said to be orthogonal written $x \perp y$, iff $x \leq y'$ (or equivalently $y \leq x'$).

Note: It is clear that $x \perp y$ iff $y \perp x$.

A subset $\{x_1, x_2, ..., x_n\}$ of an orthoposet $P$ is called an orthogonal family
iff \( x_i \perp x_j \) for \( i \neq j, 1 \leq i \leq n, 1 \leq j \leq n \).

**Theorem 1.9** [3] (Generalized De Morgan Laws). Let \((P, \leq, ')\) be an orthocomplemented poset, and let \( \{e_\alpha\}_{\alpha \in A} \) be a subset of \( P \).

(i) If either \( \bigvee_{\alpha \in A} e_\alpha \) or \( \bigwedge_{\alpha \in A} e'_\alpha \) exists, then they both do, and moreover,

\[
(\bigvee_{\alpha \in A} e_\alpha)' = \bigwedge_{\alpha \in A} e'_\alpha.
\]

(1.1)

(ii) If either \( \bigwedge_{\alpha \in A} e_\alpha \) or \( \bigvee_{\alpha \in A} e'_\alpha \) exists, then they both do, and moreover,

\[
(\bigwedge_{\alpha \in A} e_\alpha)' = \bigvee_{\alpha \in A} e'_\alpha.
\]

(1.2)

**Proof.** (i) Suppose, first, that \( \bigvee_{\alpha \in A} e_\alpha \) exists and let \( z = (\bigvee_{\alpha \in A} e_\alpha)' \). Then \( z' = \bigvee_{\alpha \in A} e_\alpha \) so that \( e_\alpha \leq z' \forall \alpha \in A \), then \( z \leq e'_\alpha \forall \alpha \in A \) and hence \( z \) is a lower bound for the set \( \{e'_\alpha : \alpha \in A\} \). Let \( w \) be a lower bound for the set \( \{e'_\alpha : \alpha \in A\} \). Then \( w \leq e'_\alpha \forall \alpha \in A \) and hence \( e_\alpha \leq w' \forall \alpha \in A \); that is, \( w' \) is an upper bound for the set \( \{e_\alpha : \alpha \in A\} \). It follows that \( z' \leq w' \), hence \( w \leq z \), and so \( z \) is the greatest lower bound of the set \( \{e'_\alpha : \alpha \in A\} \). Therefore (1.1) is hold. A similar argument yields the same result if we suppose the existence of \( \bigwedge_{\alpha \in A} e'_\alpha \).

(ii) A similar to the proof of part (i). \( \square \)

**Definition 1.10** [3] By an orthomodular poset (abbreviated OMP) we mean an orthocomplemented poset \((P, \leq)\) that satisfies the following:

(i) If \( a, b \in P, a \perp b \), then \( a \lor b \in P \)

(ii) If \( x, y \in P \) with \( x \leq y \), then \( y = x \lor (y \land x') \).

**Note:** The relationship in (ii) is called the orthomodular identity abbreviated OMI. A poset \((P, \leq)\) is called complete (respectively, \(\sigma\)-complete) iff every subset (countable subset) has a join and a meet. If \((P, \leq, ')\) is an or-
thocomplemented distributive lattice, then it is called a Boolean algebra, and if also \((P, \leq, \prime)\) is \(\sigma\)-complete, then it is called a Boolean \(\sigma\)-algebra.

**Definition 1.11** [3] Let \(L\) be an OMP and let \(A \subseteq L\) be such that

(i) \(x, y \in A \Rightarrow x \vee y \in A\);

(ii) \(x \in A \Rightarrow x^{\prime} \in A\);

(iii) \(A\) is distributive.

Then \(A\) is called a Boolean subalgebra of \(L\). If \(A\) has the property that \(\{x_i : i \in \mathbb{N}\} \subseteq A\) implies \(\bigvee_{i=1}^{\infty} x_i \in A\) then \(A\) is called a Boolean \(\sigma\)-subalgebra of \(L\).

Note that by De Morgan law and (i), \(x, y \in A \Rightarrow x \wedge y \in A\).

An orthomodular lattice (abbreviated OML) is an OMP which is also a lattice.

**Example 1.12** Let \(X = \{1, 2, 3, 4, 5, 6, 7, 8\}\) and let \(L := \{A \subseteq X : \text{Card}(A) \text{ is even}\}\), where \(\text{Card}(A)\) is the number of elements in \(A\). Partially order \(L\) by set-theoretic inclusion, and define a mapping \(A \rightarrow A^{c} : L \rightarrow L\) by \(A^{c} := X \setminus A\). Clearly \((L, \leq, ^{c})\) is an orthoposet. Also, for \(A, B \in L, A \leq B^{c}(A \subseteq B^{c}) \Rightarrow A \cap B = \emptyset \Rightarrow A \cup B \in L\). It follows that the join of orthogonal (i.e., disjoint) elements exists and equals their union. Thus if \(A, B \in L\) and \(A \leq B\), then \(B^{c} \vee B \in L\), and \(B^{c} \vee A \in L \Rightarrow B^{c} = B \cap A^{c} = B \wedge A^{c}\). Hence \(A \vee (B \wedge A^{c})\) exist and \(A \vee (B \wedge A^{c}) = A \cup (B \cap A^{c}) = A \cup (B \setminus A) = B\). Therefore, \(L\) is an OMP. But \(L\) is not a lattice, since \(A = \{1, 2\} \in L\) and \(B = \{2, 3\} \in L\), but \(A \vee B = B \cup A = \{1, 2, 3\} \in L\).

**Definition 1.13** [3] By a sub-OMP \(M\) of an OMP \(L\) we mean a subset \(M \subseteq L\) such that

(i) \(x, y \in M \Rightarrow x \leq y \Leftrightarrow x \leq M y\),

(ii) \(^{\prime}\) is the orthocomplemented in \(M\), and

(iii) if \((x_i) \subseteq M\) is a finite orthogonal elements, then \(\bigvee_{i=1}^{M} x_i = \bigvee_{i=1}^{L} x_i\).
An OMP $L$ is called a $\sigma$-OMP if every countable orthogonal subset of $L$ has supremum in $L$. If $L$ is a $\sigma$-OMP, then $M$ is a $\sigma$-subOMP if it satisfies (i), (ii) and the following condition

(iv) if $(x_i) \subseteq M$, is a countable pairwise orthogonal elements, then
\[ \bigvee_{i \in \mathbb{N}} x_i = \bigvee_{i \in \mathbb{N}} x_i. \]

The following Theorem was proved in [3, p. 96].

**Theorem 1.14** If $A$ is a sub-OMP of $L$ and if $A$ is a Boolean algebra (with respect to $\,'$, $\lor^A$, $\land^A$), then $A$ is a Boolean subalgebra (i.e., $\lor^A$ will coincide with $\lor^L$ and $\land^A$ will coincide with $\land^L$.) In particular if $L$ is a $\sigma$-OMP and if $A$ is a Boolean $\sigma$-algebra that satisfies (i), (ii) and (iv) of Definition 1.13, then $A$ is a Boolean $\sigma$-subalgebra of $L$. In this case $\bigvee_{i \in \mathbb{N}} x_i = \bigvee_{i \in \mathbb{N}} x_i$ if $\{x_i : i \in \mathbb{N}\} \subseteq A$.

### 1.4 Spectral Resolutions

Recall that the collection $\mathcal{B}$ of Borel sets is the smallest $\sigma$-algebra of $\mathbb{R}$ which contains all open subsets of $\mathbb{R}$.

**Definition 1.15** [3] Let $B$ be a Boolean $\sigma$-algebra and let $\mathcal{B}$ be the $\sigma$-algebra of Borel subsets of $\mathbb{R}$. By an $B$-valued Borel measure we mean a function $A : \mathcal{B} \to B$ satisfying:

(i) $E, F \in \mathcal{B}, E \cap F = \emptyset \Rightarrow A(E) \perp A(F)$;

(ii) if $\{E_i\}_{i \in \mathbb{N}}$ is a countable family of mutually disjoint Borel sets, then
\[ A\left(\bigcup_{i=1}^{\infty} E_i\right) = \bigvee_{i=1}^{\infty} A(E_i); \tag{1.3} \]

(iii) $A(\emptyset) = 0$ and $A(\mathbb{R}) = 1$. 

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If $L$ is a $\sigma$-OMP, we say that $A : \mathcal{B}(\mathbb{R}) \to L$ is an observable on $L$ if it is satisfies (i), (ii) and (iii) of Definition 1.15.

The following Theorem was proved in [3, page 75].

**Theorem 1.16** Let $L$ be a $\sigma$-OMP. If $A$ is an observable on $L$, then the range of $A$ is a Boolean $\sigma$-subalgebra of $L$.

**Definition 1.17** [3] Let $L$ be any bounded poset. By real (resp., rational) spectral resolution in $L$ we mean a function $e : \mathbb{R} \to L$ (resp., $e : \mathbb{Q} \to L$) such that the following conditions are satisfied:

(i) $\lambda \leq \mu$ in $\mathbb{R}$ (resp., $\mathbb{Q}$) $\Rightarrow e_\lambda \leq e_\mu$ in $L$;

(ii) $\bigwedge_{\lambda} e_\lambda = 0$;

(iii) $\bigvee_{\lambda} e_\lambda = 1$;

(iv) $\bigwedge_{\mu < \lambda} e_\lambda = e_\mu \forall \mu \in \mathbb{R}$ (resp., $\forall \mu \in \mathbb{Q}$).

**Lemma 1.18** [3] Let $B$ be a Boolean $\sigma$-algebra. Then there is a one-to-one correspondence between real and rational spectral resolutions on $B$, as follows:

(1) If $e : \mathbb{R} \to B$ is a real spectral resolution, then the rational spectral resolution $f$ associated with it is given by $f := e|_{\mathbb{Q}}$.

(2) If $f : \mathbb{Q} \to B$ is a rational spectral resolution, then the real spectral resolution $e$ associated with it is given by

$$e_\lambda := \bigwedge\{f_\mu : \mu \in \mathbb{Q}, \lambda \leq \mu\}, \lambda \in \mathbb{R}.$$ 

**Proof.** (1) Note that $B$ is a Boolean $\sigma$-algebra; hence $\bigwedge_{\lambda \in \mathbb{Q}} e_\lambda$ and $\bigvee_{\lambda \in \mathbb{Q}} e_\lambda$ exist, since the set of all rational numbers is countable and hence $\bigwedge_{\lambda \in \mathbb{Q}} f_\lambda$ and $\bigvee_{\lambda \in \mathbb{Q}} f_\lambda$ exist since $f_\lambda = e_\lambda$ whenever $\lambda \in \mathbb{Q}$. We want to check the conditions
of Definition 1.17

(i) If \( \lambda \leq \mu \) in \( \mathbb{Q} \), then \( e_{\lambda} \leq e_{\mu} \) and so \( f_{\lambda} \leq f_{\mu} \).

(ii) Let \( t = \bigwedge_{\lambda \in \mathbb{Q}} f_{\lambda} \). Then \( t \leq f_{\lambda} \ \forall \lambda \in \mathbb{Q} \), and \( f_{\lambda} = e_{\lambda} \) whenever \( \lambda \in \mathbb{Q} \), so we get \( t \leq e_{\lambda} \ \forall \lambda \in \mathbb{Q} \). But for each irrational number \( \mu \), there exists a rational number \( \lambda \) such that \( \lambda \leq \mu \) (by the Archimedean property). It follows from Definition 1.17 that \( t \leq e_{\lambda} \leq e_{\mu} \); that is, \( t \leq e_{\mu} \ \forall \mu \in \mathbb{R} \setminus \mathbb{Q} \); hence \( t \leq e_{\mu} \ \forall \mu \in \mathbb{R} \). But \( \bigwedge_{\lambda \in \mathbb{R}} e_{\lambda} = 0 \), so we get \( t = 0 \). Therefore \( \bigwedge_{\lambda \in \mathbb{Q}} f_{\lambda} = 0 \).

(iii) Let \( u = \bigvee_{\lambda \in \mathbb{Q}} f_{\lambda} \). Then \( f_{\lambda} \leq u \ \forall \lambda \in \mathbb{Q} \) and \( f_{\lambda} = e_{\lambda} \) whenever \( \lambda \in \mathbb{Q} \), so we get \( e_{\lambda} \leq u \ \forall \lambda \in \mathbb{Q} \). By the Archimedean property, for each irrational number \( \lambda \) there exists a rational number \( \lambda \) such that \( \lambda > \mu \). It follows that \( e_{\mu} \leq e_{\lambda} \leq u \); that is, \( e_{\mu} \leq u \ \forall \mu \in \mathbb{R} \setminus \mathbb{Q} \); hence \( e_{\mu} \leq u \ \forall \mu \in \mathbb{R} \). But \( \bigvee_{\lambda \in \mathbb{R}} e_{\lambda} = 1 \), which implies that \( u = 1 \); hence \( \bigvee_{\lambda \in \mathbb{Q}} f_{\lambda} = 1 \).

(iv) Let \( w = \bigwedge_{\mu < \lambda} f_{\lambda} \), \( \lambda, \mu \in \mathbb{Q} \). Then \( w \leq f_{\lambda} \) whenever \( \mu < \lambda, \lambda, \mu \in \mathbb{Q} \), also, by the Archimedean property, for each irrational number \( \lambda_{1} > \mu \) we can find a rational number \( \lambda \) such that \( \lambda_{1} > \lambda > \mu \), and hence by part (i) \( w \leq f_{\lambda} = e_{\lambda} \leq e_{\lambda_{1}} \), so we have \( w \leq \bigwedge_{\mu < \lambda} e_{\lambda} = e_{\mu}, \ \lambda \in \mathbb{R} \).

Now we want to prove that \( e_{\mu} \leq w \). If \( \lambda \in \mathbb{R} \), then \( e_{\mu} = \bigwedge_{\mu < \lambda} e_{\lambda} \leq \bigwedge_{\mu < \lambda} e_{\lambda} \) whenever \( \lambda \in \mathbb{Q} \); but \( e_{\lambda} = f_{\lambda} \) whenever \( \lambda \in \mathbb{Q} \), so we get \( \bigwedge_{\lambda < \mu} e_{\lambda} = \bigwedge_{\lambda < \mu} f_{\lambda} = w \). Thus \( e_{\mu} \leq w \), and therefore \( w = e_{\mu} = f_{\mu} \).

(2) If \( f : \mathbb{Q} \to B \) is a rational spectral resolution we want to prove that \( e_{\lambda} = \bigwedge \{ f_{\mu} : \mu \in \mathbb{Q}, \lambda \leq \mu \}, \lambda \in \mathbb{R} \) is a spectral resolution.

(i) If \( \alpha \leq \beta \), then \( \{ f_{\mu} : \mu \in \mathbb{Q}, \beta \leq \mu \} \subseteq \{ f_{\mu} : \mu \in \mathbb{Q}, \alpha \leq \mu \} \), so that we
get \( \bigwedge \{ f_\mu : \mu \in \mathbb{Q}, \alpha \leq \mu \} \leq \bigwedge \{ f_\mu : \mu \in \mathbb{Q}, \beta \leq \mu \} \); that is, \( e_\alpha \leq e_\beta \).

(ii) Note that from the definition of \( e_\lambda \), if \( \lambda \in \mathbb{Q} \), then \( e_\lambda = f_\lambda \). Let \( t \in B \) be such that \( t \leq e_\lambda \ \forall \lambda \in \mathbb{R} \). Then \( t \leq e_\lambda \ \forall \lambda \in \mathbb{Q} \) which implies that \( t \leq \bigwedge_{\lambda \in \mathbb{Q}} f_\lambda = 0 \), since \( (f_\lambda)_{\lambda \in \mathbb{Q}} \) is a spectral resolution; that is \( t = 0 \). Therefore \( \bigwedge_{\lambda \in \mathbb{R}} e_\lambda = 0 \).

(iii) Let \( t \in B \) be such that \( e_\lambda \leq t \ \forall \lambda \in \mathbb{R} \). Then \( e_\lambda \leq t \ \forall \lambda \in \mathbb{Q} \), but \( e_\lambda = f_\lambda \) whenever \( \lambda \in \mathbb{Q} \). So we have \( f_\lambda \leq t \ \forall \lambda \in \mathbb{Q} \), which implies that \( 1 = \bigvee_{\lambda \in \mathbb{Q}} f_\lambda \leq t \); hence \( t = 1 \). Therefore \( \bigvee_{\lambda \in \mathbb{R}} e_\lambda = 1 \).

(iv) We want to prove that \( \bigwedge_{\theta < \mu} e_\mu = e_\theta \).

Case(1): \( \theta \in \mathbb{Q} \).

From the definition of \( e_\theta \),

\[
e_\theta = f_\theta = \bigwedge_{\theta < \mu} f_\mu = \bigwedge_{\theta < \mu} e_\mu, \mu \in \mathbb{Q}. \quad (1.4)
\]

If \( \theta < \mu, \mu \in \mathbb{R} \) then from part (i), \( e_\theta \leq e_\mu, \mu \in \mathbb{R} \); that is, \( e_\theta \) is lower bound of the set \( \{ e_\mu : \theta < \mu, \mu \in \mathbb{R} \} \). Let \( t \leq e_\mu, \mu > \theta, \mu \in \mathbb{R} \). Then \( t \leq e_\mu, \ \forall \mu \in \mathbb{Q}, \theta < \mu \); hence \( t \leq \bigwedge_{\theta < \mu} e_\mu = e_\theta \) from (1.4). Therefore

\[
e_\theta = \bigwedge_{\theta < \mu} e_\mu, \mu \in \mathbb{R}, \theta \in \mathbb{Q}.
\]

Case(2): \( \theta \in \mathbb{R} \setminus \mathbb{Q} \).

From the definition of \( e_\theta \) we get,

\[
e_\theta = \bigwedge_{\theta \leq \mu} f_\mu = \bigwedge_{\theta < \mu} f_\mu = \bigwedge_{\theta < \mu} e_\mu, \mu \in \mathbb{Q}. \quad (1.5)
\]

By the same proof of case(1) we get \( e_\theta = \bigwedge_{\theta < \mu} e_\mu, \mu \in \mathbb{R}, \theta \in \mathbb{R} \setminus \mathbb{Q} \). \( \square \)

**Definition 1.19** [3] Given Boolean algebras \( A \) and \( B \), a map \( \psi : A \to B \) is a Boolean homomorphism if \( \psi \) satisfies the following:
(i) $\psi(0) = 0, \psi(1) = 1,$

(ii) $\psi(a') = (\psi(a))' \forall a \in A,$

(iii) $\psi(a \lor b) = \psi(a) \lor \psi(b) \forall a, b \in A.$

Note that

$$\psi(a \land b) = (\psi(a') \lor b')' = (\psi(a')')' \land (\psi(b')')' = \psi(a) \land \psi(b).$$

If $\psi$ is a surjective homomorphism then $\psi$ is called a Boolean epimorphism. If $\psi$ is bijective, then $\psi$ is a Boolean isomorphism. If the condition (iii) of the Definition 1.19 holds for a countable join, then $\psi$ is called a Boolean $\sigma$-homomorphism. Let $L$ and $M$ be Boolean $\sigma$-algebras. If there exists a bijective Boolean $\sigma$-homomorphism $\psi : L \rightarrow M$, then we write $L \cong M$

**Definition 1.20 [3]** Let $L$ be a $\sigma$-lattice. A nonempty subset $I$ is an $\sigma$-ideal iff

(i) $(a_i)_{i \in \mathbb{N}} \subseteq I \Rightarrow \bigvee_{i \in \mathbb{N}} a_i \in I,$

(ii) $a \in I, x \in L \Rightarrow a \land x \in I.$

Given a Boolean $\sigma$-algebra $B$ and a proper $\sigma$-ideal $I$ of $B$. Define the equivalence relation $\sim$ on $B$ as the follow:

$$a \sim b \iff (a \land b') \lor (a' \land b) \in I,$$

then the set of equivalence classes denoted by $B/I$, is a Boolean $\sigma$-algebra [27, page 396].

**Theorem 1.21 [3] (Loomis).** Let $B$ be a Boolean $\sigma$-lattice. Then there exists a set $X$, and a Boolean $\sigma$-sublattice $S$ of the power set of $X$ and a Boolean ideal $I$ in $S$ such that $B \cong S/I$.

Notation : Recall that a measurable space is a pair $(X, \mathcal{M})$ consisting of a set $X$ and a subset $\mathcal{M}$ of the power set of $X$ such that $\mathcal{M}$ is closed under countable unions and complements, and $\emptyset \in \mathcal{M}$. 

12
Chapter 2

Effect Algebras

In this chapter, we introduce an overview of $\sigma$-OMPs, $\sigma$-orthoalgebras and $\sigma$-effect algebras that are needed for this thesis. We shall present some of the basic and important facts that link these structures together.

2.1 Known Results

Definition 2.1 [11] An effect algebra is a set $L$ with two particular elements 0, 1, and a partial binary operation $\oplus$ on $L$ such that the following are satisfied $\forall p, q, r \in L$:

(EA1) (Commutativity) If $p \oplus q$ is defined, then $q \oplus p$ is defined and $p \oplus q = q \oplus p$.

(EA2) (Associativity) If $q \oplus r$ is defined and $p \oplus (q \oplus r)$ is defined, then $p \oplus q$ is defined, $(p \oplus q) \oplus r$ is defined, and $p \oplus (q \oplus r) = (p \oplus q) \oplus r$.

(EA3) (Orthocomplementation) For every $p \in L$ there exists a unique $q \in L$ such that $p \oplus q$ is defined and $p \oplus q = 1$.

(EA4) (Zero-One Law) If $1 \oplus p$ is defined, then $p = 0$.

When we write an equation such as $p \oplus r = q$ in an effect algebra, we assert that both $p \oplus r$ is defined and that $p \oplus r = q$. From now on, unless otherwise stated, $L = (L, \oplus, 0, 1)$ will denote a fixed effect algebra.

Definition 2.2 [11] Let $p, q \in L$.

(i) We say that $p$ is orthogonal to $q$ and write $p \perp q$ iff $p \oplus q$ is defined.
(ii) We say that \( p \) is less than or equals to \( q \) and write \( p \leq q \) iff there exists an element \( r \in L \) such that \( p \perp r \) and \( p \oplus r = q \).

(iii) The unique element \( q \) such that \( p \perp q \) and \( p \oplus q = 1 \) is written as \( p' := q \) and called the orthosupplement of \( p \). (We use the notation := to mean “equals by definition”).

The following is an example of an effect algebra.

**Example 2.3 [9]** Let the closed interval \([0, 1]\) be ordered in the natural way, and for two numbers \( a, b \in [0, 1] \), we define \( a \oplus b \) iff \( a + b \leq 1 \) and we write \( a \oplus b := a + b \). Then \(([0, 1], \oplus, 0, 1)\) is an effect algebra.

**Lemma 2.4 [11]** Let \( p, q \in L \). Then

(i) \( p \perp q \Rightarrow q \perp p \),
(ii) \( p'' = p \),
(iii) \( 1' = 0 \) and \( 0' = 1 \),
(iv) \( p \oplus p' = 1 \),
(v) \( p \perp 0 \) and \( p \oplus 0 = p \),
(vi) \( p \perp 1 \Leftrightarrow p = 0 \),
(vii) \( p \oplus q = 0 \Rightarrow p = q = 0 \).

**Proof.** (i), (ii), (iii), and (iv) are obvious.

(v) \( : 1 = 1 \oplus 1' = (p' \oplus p) \oplus 0 = p' \oplus (p \oplus 0) \); hence, \( p \oplus 0 = p'' = p \).

(vi) : \( p \perp 1 \Rightarrow p = 0 \) by the zero-one law. The converse follows from part v.

(vii) : If \( p \oplus q = 0 \), then \( 1 = 1 \oplus 0 = 1 \oplus (p \oplus q) = (1 \oplus p) \oplus q \), so \( 1 \perp p \), and it follows that \( p = 0 \). By symmetry, \( q = 0 \). \( \square \)

**Theorem 2.5 [11]** Let \( p, q \in L \). Then

(i) \( p \perp q \Rightarrow p \perp (p \oplus q)' \) and \( p \oplus (p \oplus q)' = q' \).
(ii) \( p \perp q \Leftrightarrow p \leq q' \).
(iii) \( p \leq q \Rightarrow q' \leq p' \).

(iv) \( p \leq q \Rightarrow p \perp (p \oplus q)' \) and \( p \oplus (p \oplus q)' = q \).

**Proof.** (i): Let \( r := (p \oplus q)' \). Then, \( 1 = (p \oplus q) \oplus r = (q \oplus p) \oplus r = q \oplus (p \oplus r) \), so \( q' = p \oplus r \), therefore \( q' = p \oplus (p \oplus q)' \).

(ii): If \( p \perp q \), then by part (i) \( q' = p \oplus (p \oplus q)' \) so that \( p \leq q' \). Conversely, if \( p \leq q' \), then there exists \( r \in L \) with \( p \oplus r = q' \); hence, \( 1 = q' \oplus q = (p \oplus r) \oplus q = (r \oplus p) \oplus q = r \oplus (p \oplus q) \). It follows that \( p \perp q \).

(iii): Assume \( p \leq q \Rightarrow p \leq (q')' \Rightarrow p \perp q' \Rightarrow q' \perp p \Rightarrow q' \leq p' \).

(iv): Assume \( p \leq q \Rightarrow p \leq (q')' \Rightarrow p \perp q' \), so by part (i), we get \( p \oplus (p \oplus q')' = (q')' = q \). \( \square \)

Part (iv) of Theorem 2.6 is called the *orthomodular identity*.

**Theorem 2.6** [11] (Cancellation Laws). Let \( p, q, r \in L \) with \( p, q \perp r \). Then

(i) \( p \oplus r = q \oplus r \Rightarrow p = q \).

(ii) \( p \oplus r \leq q \oplus r \Rightarrow p \leq q \).

Proof. (i): Assume that \( p \oplus r = q \oplus r \), and let \( s := (p \oplus r)' = (q \oplus r)' \). Then \( (p \oplus r) \oplus s = 1 = (q \oplus r) \oplus s, p \oplus (r \oplus s) = q \oplus (r \oplus s) = 1 \), and it follows that \( p = (r \oplus s)' = q \).

(ii): Assume that \( p \oplus r \leq q \oplus r \). Then there exists \( s \in L \) with \( (p \oplus r) \oplus s = q \oplus r \); then \( (p \oplus s) \oplus r = q \oplus r \), so by part (i), \( p \oplus s = q \); hence \( p \leq q \). \( \square \)

**Theorem 2.7** [11] An effect algebra \( L \) is partially ordered by \( \leq \), and \( 0 \leq p \leq 1 \) holds \( \forall p \in L \).

Proof. It is clear that \( \leq \) is reflexive. To prove the anti-symmetry of \( \leq \), suppose that \( a, b \in L \) with \( a \leq b \) and \( b \leq a \). Then there are elements \( p, q \in L \) with \( a \oplus p = b \) and \( b \oplus q = a \). Hence, \( a \oplus 0 = a = b \oplus q = (a \oplus p) \oplus q = a \oplus (p \oplus q) \), so \( p \oplus q = 0 \) by the cancellation law. Therefore, \( p = q = 0 \), by part (vii) of Lemma 2.4, and it follows that \( a = b \). It remains to prove that \( \leq \) is transitive. Let \( a, b, c \in L \) be such that \( a \leq b \) and \( b \leq c \). Then there exist \( p, q \in L \) such that
a ⊕ p = b and b ⊕ q = c, then \((a ⊕ p) ⊕ q = c\), so \(a ⊕ (p ⊕ q) = c\) it follows that \(a ≤ c\). □

**Theorem 2.8** An effect algebra \((L, ⊕, 0, 1, 'l)\) is an OMP iff \(a ⊕ b = a ∨ b\) whenever \(a ≤ b'\).

**Proof.** Assume \(L\) is an effect algebra, and \(a ⊕ b = a ∨ b\) whenever \(a ≤ b'\). To show that \(L\) is an OMP, note first that, \((L, 0, 1, 'l)\) is an orthocomplemented poset. Indeed for any \(x, y ∈ L\), if \(x ≤ y\), then by Theorem 2.5 \(y' ≤ x'\). Also by Lemma 2.4, for each \(x ∈ L\), \(x'' = x, x ⊕ x' = 1\). Then \(x ∨ x' = 1\), and hence \(x ∧ x' = 0\) by De Morgan law and Lemma 2.4(iii). Therefore \((L, 0, 1, 'l)\) is an orthocomplemented poset. Now we want to prove the OMI for OMPs. By OMI of effect algebra; if \(p ≤ q\), then \(p ⊕ (p ⊕ q)' = q\), and hence \(p ∨ (p ∨ q)' = q\) or \(p ∨ (p' ∨ q) = q\) by De Morgan law; so the OMI for OMPs holds. Therefore \((L, 0, 1, 'l)\) is an OMP.

To prove the converse, assume that \((L, ⊕, 0, 1, 'l)\) is an OMP; that is, an orthocomplemented poset, in particular \(p ∧ p' = 0\), for each \(p ∈ L\).

**Claim:** If \(p ⊥ q, r ∈ L\), and \(p, q ≤ r ≤ p ⊕ q\), then \(r = p ⊕ q\).

To see this, since \(p, q ≤ r\), then there exist \(s, t ∈ L\), such that \(p ⊕ s = r\), and \(q ⊕ t = r\). Also since \(r ≤ p ⊕ q\), then there exists \(u ∈ L\), such that \(r ⊕ u = p ⊕ q\), so that we get \((q ⊕ t) ⊕ u = p ⊕ q\). It follows that \(u ⊕ (q ⊕ t) = p ⊕ q\); that is, \((u ⊕ t) ⊕ q = p ⊕ q\); hence \(u ⊕ t = p\), by the cancellation law; that is \(u ≤ p\). Similarly \(u ≤ q\). Since \(p ⊥ q\), we have \(q ≤ p'\); so that \(u ≤ p, p'\), and hence \(u ≤ p ∧ p'\). But \(p ∧ p' = 0\) yields \(u = 0\), and therefore \(r = p ⊕ q\).

Now, since \(p ⊥ q\), and \(L\) is an OMP, then \(p ∨ q\) exists and \(p, q ≤ p ∨ q ≤ p ⊕ q\). Hence, by the above claim, \(p ∨ q = p ⊕ q\). □

**Theorem 2.9** Let \(L\) be an OMP, and define \(a ⊕\) on \(L\) as a follows:

for \(p, q ∈ L\) with \(p ≤ q', p ⊕ q := p ∨ q\). Then \((L, ⊕, 0, 1)\) is an effect algebra.

**Proof.** We want to prove that \(⊕\) satisfy the conditions EA1, EA2, EA3 and EA4.

(EA1) If \(p ⊕ q\) is defined, then \(p ⊕ q = p ∨ q = q ∨ p = q ⊕ p\); hence \(q ⊕ p\) is defined and \(p ⊕ q = q ⊕ p\).
If $q \oplus r$ and $p \oplus (q \oplus r)$ are defined then $q \oplus r \leq p'$; that is $q \leq q \lor r \leq p'$, so that $q \leq p'$ which implies $p \lor q$ exist and $p \lor q = p \oplus q$. Since $q \oplus r$ is defined, then $q \leq r'$ and since $p \leq r'$, then $p \lor q \leq r'$ since $p \lor q$ is defined. Hence $(p \oplus q) \oplus r$ is defined.

Finally using Theorem 1.3(c) we have

$$p \oplus (q \oplus r) = p \lor (q \lor r) = (p \lor q) \lor r = (p \oplus q) \oplus r.$$  

(EA3) For any $p \in L$, there exists a unique $p' \in L$, such that $p \lor p' = 1$ since $L$ is an OMP hence $p \oplus p'$ is defined and $p \oplus p' = p \lor p' = 1$.

(EA4) If $1 \oplus p$ is defined, then $p \leq 1'$, this implies $p = 0$.

Therefore, $(L, \oplus, 0, 1)$ is an effect algebra.  \[\Box\]

The following is an example of an effect algebra that is not an OMP.

**Example 2.10** [13] (Foulis and Greech). Let $L = \{0, 1, a, b, c, a', b', c'\}$ be the effect algebra with the following $\oplus$ table. In this table we do not include 0 and 1 since they have trivial sums and $a$- means that the corresponding $\oplus$ is not defined.

$$\begin{array}{ccccccc}
& a & b & c & a' & b' & c' \\
\hline
a & - & c' & b' & 1 & - & - \\
b & c' & b' & a' & - & 1 & - \\
c & b' & a' & - & - & - & 1 \\
a' & 1 & - & - & - & - & - \\
b' & - & 1 & - & - & - & - \\
c' & - & - & 1 & - & - & - \\
\end{array}$$

Then $a \oplus b = c'$, but $a \lor b$ does not exist. Indeed from above table it is clearly that the only upper bounds of $a$, $b$ is $c'$ and $b'$, but $c' \not\leq b'$ and $b' \not\leq c'$. \[\Box\]

**Definition 2.11** [11] A subset $A \subseteq L$ is called a sub-effect algebra of $L$ iff $0, 1 \in A, A$ is closed under $p \rightarrow p'$, and, for all $p, q \in A, p \perp q \Rightarrow p \oplus q \in A$.

A sub-effect algebra $A$ of an effect algebra $L$ is an effect algebra in its own right.

**Definition 2.12** [11] An orthoalgebra [11, 14] is an effect algebra $L$ in which the zero-one law (part (EA4) of Definition 2.1) is replaced by the stronger consistency law:
Therefore, every orthoalgebra is an effect algebra. In Example 2.10 $L$, is an effect algebra that is not an orthoalgebra. Indeed, $b \perp b$, but $b \neq 0$.

**Definition 2.13** [17] Let $L$ be an effect algebra. We say that $a \in L$ is **sharp** if $a \land a' = 0$.

If $a \oplus a$ is defined and $a$ is sharp, then $a = 0$. Indeed, since $a \oplus a$ exists, then $a \leq a'$, also $a \leq a$, so $a \leq a \land a' = 0$. Hence $a = 0$. Therefore, each sharp effect algebra is an orthoalgebra.

Cancellation property guarantees that in every effect algebra $L$ the partial binary operation $\ominus$ and the relation $\leq$ can be defined by

$$a \leq c \text{ and } c \ominus a = b \iff a \oplus b \text{ exist and } a \oplus b = c.$$ \hspace{1cm} (ED)

Indeed, if $a \leq c$ and $c \ominus a = b_1$ and $c \ominus a = b_2$, then $a \ominus b_1 = c$ and $a \ominus b_2 = c$; that is $a \ominus b_1 = a \ominus b_2$, so by cancellation law we get $b_1 = b_2$. From OMI if $a \leq b$, then $b = a \ominus (a \ominus b')'$ and hence

$$b \ominus a = (a \ominus b')'.$$ \hspace{1cm} (†)

**Theorem 2.14** [6, 7] Let $p, q, r$ be elements in an effect algebra $L$. Then

(i) $p \leq q \iff q = p \oplus (q \ominus p)$,

(ii) if $p \leq q$, then $p = q \iff q \ominus p = 0$,

(iii) if $p \leq q$, then $p = q \ominus (q \ominus p)$,

(iv) if $p \leq q \leq r$, then $(r \ominus q) \ominus (q \ominus p) = (r \ominus p)$.

**Proof.** Parts (i), (ii) and (iii) follow from the cancellation law and the orthomodular identity.

(iv) Since $p \oplus q' \geq q' \Rightarrow (p \oplus q')' \leq q$. Also $q \leq q \ominus r'$, so by transitivity of $\leq$, $(p \ominus q')' \leq q \ominus r'$. Hence, Lemma 2.5(ii) implies that $(p \oplus q')' \perp (q \ominus r')'$. It follows that $(q \ominus p) \perp (r \ominus q)$, i.e., $(q \ominus p) \ominus (r \ominus q)$ is defined. Since $q \leq r$, then from part (i), $r = q \ominus (r \ominus q)$. Also $p \leq r$, $p \leq q$, then $r = p \ominus (r \ominus p)$ and
\[ q = p \oplus (q \ominus p). \text{Hence } p \oplus (r \ominus p) = p \oplus (q \ominus p) \oplus (r \ominus q). \text{Therefore by the cancellation law, } r \ominus p = (q \ominus p) \oplus (r \ominus q). \]

\textbf{Lemma 2.15 [7]} Let \( a, b, c \) be elements in an effect algebra \( L \) be such that \( a \leq b \leq c \). Then

(i) \( b \ominus a \leq c \ominus a \), and

(ii) \( c \ominus b \leq c \ominus a \).

\textbf{Proof.} (i) Since \( a \leq b \) and \( a \leq c \), then there exist \( w, v \in L \) such that \( a \oplus w = b \) and \( a \oplus v = c \). But \( b \leq c \), so we get \( a \oplus w \leq a \oplus v \); hence by the cancellation law, \( w \leq v \). Therefore \( b \ominus a \leq c \ominus a \).

(ii) Since \( a \leq c, b \leq c, \) and \( a \leq b \), there exist \( u, v, w \in L \) such that \( a \oplus u = c, b \oplus v = c, \) and \( a \oplus w = b \), so we get \( (a \oplus w) \oplus v = c = a \oplus u \); that is \( a \oplus (w \oplus v) = a \oplus u \), so \( w \oplus v = u \) by the cancellation law; hence \( v \leq u \). Therefore \( c \ominus b \leq c \ominus a \). \( \Box \)

\textbf{Theorem 2.16} Let \( a, b, c \) be elements in an effect algebra \( L \). Then

(i) \( a \ominus 0 = a \), for any \( a \in L \).

(ii) If \( a \leq b \leq c \), then \( c \ominus b \leq c \ominus a \) and \( (c \ominus a) \ominus (c \ominus b) = (b \ominus a) \).

\textbf{Proof.} (i) \( a \leq 1 = 0' \) for every \( a \in L \), then \( a \ominus 0 \) is defined for every \( a \in L \). Also by Lemma 2.4(v) \( a \ominus 0 = a \). Then by (ED) \( a \ominus 0 = a \).

(ii) By Lemma 2.15(ii) \( c \ominus b \leq c \ominus a \), then there exist \( r \in L \) such that \( (c \ominus b) \oplus r = c \ominus a \) and hence \( ((c \ominus b) \oplus r) \oplus a = c \), so by the associativity of \( \oplus \) \( (c \ominus b) \oplus (r \ominus a) = c \). Since \( b \leq c \), then \( c = b \oplus (c \ominus b) \) by Theorem 2.14(i). Hence \( (c \ominus b) \oplus (r \ominus a) = c = (c \ominus b) \oplus b \), then by the cancellation law \( r \ominus a = b \). Therefore \( r = b \ominus a \). \( \Box \)

\textbf{Lemma 2.17 [7]} Let \( a, b, c \) be elements in an effect algebra \( L \). Then

(i) \( a \ominus a = 0 \).

(ii) \( a \leq b \) implies \( b \ominus a = 0 \iff b = a \).
(iii) $a \leq b$ implies $b \ominus a = b \iff a = 0$.

(iv) $a \leq b \leq c \Rightarrow b \ominus a \leq c \ominus a$ and $(c \ominus a) \ominus (b \ominus a) = c \ominus b$.

(v) $a \leq b \Rightarrow b \ominus a \leq b$ and $b \ominus (b \ominus a) = a$.

(vi) $a \leq c$ and $b \leq c \ominus a \Rightarrow (c \ominus a) \ominus b = (c \ominus b) \ominus a$.

(vii) $a \leq b \leq c \Rightarrow a \leq c \ominus (b \ominus a)$ and $(c \ominus (b \ominus a)) \ominus a = c \ominus b$.

**Proof.** Parts (i)-(iii) follows directly from Theorem 2.16. For (iv), from Lemma 2.15(i), we have $b \ominus a \leq c \ominus a$ and Theorem 2.16(ii), Theorem 2.14(iii) yield

\[(c \ominus a) \ominus (b \ominus a) = (c \ominus a) \ominus ((c \ominus a) \ominus (c \ominus b)) = c \ominus b.\]

(v) Note that $0 \leq a \leq b$, then by Lemma 2.15(ii) $b \ominus a \leq b \ominus 0 = b$. By Theorem 2.14(iii) $b \ominus (b \ominus a) = a$

(vi) Observe that $b \leq c \ominus a \leq c$ and Theorem 2.16(ii) yield $(c \ominus a) \ominus b = (c \ominus b) \ominus [c \ominus (c \ominus a)] = (c \ominus b) \ominus a$

(vii) From (v), $b \ominus a \leq b \leq c$. Then by Lemma 2.15(i), $a \leq c \ominus (b \ominus a)$ and by part (vi), part (iv) we have

\[(c \ominus (b \ominus a)) \ominus a = (c \ominus a) \ominus (b \ominus a) = c \ominus b. \,

\]

\[\square\]

### 2.2 Compatibility in Effect Algebras

**Definition 2.18** [17] For an effect algebra $(L, \oplus, 0, 1)$ two elements $a, b \in L$ are *compatible* (written $a \leftrightarrow b$) if there exist $a_1, b_1, c \in L$ such that $a = a_1 \oplus c$, $b = b_1 \oplus c$, and $a_1 \oplus b_1 \oplus c$ is defined.

The following definition for the compatibility of two elements in an effect algebra is given in terms of $\ominus$.

**Definition 2.19** [7] For an effect algebra $(L, \oplus, 0, 1)$ two elements $a, b \in L$ are *compatible* if there exist $u, v \in L$ such that $v \leq a \leq u$, $v \leq b \leq u$ and $u \ominus a = b \ominus v$ (evidently then $u \ominus b = a \ominus v$).

We shall prove that the above two definitions of compatibility are equivalent.

**Theorem 2.20** Let $L$ be an effect algebra. Then the following assertions are equivalent:

\[\square\]
(i) $a \leftrightarrow b$ (in the sense of Definition 2.19).

(ii) $\exists v \in L$ such that $v \leq a$, $v \leq b$ and $a \ominus v \leq b'$.

(iii) $a \leftrightarrow b$ (in the sense of Definition 2.18).

**Proof.** (i) $\Rightarrow$ (ii): Assume $a \leftrightarrow b$ in the sense of Definition 2.19. Then there exist $u, v \in L$ such that $v \leq a \leq u$, $v \leq b \leq u$, and $u \ominus a = b \ominus v$. Since $b \leq u \leq 1$, then by Lemma 2.15(i) $a \ominus v = u \ominus b \leq 1 \ominus b = b'$.

(ii) $\Rightarrow$ (iii): Assume (ii) holds. Then there exist $a_1, b_1 \in L$ such that $a = a_1 \ominus v$, $b = b_1 \ominus v$, and $(a \ominus v) \ominus b$ is defined. Hence $a_1 \ominus b$ is defined; that is $a_1 \ominus b_1 \ominus v$ is defined. Therefore $a \leftrightarrow b$ in the sense of Definition 2.18.

(iii) $\Rightarrow$ (ii): Assume (iii) holds. Then there exist $a_1, b_1, c$ such that $a = a_1 \ominus c$, $b = b_1 \ominus c$, and $a_1 \oplus b_1 \ominus c$ is defined. Let $u = a_1 \oplus b_1 \ominus c$, $v = c$. Then clearly $v \leq a \leq u$, $v \leq b \leq u$ and

$$u \ominus a = ((a_1 \ominus c) \ominus b_1) \ominus (a_1 \ominus c) = b_1 = b \ominus c = b \ominus v.$$ 

Therefore (i) holds. $\square$

It follows from the above theorem that the definition of compatibility in term of $\ominus$ is equivalent to the definition of compatibility in term of $\ominus$.

If an effect algebra $(L, \oplus, 0, 1)$ is a lattice under the partial order $\leq$ induced by the partial operation $\oplus$, then it is called a lattice effect algebra.

Theorem 2.21 and Theorem 2.22 will be used in the proof of Theorem 2.23, and they appear in [7] without proof.

**Theorem 2.21** Let $L$ be a lattice effect algebra. Then $\forall a, b, c \in L$, we have:

(i) If $c \leq a$, $c \leq b$, then $(a \lor b) \ominus c = (a \ominus c) \lor (b \ominus c)$.

(ii) $((a \lor b) \ominus a) \land ((a \lor b) \ominus b) = 0$.

(iii) $(a \ominus (a \land b)) \land (b \ominus (a \land b)) = 0$.

**Proof.** (i): Since $c \leq a \leq a \lor b$, then using Lemma 2.15(i), we have $a \ominus c \leq (a \lor b) \ominus c$; also $c \leq b \leq a \lor b$ implies that $b \ominus c \leq (a \lor b) \ominus c$, we conclude that
\[(a \oplus c) \lor (b \oplus c) \leq (a \lor b) \oplus c.\]

To get the result we want to prove that \((a \oplus c) \lor (b \oplus c) \geq (a \lor b) \oplus c\).

**Claim:** \(((a \oplus c) \lor (b \oplus c)) \oplus c\) exist and \((a \oplus c) \lor (b \oplus c) \geq (a \lor b) \oplus c\)

Indeed, since \(c \leq a \leq 1\). Then using Lemma 2.15(i), we have \(a \oplus c \leq 1 \ominus c = c'\); also \(c \leq b \leq 1\) implies that \(b \ominus c \leq 1 \ominus c = c'\). Hence \((a \ominus c) \lor (b \ominus c) \leq c'\); that is \(((a \ominus c) \lor (b \ominus c)) \oplus c\) exist. It is clearly \(a \ominus c \leq (a \ominus c) \lor (b \ominus c)\) implies that \((a \ominus c) \oplus c \leq ((a \ominus c) \lor (b \ominus c)) \oplus c\). Hence Theorem 2.14(i), implies that \(a \leq ((a \ominus c) \lor (b \ominus c)) \oplus c\). Similarly \(b \leq ((a \ominus c) \lor (b \ominus c)) \oplus c\); hence \(a \lor b \leq ((a \ominus c) \lor (b \ominus c)) \oplus c\).

Therefore
\[(a \lor b) \oplus c \leq (a \ominus c) \lor (b \ominus c).\]

From above claim \((a \lor b) \oplus c = (a \ominus c) \lor (b \ominus c)\).

(ii) Let \(t \in L\) be such that \(t \leq (a \lor b) \ominus a\), \(t \leq (a \lor b) \ominus b\); that is, \(a \leq (a \lor b) \ominus t\) and \(b \leq (a \lor b) \ominus t\). Hence \(a \lor b \leq (a \lor b) \ominus t\), so we get \(t \leq (a \lor b) \ominus (a \lor b) = 0\). Therefore \(t = 0\), and so \(((a \lor b) \ominus a) \land ((a \lor b) \ominus b) = 0\).

(iii) Let \(t \in L\) be such that \(t \leq a \ominus (a \land b), \ t \leq b \ominus (a \land b)\). Then \(t \ominus (a \land b) \leq a \land b \ominus (a \land b) \leq b; hence t \ominus (a \land b) \leq a \land b = 0 \ominus (a \land b)\) implies that \(t = 0\) by the cancellation law. Therefore \((a \ominus (a \land b)) \land (b \ominus (a \land b)) = 0\). □

**Theorem 2.22** Let \(L\) be a lattice effect algebra.

(i) If \(a \leq c, b \leq c\), then \(c \ominus (a \lor b) = (c \ominus a) \land (c \ominus b)\).

(ii) If \(a \leq c, b \leq c, c \ominus (a \land b) = (c \ominus a) \lor (c \ominus b)\).

(iii) If \(c \leq a, c \leq b\), then \((a \land b) \ominus c = (a \ominus c) \land (b \ominus c)\).

**Proof.** (i) Since \(a \leq c\) and \(b \leq c\), then \(a \lor b \leq c\) so that \(c \ominus (a \lor b)\) is defined. Since \(a \leq a \lor b \leq c\), then by Lemma 2.15(ii) \(c \ominus (a \lor b) \leq c \ominus a\). Also, \(b \leq a \lor b \leq c\), implies that \(c \ominus (a \lor b) \leq c \ominus b\); so that \(c \ominus (a \lor b)\) is a lower bound of the set \(\{c \ominus a, c \ominus b\}\). It remains to prove that \(c \ominus (a \lor b)\) is the greatest lower bound of the set \(\{c \ominus a, c \ominus b\}\). To this end, let \(l\) be a lower bound of the set \(\{c \ominus a, c \ominus b\}\). Then \(l \leq c \ominus a\) and \(l \leq c \ominus b\); hence \(a \leq c \ominus l\) and \(b \leq c \ominus l\), and so \(a \lor b \leq c \ominus l\). It follows that \(l \leq c \ominus (a \lor b)\), and therefore \(c \ominus (a \lor b) = (c \ominus a) \land (c \ominus b)\).

(ii) Note that, since \(a \land b \leq a \leq c\) and \(a \land b \leq b \leq c\). By Lemma 2.15(ii), we have
\[ c \triangleleft a \leq c \triangleleft (a \wedge b) \text{ and } \]

\[ c \triangleleft b \leq c \triangleleft (a \wedge b). \]

It follows that

\[ (c \triangleleft a) \lor (c \triangleleft b) \leq c \triangleleft (a \wedge b) \quad (\ast) \]

Also By Lemma 2.17(v), we have

\[ (c \triangleleft a) \lor (c \triangleleft b) \leq c \]

\[ \quad (\ast\ast) \]

Now, let \( t = (c \triangleleft a) \lor (c \triangleleft b) \). By \((\ast\ast)\), we have

\[ c \triangleleft a \leq t \leq c \]

by Lemma 2.15(ii) and Theorem 2.14(iii), we have

\[ c \triangleleft t \leq c \triangleleft (c \triangleleft a) = a. \]

Similarly \( c \triangleleft t \leq b \). Hence \( c \triangleleft t \leq a \wedge b \leq c \). Again using Lemma 2.15(ii), Theorem 2.14(iii), and \((\ast)\), we have

\[ c \triangleleft (a \wedge b) \leq c \triangleleft (c \triangleleft t) = t \leq c \triangleleft (a \wedge b). \]

Therefore \( t = c \triangleleft (a \wedge b) \).

(iii) Since \( c \leq a \wedge b \leq a \), we have \((a \wedge b) \ominus c \leq a \ominus c \) by Lemma 2.15(i). Similar \( c \leq a \wedge b \leq b \) implies that \((a \wedge b) \ominus c \leq b \ominus c \). Thus \((a \wedge b) \ominus c \) is a lower bound of the set \( \{a \ominus c, b \ominus c\} \). Let \( l \) be a lower bound of the set \( \{a \ominus c, b \ominus c\} \). Then \( l \leq a \ominus c \) and \( l \leq b \ominus c \); hence \( l \ominus c \leq a \) and \( l \ominus c \leq b \), by \((\text{ED})\) and associativity of \( \ominus \). We conclude that \( l \ominus c \leq a \wedge b \), and hence \( l \leq (a \wedge b) \ominus c \). Therefore \((a \wedge b) \ominus c = (a \ominus c) \wedge (b \ominus c) \). \( \square \)

The proof of \((i) \Rightarrow (ii)\) in the next theorem, is found in [7].

**Theorem 2.23** Let \( L \) be a lattice effect algebra. Then the following assertions are equivalent:

(i) \( a \leftrightarrow b \).

(ii) \((a \lor b) \ominus b = a \ominus (a \wedge b)\).

(iii) \( b \ominus (a \ominus (a \wedge b)) \) exists.
Proof. (i) ⇒ (ii): Assume (i) holds. Then there exist \( c, d \in L \) such that 
\[ d \leq a \leq c, \quad d \leq b \leq c \quad \text{and} \quad c \ominus a = b \ominus d. \]
From the inequalities \( a \leq a \lor b \leq c \) and 
\[ b \leq a \lor b \leq c \]
\] it follows from Lemma 2.15(i) that \((a \lor b) \ominus a \leq c \ominus a = b \ominus d \leq b\) and similarly \((a \lor b) \ominus b \leq a\). Then 
\[ a \ominus ((a \lor b) \ominus b) = ((a \lor b) \ominus ((a \lor b) \ominus a)) \ominus ((a \lor b) \ominus b) = b \ominus ((a \lor b) \ominus a) \leq b \]
by Theorem 2.21(ii). Hence Lemma 2.17(ii) implies that \( w = a \ominus ((a \lor b) \ominus b) \). Then \( w \leq a, \) and \( w \leq b, \) which implies \( w \leq a \land b \) and 
\[ a \ominus w = (a \lor b) \ominus b \quad \text{and} \quad b \ominus w = ((a \lor b) \ominus ((a \lor b) \ominus b)) \ominus w = ((a \lor b) \ominus ((a \lor b) \ominus b)) \ominus (a \ominus ((a \lor b) \ominus b)) = (a \lor b) \ominus a, \]
by Lemma 2.17(iv). By Theorem 2.22(iii), we have \((a \land b) \ominus w = (a \ominus w) \land (b \ominus w) = ((a \lor b) \ominus b) \land ((a \lor b) \ominus a) = 0, \)
by Theorem 2.21(ii). Hence Lemma 2.17(ii) implies that \( w = a \land b \), which gives 
\[ a \land b = a \ominus ((a \lor b) \ominus b); \]
that is, \((a \lor b) \ominus b = a \ominus (a \land b). \]

(ii) ⇒ (iii): This part is trivial.

(iii) ⇒ (i): Assume \( b \oplus (a \ominus (a \land b)) \) exists. Then it is clear that \( b \leq b \oplus (a \ominus (a \land b)). \) Also \( a \leq b \oplus (a \ominus (a \land b)). \) Indeed, by Theorem 2.14(i), 
\[ a = (a \land b) \oplus (a \ominus (a \land b)) \leq b \oplus (a \ominus (a \land b)). \]
Now, let \( u = b \oplus (a \ominus (a \land b)), \) \( v = a \land b. \) Then \( v \leq a \leq u, v \leq b \leq u, \) and \( u \ominus b = a \ominus v. \) Therefore \( a \leftrightarrow b. \) □

**Lemma 2.24** For elements \( a, b, c \) of an effect algebras \((L, \oplus, 0, 1)\), we have the following:

(i) If \( a \oplus b \) and \( a \lor b \) exist, then \( a \land b \) exists and
\[ a \oplus b = (a \lor b) \oplus (a \land b). \]

(ii) If \( L \) is a lattice and \( a \oplus b, a \oplus c \) are defined, then \( a \oplus (b \lor c) \) exists and 
\[ a \oplus (b \lor c) = (a \oplus b) \lor (a \oplus c). \]

**Proof.** (i) Since \( a \leq a \lor b \leq a \oplus b, \) then, by Lemma 2.15(ii), \((a \oplus b) \ominus (a \lor b) \leq (a \ominus b) \ominus a; \)
hence \((a \ominus b) \ominus (a \lor b) \leq b. \) Similarly \( b \leq a \lor b \leq a \oplus b, \) implies that \((a \ominus b) \ominus (a \lor b) \leq (a \oplus b) \ominus b; \)
hence \((a \ominus b) \ominus (a \lor b) \leq a. \) Thus \((a \ominus b) \ominus (a \lor b) \) is lower bound of the set \{\( a, b \)\}. Let \( t \) be a lower bound of \{\( a, b \)\}. Then \( t \leq a, t \leq b, \) so we get \( t \leq b \leq a \ominus b; \) hence by Lemma 2.15(ii), \((a \ominus b) \ominus b \leq (a \ominus b) \ominus t, \)
so that \( a \leq (a \ominus b) \ominus t. \) Similarly \( b \leq (a \ominus b) \ominus t. \) It
follows that \( a \lor b \leq (a \oplus b) \ominus t \), so we get \( t \leq (a \oplus b) \ominus (a \lor b) \); hence 
\((a \oplus b) \ominus (a \lor b) = a \land b \). Therefore \( a \oplus b = (a \lor b) \ominus (a \land b) \).

(ii) First, since \( a \oplus b \) and \( a \ominus c \) is defined, then \( a \leq b' \) and \( a \leq c' \); hence \( a \leq b' \land c' \), so that \( a \leq (b \lor c)' \), thus \( a \ominus (b \lor c) \) exists. Next, we have \( b \leq b \lor c \) implies that \( a \ominus b \leq a \ominus (b \lor c) \), and \( a \ominus c \leq a \ominus (b \lor c) \). Hence \( a \ominus (b \lor c) \) is an upper bound of \( \{a \ominus b, a \ominus c\} \). It remains to prove \( a \ominus (b \lor c) \) is the least upper bound of \( \{a \ominus b, a \ominus c\} \). Let \( u \) be any upper bound of \( \{a \ominus b, a \ominus c\} \). Then 
\( a \ominus b, a \ominus c \leq u \) and so \( b, c \leq u \ominus a \). Hence \( b \lor c \leq u \ominus a \), and therefore \( a \ominus (b \lor c) \leq u \). It follows that \( a \ominus (b \lor c) = (a \ominus b) \lor (a \ominus c) \). \( \square \)

**Theorem 2.25** [26] Let \((L, \ominus, 0, 1)\) be a lattice effect algebra and let \( x, y, z \in L \) be such that \( x \leftrightarrow z \) and \( y \leftrightarrow z \). Then

(i) \( x \lor y \leftrightarrow z \);

(ii) if \( x \leq y \), then \( y \ominus x \leftrightarrow z \);

(iii) \( x' = 1 \ominus x \leftrightarrow z \);

(iv) \( x \land y \leftrightarrow z \);

(v) \( x \oplus y \leftrightarrow z \).

**Proof.** By assumptions and Theorem 2.23, \( x \ominus (z \ominus (x \land z)) \) and \( y \ominus (z \ominus (y \land z)) \) exist.

(i) Note that \( x \perp (z \ominus ((x \lor y) \land z)) \). Indeed, \( x \land z \leq (x \lor y) \land z \leq z \), so by Lemma 2.15(ii), we get \( z \ominus ((x \lor y) \land z) \leq z \ominus (x \land z) \). But since \( x \ominus (z \ominus (x \land z)) \) exists, then we have \( (z \ominus (x \land z)) \leq x' \); that is \( z \ominus ((x \lor y) \land z) \leq x' \); hence \( x \ominus (z \ominus ((x \lor y) \land z)) \) exists. Similarly \( y \ominus (z \ominus ((x \lor y) \land z)) \) exists. Now \( (x \ominus (z \ominus ((x \lor y) \land z))) \lor (y \ominus (z \ominus ((x \lor y) \land z))) \) exists, since \( L \) is a lattice, so the last term equals \( (x \lor y) \ominus (z \ominus ((x \lor y) \land z)) \) by Lemma 2.24(ii). Therefore \( x \lor y \leftrightarrow z \) by Theorem 2.23.

(ii) if \( x \leq y \), then \( x \land z \leq y \land z \) and \( x \lor z \leq y \lor z \). It follows that there exists \( w \in L \) such that

\[
(x \land z) \ominus w = y \land z
\]

(*)
and by Theorem 2.23, since \( x \leftrightarrow z \), we get \( x \oplus (z \ominus (x \land z)) = x \lor z \leq y \lor z = y \oplus (z \ominus (y \land z)) = (y \land z) \oplus (y \ominus (y \land z)) \oplus (z \ominus (y \land z)) \), by Theorem 2.14(i). Thus

\[
(x \land z) \oplus (x \ominus (x \land z)) \oplus (z \ominus (x \land z)) = x \oplus (z \ominus (x \land z)) = x \lor z \leq y \lor z = y \oplus (z \ominus (y \land z)) = (y \land z) \oplus (y \ominus (y \land z)) \oplus (z \ominus (y \land z)).
\]

Hence by the cancellation law we have

\[
(x \ominus (x \land z)) \leq (y \ominus (y \land z))
\]
since \( (x \land z) \oplus (z \ominus (x \land z)) = (y \land z) \oplus (z \ominus (y \land z)) \). The last inequality implies that there is \( e \in L \) such that \( (x \land z) \oplus e = y \ominus (y \land z) \). This and (*) yield that

\[
y = (x \ominus (x \land z)) \oplus e \oplus (y \land z) = w \oplus e \oplus x = w \ominus e \oplus (x \land z) \ominus (x \ominus (x \land z)).
\]

Since \( y \oplus (z \ominus (y \land z)) \) exists, we have

\[
y \oplus (z \ominus (y \land z)) = w \oplus e \oplus (x \land z) \ominus (x \ominus (x \land z)) \ominus (z \ominus (y \land z)). \quad (**)
\]

Now

\[
y \ominus x = w \ominus e
\]

and

\[
z = (y \land z) \oplus (z \ominus (y \land z)) = w \oplus (x \land z) \ominus (z \ominus (y \land z)).
\]

Thus we conclude that \( y \ominus x \leftrightarrow z \), since \( w \ominus e \ominus (x \land z) \ominus (z \ominus (y \land z)) \) exists by (**).

(iii) \( 1 \leftrightarrow z \). Indeed, \( 1 = z \ominus z', z = z \ominus 0 \) and \( z \ominus z' \ominus 0 \) exist. Since \( x \leftrightarrow z \) and \( x \leq 1 \), then by part (ii), \( x' = 1 \ominus x \leftrightarrow z \).

(iv) By (iii), \( x' \leftrightarrow z \) and \( y' \leftrightarrow z \), which by (i), implies that \( x' \lor y' \leftrightarrow z \). Hence, by (iii), we have

\[
x \land y = (x' \lor y')' \leftrightarrow z.
\]

(v) Assume that \( x \oplus y \) exists, then \( x \oplus y = (x' \ominus y)' \). By (iii), we have \( x' \leftrightarrow z \), and, by (ii), since \( y \leq x' \) and \( y \leftrightarrow z \), we have \( x' \ominus y \leftrightarrow z \). So, by (iii), we have

\[
x \oplus y = (x' \ominus y)' \leftrightarrow z. \quad \square
\]

**Theorem 2.26** [17] Let \( L \) be a lattice effect algebra. Assume \( b \in L \) and \( A \subseteq L \) are such that \( \lor A \) exist in \( L \) and \( b \leftrightarrow a \forall a \in A \). Then
(i) \( b \leftrightarrow \bigvee A \).

(ii) \( \bigvee \{b \land a : a \in A\} \) exist in \( L \) and equal \( b \land (\bigvee A) \).

**Proof.** (i) Since \( b \leftrightarrow a \forall a \in A \), then \( a \oplus (b \oplus (a \land b)) \) exist \( \forall a \in A \) by Theorem 2.23. Then for every \( a \in A \)

\[
a \leq (b \oplus (a \land b))'.
\]

Since \( a \land b \leq (\bigvee A) \land b \leq b \), then by Lemma 2.15(ii), we have \( b \oplus ((\bigvee A) \land b) \leq b \oplus (a \land b) \), implies that,

\[
(b \oplus (a \land b))' \leq (b \oplus ((\bigvee A) \land b))'
\]

for every \( a \in A \). From (*) and (**) we have

\[
a \leq (b \oplus (a \land b))' \leq (b \oplus ((\bigvee A) \land b))'
\]

for every \( a \in A \); hence \( \bigvee A \leq (b \oplus ((\bigvee A) \land b))' \); that is (\( \bigvee A \oplus (b \oplus ((\bigvee A) \land b)) \)) exist. Therefore \( \bigvee A \leftrightarrow b \) by Theorem 2.23.

(ii) **Claim:** \( \land \{b \ominus (a \land b) : a \in A\} = b \ominus (b \land (\bigvee A)). \)

First \( b \ominus (b \land (\bigvee A)) \) is a lower bound of \( \{b \ominus (a \land b) : a \in A\} \). Indeed, \( a \land b \leq b \land (\bigvee A) \leq b \forall a \in A \), then by Lemma 2.15(ii) we have

\[
b \ominus (b \land (\bigvee A)) \leq b \ominus (a \land b)
\]

for every \( a \in A \); that is \( b \ominus (b \land (\bigvee A)) \) is lower bound of \( \{b \ominus (a \land b) : a \in A\} \). It remains to prove that \( b \ominus (b \land (\bigvee A)) \) is the greatest lower bound of \( \{b \ominus (a \land b) : a \in A\} \). Let \( d \) be a lower bound of \( \{b \ominus (a \land b) : a \in A\} \), since \( b \rightarrow a \forall a \in A \), then \( (a \lor b) \cdot a = b \ominus (a \land b) \forall a \in A \) by Theorem 2.23. Now \( a \leq a \lor b \leq b \lor (\bigvee A) \), then by Lemma 2.15(i), \( (a \lor b) \cdot a \leq (b \lor (\bigvee A)) \cdot a \), but \( (a \lor b) \cdot a = b \ominus (a \land b) \), so we get \( b \ominus (a \land b) \leq (b \lor (\bigvee A)) \cdot a \forall a \in A \), so that

\[
d \leq (b \lor (\bigvee A)) \cdot a
\]

then we have \( a \leq (b \lor (\bigvee A)) \cdot d \forall a \in A \); hence \( \bigvee A \leq (b \lor (\bigvee A)) \cdot d \) so we have

\[
d \leq (b \lor (\bigvee A)) \cdot \bigvee A
\]
By part(i), $b \leftrightarrow \bigvee A$, then by Theorem 2.23

$$(b \vee (\bigvee A)) \ominus A = b \ominus (b \land (\bigvee A))$$

which implies that

$$d \leq b \ominus (b \land (\bigvee A)).$$

Therefore $\bigwedge\{b \ominus (a \land b) : a \in A\} = b \ominus (b \land (\bigvee A))$ which is complete proof of Claim. Since $a \leq \bigvee A \forall a \in A$, implies that $b \land a \leq b \land (\bigvee A) \forall a \in A$; that is $b \land (\bigvee A)$ is an upper bound of $\{a \land b : a \in A\}$. Let $e$ be an upper bound of $\{a \land b : a \in A\}$, then $a \land b \leq e \forall a \in A$, and so $a \land b \leq e \land b \leq b$; hence by Lemma 2.15(ii), we have $b \ominus (e \land b) \leq b \ominus (a \land b) \forall a \in A$, implies that

$$b \ominus (e \land b) \leq \bigwedge\{b \ominus (a \land b) : a \in A\}$$

and so by Claim we have $b \ominus (e \land b) \leq b \ominus (b \land (\bigvee A))$. Thus $b \land (\bigvee A) \leq e \land b \leq e$. Therefore $b \land (\bigvee A) = \bigvee\{b \land a : a \in A\}$. □

**Corollary 2.27** Let $L$ be a lattice effect algebra. Assume $b \leftrightarrow a_1$ and $b \leftrightarrow a_2$, then $b \land (a_1 \lor a_2) = (b \land a_1) \lor (b \land a_2)$.

### 2.3 $\sigma$-Orthocomplete Effect Algebras

Let $F = \{a_1, a_2, ..., a_n\}$ be a finite sequence in $L$. Recursively, we define for $n \geq 3$

$$a_1 \oplus a_2 \oplus ... \oplus a_n := (a_1 \oplus ... \oplus a_{n-1}) \oplus a_n. \quad (2.1)$$

Supposing that $a_1 \oplus ... \oplus a_{n-1}$ and $(a_1 \oplus ... \oplus a_{n-1}) \oplus a_n$ exist in $L$. From the associativity of $\oplus$ in an effect algebra, we conclude that (2.1) is correctly defined. By definition, we put $a_1 \oplus ... \oplus a_n = a_1$ if $n = 1$, $a_1 \oplus ... \oplus a_n = 0$ if $n = 0$. Then for any permutation $(i_1, ..., i_n)$ of $\{1, 2, ..., n\}$ and any $k$ with $1 \leq k \leq n$, we have

$$a_1 \oplus a_2 \oplus ... \oplus a_n = a_{i_1} \oplus ... \oplus a_{i_n}, \quad (2.2)$$

$$a_1 \oplus a_2 \oplus ... \oplus a_n = (a_1 \oplus a_2 \oplus ... \oplus a_k) \oplus (a_{k+1} \oplus a_{k+2} \oplus ... \oplus a_n). \quad (2.3)$$

We say that a finite sequence $F = \{a_1, ..., a_n\}$ in $L$ is $\oplus$-orthogonal if $a_1 \oplus$
... $\oplus a_n$ exists in $L$. In this case we say that $F$ has a $\oplus$-sum, $\bigoplus_{i=1}^{n} a_i$, defined via

$$\bigoplus_{i=1}^{n} a_i = a_1 \oplus a_2 \oplus ... \oplus a_n. \quad (2.4)$$

It is clear that two elements $a, b$ of $L$ are orthogonal; i.e., $a \perp b$, iff $\{a, b\}$ is $\oplus$-orthogonal.

An arbitrary system $G = \{a_i\}_{i \in I}$ of not necessarily different elements of $L$ is $\oplus$-orthogonal iff, for every finite subset $F$ of $I$, the system $\{a_i\}_{i \in F}$ is $\oplus$-orthogonal. If $G = \{a_i\}_{i \in I}$ is $\oplus$-orthogonal, so is any $\{a_i\}_{i \in J}$ for any $J \subseteq I$. An $\oplus$-orthogonal system $G = \{a_i\}_{i \in I}$ of $L$ has a $\oplus$-sum in $L$, written as $\bigoplus_{i \in I} a_i$, iff in $L$ there exists the join

$$\bigoplus_{i \in I} a_i := \bigvee_{F \in F(\mathbb{N})} \bigoplus_{i \in F} a_i \quad (2.5)$$

where $F$ runs over all finite subsets in $I$. In this case, we also write $\bigoplus G =$ $\bigoplus_{i \in I} a_i$. It is evident that if $G = \{a_1, ..., a_n\}$ is $\oplus$-orthogonal, then the $\oplus$-sums defined by (2.4) and (2.5) coincide.

We say that an effect algebra $L$ is $\sigma$-orthocomplete iff $\bigoplus_{i \in I} a_i$ belongs to $L$ for any countable system $\{a_i : i \in I\}$ of $\oplus$-orthogonal elements from $L$. A $\sigma$-orthocomplete effect algebra is also called a $\sigma$-effect algebra. An OMP $L$ is called $\sigma$-orthocomplete if every countable orthogonal subset of $L$ has supremum in $L$, we also call $L$ a $\sigma$-OMP.

**Notation:** For any set $X$, we let

$$\mathcal{F}(X) := \{F : F \text{ is a finite subset of } X \}. \quad 29$$

**Lemma 2.28** [14] Let $P$ be an effect algebra. If $P$ is an OMP and $\{x_1, ..., x_n\}$ $\subseteq P$ is pairwise orthogonal, then $x_1 \oplus ... \oplus x_n$ is defined, $x_1 \vee ... \vee x_n$ exists and

$$x_1 \vee ... \vee x_n = x_1 \oplus ... \oplus x_n.$$
Proof. We proceed by induction on \( n \). Since \( P \) is an OMP, \( x_1 \perp x_2 \Rightarrow x_1 \lor x_2 = x_1 \oplus x_2 \). Assume \( n > 1 \), \( x_1 \oplus \ldots \oplus x_{n-1} \) is defined, \( x_1 \lor \ldots \lor x_{n-1} \) exists and

\[
x_1 \lor \ldots \lor x_{n-1} = x_1 \oplus \ldots \oplus x_{n-1}.
\]

Since \( x_i \perp x_n \) \( \forall i \in \{1, \ldots, n-1\} \), we have \( x_i \leq x'_n \) \( \forall i \in \{1, \ldots, n-1\} \Rightarrow x_1 \lor \ldots \lor x_{n-1} \leq x'_n \Rightarrow (x_1 \oplus \ldots \oplus x_{n-1}) \perp x_n \). Hence \( x_1 \oplus \ldots \oplus x_n \in P \), \( (x_1 \oplus \ldots \oplus x_n) \lor x_n \) exists and

\[
x_1 \oplus \ldots \oplus x_n = (x_1 \oplus \ldots \oplus x_{n-1}) \lor x_n = x_1 \lor \ldots \lor x_{n-1} \lor x_n. \tag*{□}
\]

Theorem 2.29 [14] An OMP is \( \sigma \)-orthocomplete iff it is \( \sigma \)-orthocomplete as an effect algebra.

Proof. \((\Rightarrow)\): Assume that \( P \) is a \( \sigma \)-orthocomplete OMP. Let \( X \) be a countable \( \oplus \)-orthogonal subset of \( P \). Then \( \bigvee X \) exists, \( \bigvee F = \bigoplus F \) exists \( \forall F \in \mathcal{F}(X) \), and \( \bigvee X \) is an upper bound for \( \{ \bigvee F : F \in \mathcal{F}(X) \} \). Let \( u \in P \) be such that \( \bigvee F \leq u \) \( \forall F \in \mathcal{F}(X) \). Then in particular, \( x \leq u \) \( \forall x \in X \) and so \( \bigvee X \leq u \). Thus \( \bigoplus X = \bigvee_{F \in \mathcal{F}(X)} \bigoplus F = \bigvee_{F \in \mathcal{F}(X)} \bigvee F = \bigvee X \) exists in \( P \). Therefore \( P \) is \( \sigma \)-orthocomplete as an effect algebra.

\((\Leftarrow)\): Assume that \( P \) is a \( \sigma \)-orthocomplete as an effect algebra. Let \( X \) be a countable set of pairwise orthogonal elements in \( P \). Then by Lemma 2.28 \( \bigoplus F = \bigvee F \) \( \forall F \in \mathcal{F}(X) \); that is each finite subset of \( X \) is \( \oplus \)-orthogonal, so \( \bigoplus X \) exists in \( P \) and equals \( \bigvee_{F \in \mathcal{F}(X)} \bigoplus F \). So we have

\[
\bigoplus X = \bigvee_{F \in \mathcal{F}(X)} \bigoplus F = \bigvee_{F \in \mathcal{F}(X)} \bigvee F = \bigvee X
\]

exists in \( P \). Therefore \( P \) is \( \sigma \)-orthocomplete OMP. \( \tag*{□} \)

Definition 2.30 [28] A finite set \( D \subseteq L \) is called a difference set if either \( D \) is empty or there exists a strictly increasing sequence

\[
p_0 \leq p_1 \leq \ldots \leq p_{n-1} \leq p_n
\]

in \( L \) such that

\[
D = \{ p_i \ominus p_{i-1} : i = 1, 2, \ldots, n \}.
\]
In this case, \(n\) equals the cardinality of \(D\).

Singleton sets of nonzero elements of \(L\) are difference sets. If \(p, \ q\) are orthogonal pair of nonzero elements of \(L\), then the set \(\{p, \ q\}\) is a difference set.

**Definition 2.31** [28] Let \(D\) be the difference set corresponding to the strictly increasing sequence \(p_0 \leq \ldots \leq p_n\) in \(L\). We define

\[ \oplus D := p_n \ominus p_0. \]

If \(D\) is empty, we set \(\oplus D := 0\).

**Lemma 2.32** [28] Any nonempty difference set is an orthogonal set of nonzero elements.

**Proof.** Let \(D\) be a nonempty difference set corresponding to the strictly increasing sequence \((p_i)_{i=0}^n, \ n \geq 1\). Let \(c, \ d\) be elements in \(D\), say, \(c = p_i \ominus p_{i-1}\) for some \(i\), where \(1 \leq i \leq n\), and \(d = p_k \ominus p_{k-1}\) for some \(1 \leq k \leq n\). We may suppose without loss of generality that \(i < k\). Then \(p_{i-1} < p_i \leq k \leq p_{k-1} < p_k\). Since \(p_{i-1} < p_i\), and \(p_{k-1} < p_k\), then \(0 \neq p_i \ominus p_{i-1} = c\) and \(0 \neq p_k \ominus p_{k-1} = d\). Now we can get \((p_{k-1} \ominus p_i) \oplus (p_i \ominus p_{i-1}) = p_{k-1} \ominus p_{i-1}\) and \((p_k \ominus p_{k-1}) \oplus (p_{k-1} \ominus p_{i-1}) = p_k \ominus p_{i-1}\). Then \((p_k \ominus p_{k-1}) \oplus [(p_{k-1} \ominus p_i) \oplus (p_i \ominus p_{i-1})] = p_k \ominus p_{i-1}\); hence, by axiom(EA2), \((p_k \ominus p_{k-1}) \oplus (p_i \ominus p_{i-1})\) is defined. Therefore \((p_k \ominus p_{k-1}) \perp (p_i \ominus p_{i-1}); \ i.e., c \perp d\). □

**Theorem 2.33** [15] Every countable chain in a \(\sigma\)-effect algebra \(L\) has a supremum

**Proof.** Let \(C\) be a countable chain in \(L\). Consider the set

\[ D := \{b \ominus a : a, \ b \in C \cup \{0\}, \ a \leq b\}. \]

Since \(\forall a \in C, \ a = a \ominus 0\), it follows that \(C \subseteq D\).

**Claim:** \(D\) is a \(\ominus\)-orthogonal set in \(L\).

Indeed, let \(\{d_1, \ d_2, \ldots, \ d_n\} \subseteq D\). Then \(d_i = b_i \ominus a_i\), where \(a_i \leq b_i\) and \(a_i, \ b_i \in C \cup \{0\}\). Therefore, there exists a set \(\{c_1, \ c_2, \ldots, \ c_{2n}\}\) such that \(\{a_1, a_2, \ldots, a_n, \ b_1, b_2, \ldots, b_n\} = \{c_1, c_2, \ldots, c_{2n}\}\), and \(c_1 \leq c_2 \leq \ldots \leq c_{2n}\). Put \(e_i := c_i \ominus c_{i-1}, \ i = 1, 2, \ldots, 2n\) where \(c_0 := 0\). Then, as we proved in Lemma
2.32, we get $e_i \perp e_j$ for all $i, j$ with $1 \leq i, j \leq 2n$ and $i \neq j$. Also for any $k$ with $1 \leq k \leq 2n$, we have

$$c_k = e_1 \oplus e_2 \oplus \ldots \oplus e_k \in L,$$

and, if $1 \leq j \leq k \leq 2n$, then

$$c_k \ominus c_j = e_{j+1} \oplus e_{j+2} \oplus \ldots \oplus e_k \in L. \quad (*)$$

Thus, using $(*)$, for any $d_i = b_i \ominus a_i$, $i = 1, 2, \ldots, n$, there exists a finite subset $F_i$ of $\{1, \ldots, 2n\}$ such that $d_i = \bigoplus_{j \in F_i} e_j$, and therefore $\bigoplus_{i=1}^{n} d_i = \bigoplus_{k \in E} e_k \in L$ for some $E \subseteq \{1, \ldots, 2n\}$. This completes the proof of the claim.

Since $L$ is a $\sigma$-effect algebra, $a_0 := \bigoplus D$ exists in $L$. Because $C \subseteq D$, we have, for any $a \in C$, $a \leq a_0$. Now, let $u$ be an upper bound of $C$. Then, from the proof of above claim, we conclude

$$\bigoplus_{i=1}^{n} d_i = \bigoplus_{k \in E} e_k \leq \bigoplus_{i=1}^{2n} e_i = c_{2n} \leq u.$$ 

This means that $\oplus F \leq u \ \forall F \in \mathcal{F}(D)$, and so we have

$$a_0 = \bigoplus D = \bigvee_{F \in \mathcal{F}(D)} \bigoplus F \leq u.$$ 

Thus, $a_0 \leq u$ and hence $\bigvee C = a_0$. \hfill \Box

**Theorem 2.34** [15] Let $L$ be an effect algebra. The following statements are equivalent:

(i) $L$ is a $\sigma$-effect algebra.

(ii) Every increasing sequence in $L$ has a supremum in $L$.

**Proof.** (i) $\Rightarrow$ (ii): This part is a consequence of Theorem 2.33.

(ii) $\Rightarrow$ (i): Let $\{x_i\}_{i \in \omega}$ be a countable set of $\oplus$-orthogonal elements in $L$, where $\omega = \mathbb{N} \cup \{0\}$. Set $s_n := \bigoplus_{i=0}^{n} x_i$ ($n = 0, 1, \ldots$). Evidently $(s_n)_{n \in \omega}$ is increasing; so, $\bigvee_{n \in \omega} s_n$ exists. We claim that $\bigvee_{F} \bigoplus_{x_i \in F} x_i$ where $F$ runs over all finite subsets of $\omega$ exists and equals $\bigvee_{n \in \omega} s_n$. Indeed, notice first that
$F \in \mathcal{F}(\{x_i\}_{i \in \omega}) \Rightarrow F \subseteq \{x_1, x_2, \ldots, x_n\}$ for some $n \in \omega \Rightarrow \bigoplus F \leq \bigoplus\{x_1, x_2, \ldots, x_n\}$ for some $n \in \omega \Rightarrow \bigoplus F \leq \bigvee_{n \in \omega} s_n$. This shows that $\bigvee_{n \in \omega} s_n$ is an upper bound for $\{\bigoplus F : F \in \mathcal{F}(\{x_i\}_{i \in \omega})\}$. Second, we show that $\bigvee_{n \in \omega} s_n$ is the least among all such upper bounds. To this end, let $u \in L$ be such that $\bigoplus F \leq u \forall F \in \mathcal{F}(\{x_i\}_{i \in \omega})$. Then in particular, we have $s_n \leq u \forall n \in \omega$. Hence $\bigvee_{n \in \omega} s_n \leq u$, and the claim is proved. $\square$
Chapter 3
The Relation Between Observables and Spectral Resolution

If the logic \( L \) is taken to be the projection lattice of a complex Hilbert space \( \mathcal{H} \), then via the spectral theorem, the set of observables can be identified with the self-adjoint operators on \( \mathcal{H} \). In 1967, D. Catlin classified the spectra of an observable \( A \) by considering the character of \((A - \lambda)^{-1}\) on a \( \sigma \)-OMP without the aid of Hilbert space formalism [4]. Also, he showed that, there is a one to one correspondence between observables and spectral resolutions of the identity which is similar to the spectral family concept in operator theory [19, p. 492]. Our work will concentrate on extending the above-mentioned results to a \( \sigma \)-effect algebra, a generalization of a \( \sigma \)-OMP. Throughout this chapter, \((L, \oplus, 0, 1)\) is assumed to be a \( \sigma \)-effect algebra.

3.1 The Spectrum of an Observable

In this section we will show that the classification of spectra of an observable can be obtained on any \( \sigma \)-effect algebra.

**Definition 3.1 [9]** Let \( L \) be a \( \sigma \)-effect algebra. By an observable on \( L \) we mean any mapping \( x : \mathcal{B}(\mathbb{R}) \to L \) such that:

(i) \( x(\mathbb{R}) = 1 \).

(ii) \( x\left(\bigcup_{i=1}^{\infty} E_i\right) = \bigoplus_{i=1}^{\infty} x(E_i) \) whenever \( E_i \cap E_j = \emptyset \) for \( i \neq j \), \( E_i \in \mathcal{B}(\mathbb{R}) \).
We denote the set of all observables on $L$ by $\mathcal{O}$.

**Definition 3.2** [9] A real-valued mapping $\alpha$ on an effect algebra $L$ is said to be a state if

(i) $\alpha(1) = 1$, and

(ii) $\alpha(a \oplus b) = \alpha(a) + \alpha(b), a, b \in L$.

It is clear that $\alpha(0) = 0$. If for a state $\alpha : L \to [0, 1]$ we have

$$\alpha(\bigoplus_{i=1}^{\infty} a_i) = \sum_{i=1}^{\infty} \alpha(a_i)$$

whenever $\bigoplus_{i=1}^{\infty} a_i$ exists in $L$, then $\alpha$ is said to be a $\sigma$-additive state.

**Remark 3.3**

(i) If $A, B \in \mathcal{B}(\mathbb{R})$ and $A \subseteq B$, then $x(A) \leq x(B)$. Indeed, we have $B = A \cup (B - A)$ implies $x(B) = x(A) \oplus x(B - A)$; hence $x(A) \leq x(B)$.

(ii) If $A \subseteq B$, then $x(B - A) = x(B) \ominus x(A)$. Indeed, from (i) we have $x(B) = x(A) \oplus x(B - A)$. Hence $x(B) \ominus x(A) = x(B - A)$.

The following theorem is a generalization of Theorem 2.32 in [3] to $\sigma$-effect algebras.

**Theorem 3.4** Let $\alpha, \beta$ be states on $L$. If $\alpha \circ x = \beta \circ x$ for every $x \in \mathcal{O}$, then $\alpha = \beta$.

**Proof.** We first show that any element in $L$ can be expressed as $x(E)$ for some $x \in \mathcal{O}$ and some $E \in \mathcal{B}$. Let $e \in L$. Define $q_e : \mathcal{B} \to L$ by,

$$q_e(E) := \begin{cases} 
0 & \text{if } 1, 2 \notin E \\
e & \text{if } 1 \in E, 2 \notin E \\
e' & \text{if } 1 \notin E, 2 \in E \\
1 & \text{if } 1, 2 \in E.
\end{cases}$$

It is easy to check that $q_e \in \mathcal{O}$ and note that $q_e(\{1\}) = e$. Suppose $\alpha \circ x = \beta \circ x$ $\forall x \in \mathcal{O}$ and $\forall E \in \mathcal{B}$. We must show that $\alpha = \beta$. We have, then, that $\alpha(x(E)) = \beta(x(E)) \forall x \in \mathcal{O}$ and $\forall E \in \mathcal{B}$. But from the above, this is equivalent to saying $\alpha(e) = \beta(e) \forall e \in L$. So by definition of a
Definition 3.5 [9, 21] The spectrum $s(x)$ of an observable $x : \mathcal{B} \to L$ is the smallest closed subset $F$ of $\mathbb{R}$ such that $x(F) = 1$.

Recall that a space is said to satisfy the second axiom of countability if there is a countable base for its topology (see [27, page 177]).

The following Theorem is a generalization of 4.1.12 in [21] to $\sigma$-effect algebras.

Theorem 3.6 Every observable $x$ on $L$ has a spectrum.

Proof. Assume that $T = \{ F_a : a \in I \}$ is the collection of all closed subsets of $\mathbb{R}$ such that $x(F_a) = 1$. Then $T \neq \emptyset$ since $\mathbb{R} \in T$. Put $F = \bigcap_{a \in I} F_a$. Then the set $\{ \mathbb{R} \setminus F_a : a \in I \}$ is an open covering of the second countable space $\mathbb{R} \setminus F$ and, therefore, there is a countable set $\{ F_n : n \in \mathbb{N} \} \subseteq \{ F_a : a \in I \}$ such that $\bigcup_{n \in \mathbb{N}} (\mathbb{R} \setminus F_n) = \mathbb{R} \setminus F$. But we can find $(B_i)_{i \in \mathbb{N}}$ in $\mathbb{R}$ such that $B_i \cap B_j = \emptyset$ for $i \neq j$ and $B_i \subseteq \mathbb{R} \setminus F_i$ such that $\bigcup_{i \in \mathbb{N}} B_i = \bigcup_{i \in \mathbb{N}} \mathbb{R} \setminus F_i$. Hence $x(\mathbb{R} \setminus F) = x(\bigcup_{i \in \mathbb{N}} (\mathbb{R} \setminus F_n)) = x(\bigcup_{i \in \mathbb{N}} B_i) = \bigoplus_{i \in \mathbb{N}} x(B_i) = 0$. Therefore $x(F) = 1$. \hfill $\square$

Definition 3.7 [4] The point spectrum of an observable $x$ is the set

$$p(x) := \{ \lambda \in \mathbb{R} : x(\{ \lambda \}) \neq 0 \}.$$  

We can see that $p(x) \subseteq s(x)$. Indeed, if $\lambda \in p(x)$, then $x(\{ \lambda \}) \neq 0$. Now, let $F$ be a closed subset of $\mathbb{R}$ such that $x(F) = 1$. Then $x(\mathbb{R} \setminus F) = 0$; hence $\lambda \notin \mathbb{R} \setminus F$. Otherwise $\lambda \in \mathbb{R} \setminus F$ implies $\{ \lambda \} \subseteq \mathbb{R} \setminus F$; hence

$$0 \leq x(\{ \lambda \}) \leq x(\mathbb{R} \setminus F) = 0,$$

which is a contradiction. We conclude that $\lambda \in F$. Therefore $\lambda \in \bigcap \{ F : F \text{ is a closed subset of } \mathbb{R} \text{ and } x(F) = 1 \} = s(x)$.

The continuous spectrum of an observable $x$ is the set

$$c(x) := s(x) \setminus p(x).$$
All of the following theorems is a generalization of the results that appears in [4] to $\sigma$-effect algebra; the proofs we present here is slightly different from the proofs of the same results in the $\sigma$-OMP.

**Theorem 3.8** For an observable $x$, $\lambda \in s(x) \iff x(\lambda - \epsilon, \lambda + \epsilon) \neq 0 \forall \epsilon > 0$.

**Proof.** ($\Rightarrow$): Assume $\lambda \in s(x)$. If there exist $\epsilon > 0$, such that $x(\lambda - \epsilon, \lambda + \epsilon) = 0$, then $\lambda$ does not belong to the closed set $F := \mathbb{R} \setminus (\lambda - \epsilon, \lambda + \epsilon)$, where $x(F) = 1$, which is a contradiction.

($\Leftarrow$): Assume that $x(\lambda - \epsilon, \lambda + \epsilon) \neq 0 \forall \epsilon > 0$. Let $F$ be a closed set, such that $x(F) = 1$. Then $x(\mathbb{R} \setminus F) = 0$, hence $\lambda \notin \mathbb{R} \setminus F$. Otherwise, $\lambda \in \mathbb{R} \setminus F = \bigcup_{k=1}^{\infty} I_k$, where the $I_k$ are disjoint open intervals. Hence $\lambda \in I_k$ for some $k \in \mathbb{N}$, so we can find an $\epsilon > 0$ such that $(\lambda - \epsilon, \lambda + \epsilon) \subseteq I_k$, so we have $0 \leq x(\lambda - \epsilon, \lambda + \epsilon) \leq x(I_k) = 0$, which is a contradiction. $\square$

Let $f$ be a real valued function whose domain, $\text{dom} f$, is a subset of the reals. We say that $f$ is a *Borel function* providing $\text{dom} f \in \mathcal{B}$ and for each $G \in \mathcal{B}$, $f^{-1}(G) \in \mathcal{B}$. Clearly any Borel function $f$ can be extended to a Borel function $\hat{f}$ where $\text{dom} \hat{f} = \mathbb{R}$. (Just define $\hat{f}(x) = 0$ for $x \in \mathbb{R} \setminus \text{dom} f$.) If $f$ is a Borel function with $\text{dom} f = \mathbb{R}$ and if $x \in \mathcal{O}$, we define $f(x)$ to be the observable $x \circ f^{-1}$. If $\text{dom} f \neq \mathbb{R}$ we say that $f(x)$ *exists* or is defined providing it is the case that for every pair of Borel extensions $f_1$, $f_2$ of $f$ with $\text{dom}(f_1) = \text{dom} (f_2) = \mathbb{R}$ we have $f_1(x) = f_2(x)$. If $f(x)$ exists, we define $f(x)$ to be $\hat{f}(x)$ for any extension $\hat{f}$ of $f$ with $\text{dom}(\hat{f}) = \mathbb{R}$.

**Theorem 3.9** Let $x \in \mathcal{O}$ and let $f$ be a Borel function. Then $f(x)$ exists if and only if $x(\text{dom} f) = 1$.

**Proof.** If $f(x)$ exists, let $\lambda_2 \in f(\text{dom} f)$ and let $\lambda_1 \neq \lambda_2$. Define

$$f_i(\lambda) := \begin{cases} f(\lambda) & \text{if } \lambda \in \text{dom} f \\ \lambda_i & \text{if } \lambda \notin \mathbb{R} \setminus \text{dom} f \end{cases}$$
$i = 1, 2$. Each $f_i$ is a Borel extension of $f$. Thus

$$x \circ f_1^{-1}(\{\lambda_2\}) = x \circ f_2^{-1}(\{\lambda_2\}),$$

and so

$$x(f_1^{-1}(\{\lambda_2\})) = x(f_2^{-1}(\{\lambda_2\})).$$

But

$$f_1^{-1}(\{\lambda_2\}) = f^{-1}(\{\lambda_2\}) \text{ and } f_2^{-1}(\{\lambda_2\}) = (\mathbb{R} \setminus \text{dom } f) \cup f^{-1}(\{\lambda_2\}).$$

Thus

$$x(f^{-1}(\{\lambda_2\})) = x(\mathbb{R} \setminus \text{dom } f) \oplus x(f^{-1}(\{\lambda_2\})).$$

Whence

$$x(\mathbb{R} \setminus \text{dom } f) = 0.$$ 

Therefore

$$x(\text{dom } f) = 1.$$ 

Conversely, suppose that $x(\text{dom } f) = 1$ so that $x(\mathbb{R} \setminus \text{dom } f) = 0$. Let $f_1, f_2$ be Borel extensions of $f$. Then

$$f_1^{-1}(E) \cap \text{dom } f = f_2^{-1}(E) \cap \text{dom } f$$

for all $E \in \mathcal{B}$. Now $\forall E \in \mathcal{B}$, we have

$$f_i(x)(E) = x(f_i^{-1}(E))$$

$$= x(f_i^{-1}(E) \cap (\mathbb{R} \setminus \text{dom } f)) \oplus x(f_i^{-1}(E) \cap \text{dom } f)$$

Thus

$$f_1(x) = f_2(x). \quad \Box$$

**Theorem 3.10** Let $x \in \mathcal{O}$ and suppose that for a Borel function $f$, $f(x)$ is defined.

(i) $x(\text{dom } f \cap s(x)) = 1$.

(ii) $s(x) \subseteq \overline{\text{dom } f}$.

(iii) $f(x) = x \circ f^{-1}$.

(iv) If $g(f(x))$ is defined, then so is $(g \circ f)(x)$ and $g(f(x)) = (g \circ f)(x)$.

**Proof.** (i)

$$1 = x(\text{dom } f \cup s(x))$$

$$= x(\text{dom } f \setminus s(x)) \oplus x(\text{dom } f \cap s(x)) \oplus x(s(x) \setminus \text{dom } f)$$

$$= x(\text{dom } f \cap s(x)).$$
(ii) Since $x(\text{dom} f) = 1, x(\overline{\text{dom} f}) = 1$. Now by definition of $s(x)$, we get $s(x) \subseteq \overline{\text{dom} f}$.

(iii) Let $\hat{f}$ be any extension of $f$. Then for all $E \in \mathcal{B}$, we have

$$f(x)(E) = \hat{f}(x)(E) = x(f^{-1}(E)) = x(f^{-1}(E) \cap \text{dom} f) \oplus x(f^{-1}(E) \cap (\mathbb{R}\setminus\text{dom} f)) = x(f^{-1}(E) \cap \text{dom} f) = x(f^{-1}(E)).$$

(iv) This follows at once from (iii). □

**Lemma 3.11** Let $x \in \mathcal{O}$, and let $f$ be a Borel function such that $f(x)$ is defined. Then

$$s(f(x)) \subseteq \overline{f(s(x))}.$$

**Proof.** $f(x)(\overline{f(s(x))}) = x(f^{-1}\overline{f(s(x))}) \geq x(\text{dom} f \cap s(x)) = 1$. Hence $f(x)(\overline{f(s(x))}) = 1$ and by definition of $s(x), s(f(x)) \subseteq \overline{f(s(x))}$. □

**Theorem 3.12** If $f$ is continuous on $s(x)$, or if $f$ has a continuous extension to $s(x)$ and if $f(x)$ exists, then

$$s(f(x)) = \overline{f(s(x))}.$$

**Proof.** By Lemma 3.11 it suffices to prove that $\overline{f(s(x))} \subseteq s(f(x))$. If $\hat{f}$ is a continuous extension of $f$ to $s(x)$, then $\hat{f}(x) = f(x)$ and $\overline{f(s(x))} \subseteq \overline{f(s(x))}$. Thus it would suffice to show in this case that $\overline{f(s(x))} \subseteq s(\hat{f}(x))$. In other words, we can suppose that $f$ is defined and continuous on all of $s(x)$.

Let $\xi \in \overline{f(s(x))}$. Then there exists a sequence $\{\lambda_i\} \subseteq s(x)$ such that $f(\lambda_i) \to \xi$. By Theorem 3.8, we have that $\forall \delta > 0, x(\lambda_i - \delta, \lambda_i + \delta) \neq 0$. Therefore, by continuity, $\forall \epsilon > 0$, we have

$$f(x)(f(\lambda_i) - \epsilon, f(\lambda_i) + \epsilon) = x(f^{-1}((f(\lambda_i) - \epsilon, f(\lambda_i) + \epsilon)) \geq x((\lambda_i - \delta_f(\epsilon, \lambda_i), \lambda_i + \delta_f(\epsilon, \lambda_i)) \cap s(x)) \neq 0.$$

Thus, by Theorem 3.8, $f(\lambda_i) \in s(f(x)) \forall i$. Since $s(f(x))$ is closed, $\xi \in s(f(x))$. □
Definition 3.13 An observable \( x \) is said to be bounded providing that \( s(x) \) is compact.

For bounded observables, we obtain the following result, which generalizes Corollary 3.5 of [4].

Theorem 3.14 (Spectral Mapping Theorem). Let \( x \) be a bounded observable, and let \( f \) be a Borel function defined and continuous on \( s(x) \). If \( f(x) \) exists, then \( s(f(x)) = f(s(x)) \).

Proof. By continuity of \( f, f(s(x)) \) is compact; hence it is closed and bounded. Now apply Theorem 3.12. □

Definition 3.15 An observable \( x \) is said to be invertible providing \( f(x) \) exists for the function \( f(\lambda) = \frac{1}{\lambda} \).

In this case we write \( x^{-1} = f(x) \). According to Theorem 3.9, \( x \) is invertible iff \( x(\{0\}) = 0 \). In particular, if \( 0 \notin s(x) \), then \( x^{-1} \) exists. Indeed, if \( 0 \notin s(x) \), then there exist a closed set \( F \) such that \( x(F) = 1 \) and \( 0 \notin F \). Hence \( \{0\} \subseteq \mathbb{R}\setminus F \) and \( x(\mathbb{R}\setminus F) = 0 \). Therefore, by Remark 3.3(i), \( x(\{0\}) = 0 \); that is, \( x \) is invertible.

Theorem 3.16 Let \( x \in \mathcal{O} \) be invertable. Then

(i) \( (x^{-1})^{-1} \) exists and \( (x^{-1})^{-1} = x \).

(ii) If \( 0 \notin s(x) \), then \( x^{-1} \) is bounded.

(iii) If \( x \) is bounded, then \( 0 \notin s(x^{-1}) \).

Proof. (i) Let

\[
f(\lambda) := \begin{cases} 
\frac{1}{\lambda} & \text{if } \lambda \neq 0 \\
0 & \text{if } \lambda = 0 
\end{cases}
\]

Then \( x^{-1} = f(x) \). Now \( x^{-1}(\{0\}) = f(x)(\{0\}) = x(f^{-1}(\{0\})) = x(\{0\}) = 0 \). Whence \( x^{-1} \) is invertible. Thus

\[
(x^{-1})^{-1} = f(f(x)) = (f \circ f)(x) = x.
\]
(ii) Suppose that $0 \notin s(x)$. Then, by Theorem 3.8, there exists an open interval $I = (-\gamma, \gamma)$ such that $x(I) = 0$, and $0 \in I \subseteq \mathbb{R} \setminus s(x)$. Indeed, we have $C := \mathbb{R} \setminus (-\gamma, \gamma)$ is closed and $x(C) = 1$. So, by Definition 3.5, we have $s(x) \subseteq \mathbb{R} \setminus I$. Since $0 \notin s(x), f(\lambda) = \frac{1}{\lambda}$ is continuous on $s(x)$ and so, by Theorem 3.12,

$$s(x^{-1}) = f(s(x)) \subseteq f(\mathbb{R} \setminus I) = [-\frac{1}{\gamma}, \frac{1}{\gamma}].$$

Thus $s(x^{-1})$ is bounded. Also by Definition 3.5 $s(x^{-1})$ is a closed set; hence $s(x^{-1})$ is compact. Therefore $x^{-1}$ is bounded.

(iii) Suppose that $x$ is bounded. Then $s(x)$ is bounded, so we can find $k > 0$, such that $s(x) \subseteq [-k, k]$. Since $x(s(x)) = 1$, then $x([-k, k]) = 1$ and so, $x((\infty, -k) \cup (k, \infty)) = 0$; hence $x(f^{-1}(-\frac{1}{k}, \frac{1}{k})) = 0$; that is,

$$f(x)(-\frac{1}{k}, \frac{1}{k}) = 0,$$

we conclude $x^{-1}(-\frac{1}{k}, \frac{1}{k}) = 0$. By Theorem 3.8, $0 \notin s(x^{-1})$. □

If we define $f_\lambda : \mathbb{R} \to \mathbb{R}$ by $f_\lambda(\xi) = \xi - \lambda$, then it is natural to write $x - \lambda = f_\lambda(x)$. Now we show that spectra can be classified using $x - \lambda$ in exactly the same manner as is usually done in operator theory.

**Theorem 3.17** Let $x \in \mathcal{O}$.

(i) $\lambda \in \mathbb{R} \setminus s(x)$ $\iff$ $(x - \lambda)^{-1}$ exists and is bounded.

(ii) $\lambda \in p(x) \iff (x - \lambda)^{-1}$ does not exist.

(iii) $\lambda \in c(x)$ $\iff$ $(x - \lambda)^{-1}$ exists and is not bounded.

**Proof.** We first observe that $\forall \lambda$, $f_\lambda$ is continuous on $s(x)$. It follows

$$s(x - \lambda) = s(x) - \{\lambda\}.$$

(i) By Theorem 3.16, $(x - \lambda)^{-1}$ exists and is bounded

$$\iff 0 \notin s(x - \lambda) \iff 0 \notin s(x) - \{\lambda\} \iff \lambda \notin s(x).$$

(ii) $(x - \lambda)^{-1}$ fails to exist $\iff x(f_{\lambda}^{-1}(\{0\})) \neq 0 \iff x(\{\lambda\}) \neq 0 \iff \lambda \in p(x)$

(iii) By Definition 3.7 $\lambda \in c(x) \iff \lambda \in s(x) \setminus p(x) \iff (x - \lambda)^{-1}$ exists and is not bounded. □
3.2 Observables Determining Spectral Resolutions

In sections 3.2 and 3.3, we will study the relationship between observables and spectral resolutions in $L$. In particular, we will give answers to the following two questions:

(Q1) If $x$ is an observable on $L$, does $x$ determine a spectral resolution in $L$.

(Q2) Conversely, if $(e_\lambda)_{\lambda \in \mathbb{R}}$ is a spectral resolution in $L$, does there exist an observable on $L$, determining $(e_\lambda)_{\lambda \in \mathbb{R}}$.

In [4], D. Catlin showed that the answer to (Q1) is yes if $x$ is an observable on a $\sigma$-OMP. Indeed, define $e^x : \mathbb{R} \to L$ by $e^x_\lambda := x((-\infty, \lambda])$. Then it is easy to verify that $(e^x_\lambda)_{\lambda \in \mathbb{R}}$ is a spectral resolution in $L$. In this section, we will show that the answer to (Q1) is also yes when $L$ is a $\sigma$-effect algebra, a generalization of a $\sigma$-OMP. However, the proof will not be as quite easy as in the $\sigma$-OMP case.

**Definition 3.18** [10] Let $L$ be an effect algebra. Then $L$ has the Riesz-decomposition property iff, for all $x, y, z \in L$, if $y \perp z$ and $x \leq y \oplus z$, there exist $x_1, x_2 \in L$ such that $x_1 \leq y, x_2 \leq z$, and $x = x_1 \oplus x_2$.

In Definition 3.18, note that it is not necessary to assume that $x_1 \perp x_2$, since the facts that $x_1 \leq y, x_2 \leq z$, and $y \perp z$ imply that $x_1 \leq y \leq z' \leq x_2'$, whence $x_1 \perp x_2$.

**Definition 3.19** [10] A Boolean effect algebra is an orthomodular poset with the Riesz-decomposition property.

The following theorem gives an equivalent definition of a Boolean effect algebra.

**Theorem 3.20** [10] As a bounded poset, every Boolean effect algebra is a complemented distributive lattice; that is, a Boolean algebra, in which the supplement of each element coincides with its Boolean complement.

Let $L$ be an effect algebra and $B \subseteq L$ is a sub-effect algebra of $L$. For
a, b ∈ B, we mean by $a \vee^B b$ and $a \wedge^B b$ is the supremum and infimum of $a, b$ as calculated in $B$.

According to Theorem 3.20, we have

**Definition 3.21** If $B \subseteq L$ is a sub-effect algebra of $L$, then it is a *Boolean subeffect algebra* iff $(B, \vee^B, \wedge^B, ')$ is a Boolean algebra. Also $B$ is a *Boolean σ-subeffect algebra* of $L$ iff $(B, \vee^B, \wedge^B, ')$ is a σ-lattice as a Boolean algebra.

**Remark 3.22** If $L$ happens to be a σ-OMP, then the lattice operations as calculated in $B$ will coincide with the lattice operations as calculated in $L$. Indeed, it is clear that if \{b_i : i ∈ \mathbb{N}\} ⊆ $B$ is pairwise orthogonal, then by Theorem 2.29, $$\bigoplus_{i∈\mathbb{N}} b_i = \bigvee_{i∈\mathbb{N}} b_i.$$ Also, since $L$ is σ-OMP, again by Theorem 2.29, $$\bigoplus_{i∈\mathbb{N}} b_i = \bigvee_{i=1} b_i.$$ Since $'$ is the orthocomplementation in $B$, then the condition (i) and (iv) of Definition 1.13 holds. Finally, $a \leq b$ iff $a \leq_B b$, whenever $a, b ∈ B$. Indeed, if $a \leq b$, then there exist $r ∈ L$ such that $b = a ⊕ r$. Then, by the OMI and the cancellation law, $r = (a ⊕ b')' ∈ B$. Thus $a \leq_B b$. The other part is clear. Hence the condition (i) of Definition 1.13 hold. We conclude that $B$ is a σ-sub-OMP of $L$. By Theorem 1.14, $B$ is a Boolean σ-subalgebra of $L$. Therefore, the lattice operations as calculated in $B$ will coincide with the lattice operations as calculated in $L$.

**Lemma 3.23** Let $L$ be an effect algebra, and $a, b, x, y ∈ L$ be such that $a \leq x, b \leq y, x \perp y$ and $a \perp b$. If $a ⊕ b = x ⊕ y$, then $a = x$ and $b = y$.

**Proof.** Assume that $a ⊕ b = x ⊕ y$. Then by Theorem 2.14(i), $x = a ⊕ (x ⊕ a)$ and $y = b ⊕ (y ⊕ b)$. We have $x ⊕ y = a ⊕ b ⊕ (x ⊕ a) ⊕ (y ⊕ b) = x ⊕ y ⊕ (x ⊕ a) ⊕ (y ⊕ b)$. Hence by cancellation law, $(x ⊕ a) ⊕ (y ⊕ b) = 0$. Thus, using Lemma 2.4(vii), we have $x ⊕ a = 0$ and $y ⊕ b = 0$; hence $x = a$ and $y = b$.

**Lemma 3.24** If $L$ is a σ-effect algebra, if $x$ is an observable on $L$, and if $(A_i)_{i∈\mathbb{N}} ⊆ \mathcal{B}(\mathbb{R})$ is such that $A_i ⊆ A_{i+1}$ ∀$i ∈ \mathbb{N}$ and $\bigcup_{i=1}^\infty A_i = A$, then
\[ \bigvee_{i=1}^{\infty} x(A_i) = x(A). \]

**Proof.** Firstly, \( \bigvee_{i=1}^{\infty} x(A_i) \) exists by Theorem 2.34, since \( x(A_i) \leq x(A_{i+1}) \forall i \in \mathbb{N} \), by Remark 3.3(i). Since \( \forall i \in \mathbb{N}, A_i \subseteq A \Rightarrow x(A_i) \leq x(A) \), we have \( \bigvee_{i=1}^{\infty} x(A_i) \leq x(A) \). It remains to show that \( x(A) \leq \bigvee_{i=1}^{\infty} x(A_i) \). Since \( \forall i \in \mathbb{N}, A_i \subseteq A \Rightarrow x(A_i) \leq x(A) \), we have \( \bigvee_{i=1}^{\infty} x(A_i) \leq x(A) \). It remains to show that \( x(A) \leq \bigvee_{i=1}^{\infty} x(A_i) \). Since \( x(A_1) \leq x(A_2) \leq \ldots \) is a countable chain in \( L \), let \( D \) be as in the proof of Theorem 2.33. Now \( A_1, A_2 - A_1, A_3 - A_2, \ldots \) are disjoint sets, \( A_1 \cup \bigcup_{i=1}^{\infty} (A_{i+1} - A_i) = A \), and \( x \) is an observable on \( L \), so we get

\[ x(A) = x(A_1) \oplus \bigoplus_{i=1}^{\infty} (x(A_{i+1}) \ominus x(A_i)), \]

by Remark 3.3(ii). Hence, by Theorem 2.33,

\[ x(A_1) \oplus \bigoplus_{i=1}^{\infty} (x(A_{i+1}) \ominus x(A_i)) \leq \bigoplus_{i=1}^{\infty} x(A_i). \]

Therefore \( x(A) = \bigvee_{i=1}^{\infty} x(A_i). \) \( \square \)

The next lemma was proved in [24] for orthoalgebras, the same proof can be used for effect algebras.

**Lemma 3.25** Let \( L \) be an effect algebra, and let \( M \subseteq L \) be such that

(i) \( 0 \in M \), and \( a \in M \Rightarrow a' \in M \);

(ii) if \( p, q, r \) are pairwise orthogonal elements contained in \( M \), then \( p \oplus q \oplus r \)

exists in \( L \) and belongs to \( M \); and

(iii) if \( (a_i)_{i \in \mathbb{N}} \subseteq M \) is an increasing sequence, then it has a supremum in

\( M \). Then \( M \) is a \( \sigma \)-subeffect algebra of \( L \) which is a \( \sigma \)-OMP.

**Proof.** Firstly, \( M \) is subeffect algebra, since, by (i), \( 0, 1 \in M \) and if \( a \in M \), then \( a' \in M \), and if \( p, q \in M, p \perp q \), then \( p, q, 0 \) are pairwise orthogonal elements in \( M \), so by (ii), \( p \oplus q \oplus 0 = p \oplus q \) exists in \( M \). Therefore, \( M \)

is a subeffect algebra of \( L \). By (iii) and Theorem 2.34, \( M \) is a \( \sigma \)-subeffect
algebra. It remains to prove that $M$ is a $\sigma$-OMP. Let $a, b \in M$ where $a \perp b$. Then $a, b \leq a \oplus b$. Let $r \in M$ be such that $a, b \leq r$. Then $a \perp r', b \perp r'$; so by (ii), $a \oplus b \oplus r'$ exists in $M$, and hence $a \oplus b \leq r$. It follows that $a \oplus b = a \vee^M b$. Hence by Theorem 2.8, $M$ is an OMP. Also, by Theorem 2.29, $M$ is a $\sigma$-OMP. \[\square\]

Let $L$ be a $\sigma$-effect algebra and $x : \mathcal{B}(\mathbb{R}) \to L$ be an observable on $L$. If $x(E) \land (x(E))^\prime = 0$ for every $E \in \mathcal{B}(\mathbb{R})$, then $x$ is called a sharp observable.

**Theorem 3.26** Let $L$ be a $\sigma$-effect algebra and let $x$ be a sharp observable on $L$. Then the range of $x$, $R(x)$, is a Boolean $\sigma$-subeffect algebra of $L$.

**Proof.** We first prove that $R(x)$ satisfies the three conditions of Lemma 3.25.

(i) Note that $0 \in R(x)$, since $x(\emptyset) = 0$. If $a \in R(x)$, then $a = x(A)$ for some $A \in \mathcal{B}(\mathbb{R})$. Then we get \[1 = x(\mathbb{R}) = x(A \cup A^c) = x(A) \oplus x(A^c) = a \oplus x(A^c),\] so that $a' = x(A^c) \in R(x)$.

(ii) Let $a_i := x(A_i)$, where $i \in \{1, 2, 3\}$, be pairwise orthogonal elements of $R(x)$. We have
\[
\begin{align*}
A_1 &= (A_1 \cap A_2 \cap A_3) \cup (A_1 \cap A_2 \cap A_3) \cup (A_1 \cap A_2 \cap A_3) \cup (A_1 \cap A_2 \cap A_3) \\
A_2 &= (A_1 \cap A_2 \cap A_3) \cup (A_1 \cap A_2 \cap A_3) \cup (A_1 \cap A_2 \cap A_3) \cup (A_1 \cap A_2 \cap A_3) \\
A_3 &= (A_1 \cap A_2 \cap A_3) \cup (A_1 \cap A_2 \cap A_3) \cup (A_1 \cap A_2 \cap A_3) \cup (A_1 \cap A_2 \cap A_3).
\end{align*}
\]
Owing to the orthogonality of $a_1, a_2, a_3$, we obtain that
\[
x(A_1) \oplus x(A_2) = x(A_1 \cap A_2 \cap A_3) \oplus x(A_1 \cap A_2 \cap A_3) \oplus x(A_1 \cap A_2 \cap A_3) \oplus x(A_1 \cap A_2 \cap A_3)
\]
Since $x$ is sharp, we have $x(A_1 \cap A_2 \cap A_3) = x(A_1 \cap A_2 \cap A_3) = 0$ and so
\[
x(A_1) \oplus x(A_2) = x(A_1 \cap A_2 \cap A_3) \oplus x(A_1 \cap A_2 \cap A_3) \oplus x(A_1 \cap A_2 \cap A_3) \oplus x(A_1 \cap A_2 \cap A_3)
\]
Similarly,
\[
x(A_2) \oplus x(A_3) = x(A_1 \cap A_2 \cap A_3) \oplus x(A_1 \cap A_2 \cap A_3) \oplus x(A_1 \cap A_2 \cap A_3) \oplus x(A_1 \cap A_2 \cap A_3)
\]
yields $x(A_1 \cap A_2 \cap A_3) = 0$. Now we have $x(A_2) \oplus x(A_3) = x(A_2 \cap A_3) \oplus x(A_2 \cap A_3) \oplus x(A_2 \cap A_3)$ and $x(A_1) \oplus x(A_2) = x(A_1 \cap A_2 \cap A_3) \oplus x(A_1 \cap A_2 \cap A_3) \oplus x(A_1 \cap A_2 \cap A_3) \oplus x(A_1 \cap A_2 \cap A_3)$.
\[ x(A_1 \cap A_2^c \cap A_3^c) \oplus x(A_2 \cap A_1^c \cap A_3^c). \] Also
\[ x(A_1) \oplus x(A_3) = x(A_1 \cap A_2 \cap A_3) \oplus x(A_1 \cap A_2 \cap A_3^c) \oplus x(A_1 \cap A_2^c \cap A_3) \oplus x(A_1 \cap A_2^c \cap A_3^c) \oplus x(A_1 \cap A_2 \cap A_3) \oplus x(A_1 \cap A_2 \cap A_3^c) \oplus x(A_1 \cap A_2^c \cap A_3) \oplus x(A_1 \cap A_2^c \cap A_3^c) \], so that \( x(A_1 \cap A_3 \cap A_2^c) = 0 \). We conclude \( x(A_1) \oplus x(A_3) = x(A_1 \cap A_2 \cap A_3) \oplus x(A_1 \cap A_2 \cap A_3^c) \oplus x(A_1 \cap A_2^c \cap A_3) \oplus x(A_1 \cap A_2^c \cap A_3^c) \) and \( x(A_2) \oplus x(A_3) = x(A_2 \cap A_1 \cap A_3) \oplus x(A_2 \cap A_1 \cap A_3^c) \oplus x(A_2 \cap A_1^c \cap A_3) \oplus x(A_2 \cap A_1^c \cap A_3^c) \).

By Lemma 3.23, we have \( x(A_1) = x(A_1 \cap A_2 \cap A_3) \), \( x(A_2) = x(A_1 \cap A_2 \cap A_3^c) \) and \( x(A_3) = x(A_1 \cap A_2^c \cap A_3) \). Let \( B_1 = A_1 \cap A_2 \cap A_3 \), \( B_2 = A_1 \cap A_2 \cap A_3^c \) and \( B_3 = A_1 \cap A_2^c \cap A_3 \). Since \( B_1, B_2 \) and \( B_3 \) are mutually disjoint, we get
\[ x(B_1 \cup B_2 \cup B_3) = x(B_1) \oplus x(B_2) \oplus x(B_3). \] Thus \( a_1 \oplus a_2 \oplus a_3 \) exists in \( R(x) \).

(iii) Let \( a_i := x(A_i), i \in \mathbb{N} \), be an increasing sequence in \( R(x) \). Since \( L \) is a \( \sigma \)-effect algebra, \( a := \bigvee_{i=1}^{\infty} a_i \) exists in \( L \). Let \( B_n := A_1 \cup A_2 \cup \ldots \cup A_n, n \in \mathbb{N} \). We will prove, by induction, that \( x(B_n) = a_n \; \forall n \in \mathbb{N} \). Clearly \( a_1 = x(B_1) \). Assume \( a_n = x(B_n) \). Since \( B_{n+1} = B_n \cup A_{n+1} = A_n+1 \cup (B_n \cap A_{n+1}^c) \), we have
\[ x(B_{n+1}) = x(A_n+1) \oplus x(B_n \cap A_{n+1}^c). \] But \( x(B_n \cap A_{n+1}^c) \leq x(B_n) = a_n \leq a_{n+1} \), and \( x(B_n \cap A_{n+1}^c) \leq x(A_{n+1}^c) = a_{n+1}^c \), so that \( x(B_n \cap A_{n+1}^c) \leq a_{n+1} \). Hence \( x(B_{n+1}) = x(A_{n+1}) = a_{n+1} \). This completes the induction. Since \( \bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n \), we get \( a = \bigvee_{i=1}^{\infty} a_i = \bigvee_{i=1}^{\infty} x(A_i) = \bigvee_{i=1}^{\infty} x(B_i) = x(\bigcup_{i=1}^{\infty} B_i) \), which belongs to \( R(x) \).

Now, by Lemma 3.25, \( R(x) \) is a \( \sigma \)-subeffect algebra which is also a \( \sigma \)-OMP. By Theorem 1.16, \( R(x) \) is a Boolean \( \sigma \)-algebra. Therefore, \( R(x) \) is a Boolean \( \sigma \)-subeffect algebra. \( \Box \)

The following theorem is a generalization of the result in [4, p. 295] to \( \sigma \)-effect algebras.

**Theorem 3.27** If \( x \) is an observable on \( L \) where \( L \) is a \( \sigma \)-effect algebra, then \( x((-\infty, \lambda]), \lambda \in \mathbb{R} \), is a spectral resolutions in \( L \).

**Proof.** We will verify the conditions of Definition 1.17.

(i) If \( \lambda \leq \mu \) in \( \mathbb{R} \), then \((-\infty, \lambda] \subseteq (-\infty, \mu] \); hence, by Remark 3.3(i), we get \( x((-\infty, \lambda]) \leq x((-\infty, \mu]) \); that is, \( e_\lambda \leq e_\mu \), where \( e_\lambda := x((-\infty, \lambda]), \lambda \in \mathbb{R} \).
(ii) Since \((-\infty, n] \subseteq (-\infty, n+1] \forall n \in \mathbb{N}\), then, by Lemma 3.24,
\[\bigvee_{n=1}^{\infty} x(-\infty, n] = x(\bigcup_{n=1}^{\infty} (-\infty, n]) \]
\[= x(\mathbb{R}) = 1.\]

Now let \(t\) be an upper bound of \((e_\lambda)_{\lambda \in \mathbb{R}}\). Then \(e_n \leq t \forall n \in \mathbb{N}\) implies that
\[1 = \bigvee_{n=1}^{\infty} e_n \leq t;\] hence \(t = 1\), and therefore \(\bigvee_{\lambda \in \mathbb{R}} e_\lambda = 1.\)

(iii) Since \((-n, \infty) \subseteq (-n+1, \infty) \forall n \in \mathbb{N}\), then, by Remark 3.3(i), we have \(x(-n, \infty) \leq x(-(n+1), \infty) \forall n \in \mathbb{N}\). Hence, by Lemma 3.24, we have
\[\bigvee_{n=1}^{\infty} x(-n, \infty) = x(\bigcup_{n=1}^{\infty} (-\infty, \infty)) = x(\mathbb{R}) = 1;\] that is, \(\bigvee_{n=1}^{\infty} e_{-n} = 1.\) Using De Morgan Laws, we get \(\bigwedge_{n=1}^{\infty} e_{-n} = 0.\) Now let \(t\) be a lower bound of \((e_\lambda)_{\lambda \in \mathbb{R}}\). Then \(t \leq e_{-n} \forall n \in \mathbb{N}\) implies that \(t \leq \bigwedge_{n=1}^{\infty} e_{-n} = 0;\) that is, \(t = 0.\) Therefore \(\bigwedge_{\lambda \in \mathbb{R}} e_\lambda = 0.\)

(iv) If \(\mu \in \mathbb{R}\). Then \((\mu + \frac{1}{n}, \infty) \subseteq (\mu + \frac{1}{n+1}, \infty) \forall n \in \mathbb{N}\); so, by Lemma 3.24,
\[\bigvee_{n=1}^{\infty} x(\mu + \frac{1}{n}, \infty) = x(\bigcup_{n=1}^{\infty} (\mu + \frac{1}{n}, \infty)) = x(\mu, \infty).\] Now by De Morgan Law
\[\bigwedge_{n=1}^{\infty} x((-\infty, \mu + \frac{1}{n})) = x(-\infty, \mu)\]; that is, \(\bigwedge_{n=1}^{\infty} e_{\mu + \frac{1}{n}} = e_\mu.\) Finally, let \(t\) be a lower bound of \((e_\lambda)_{\mu < \lambda}\) where \(\lambda \in \mathbb{R}\). Then \(t \leq \bigwedge_{n=1}^{\infty} e_{\mu + \frac{1}{n}},\) hence \(t \leq e_\mu.\) Therefore, \(\bigwedge_{\mu < \lambda} e_\lambda = e_\mu.\) \(\square\)

3.3 Spectral Resolutions Determining an Observable

In this section, we will investigate the answer to \((Q_2)\): if \((e_\lambda)_{\lambda \in \mathbb{R}}\) is a spectral resolution in \(L,\) where \(L\) is a \(\sigma\)-effect algebra, does there exist an observ-
able on \( L \), determining \((e_\lambda)_{\lambda \in \mathbb{R}}\). We will show that the answer is yes, providing \((e_\lambda)_{\lambda \in \mathbb{R}} \subseteq B\) where \( B \) is a Boolean \( \sigma \)-subeffect algebra of \( L \).

Until the end of this section, \( B \) will be a Boolean \( \sigma \)-algebra. By Loomis Theorem (see Theorem 1.21), there exists a measurable space \((X, \mathcal{M})\) and a \( \sigma \)-ideal \( \mathcal{K} \subseteq \mathcal{M} \) such that 
\[
B \cong \mathcal{M} / \mathcal{K}.
\]
Let \( \phi : \mathcal{M} \to \mathcal{M} / \mathcal{K} \) be the natural \( \sigma \)-epimorphism, where \( \phi(M) = [M] \forall M \in \mathcal{M} \). Since \( B \cong \mathcal{M} / \mathcal{K} \), there exists a Boolean \( \sigma \)-isomorphism \( \theta : \mathcal{M} / \mathcal{K} \to B \). Hence \( \theta \circ \phi \) is an \( \sigma \)-epimorphism from \( \mathcal{M} \) onto \( B \). Set \( \eta := \theta \circ \phi \). Clearly \( \eta \) is a \( \sigma \)-epimorphism. Let \( e : \mathbb{R} \to B \) be a real spectral resolution and let \( f : \mathbb{Q} \to B \) be the restriction of \( e \) to \( \mathbb{Q} \) as in Theorem 1.16.

**Definition 3.28** [4] For each rational number \( \lambda \in \mathbb{Q} \), choose a set \( \tilde{F}_\lambda \in \mathcal{M} \) such that \( \eta(\tilde{F}_\lambda) = f_\lambda \). This can be done because \( \eta \) is onto. Define
\[
\bar{F}_\lambda := \bigcap_{\rho < \lambda} \tilde{F}_\rho, \rho \in \mathbb{Q}, \lambda \in \mathbb{Q}.
\]

**Remarks 3.29**

(i) The Definition of \( \bar{F}_\lambda \) yields that \( \bar{F}_\lambda \subseteq \bar{F}_\mu \) whenever \( \lambda \leq \mu \). Also \( \eta(\bar{F}_\lambda) = f_\lambda \) for each \( \lambda \in \mathbb{Q} \). Define
\[
\hat{F}_\lambda := \bar{F}_\lambda \setminus \bigcap_{\sigma \in \mathbb{Q}} \bar{F}_\sigma, \lambda \in \mathbb{Q}.
\]

(ii) If \( \lambda, \mu \in \mathbb{Q} \) with \( \lambda \leq \mu \), it then follows that \( \hat{F}_\lambda \subseteq \hat{F}_\mu \). Also, we have
\[
\bigcap_{\lambda \in \mathbb{Q}} \hat{F}_\lambda = \emptyset \text{ and } \eta(\hat{F}_\lambda) = f_\lambda \forall \lambda \in \mathbb{Q} \text{. Indeed, if } \bigcap_{\lambda \in \mathbb{Q}} \hat{F}_\lambda \neq \emptyset \text{ then there is an element } x \in X \text{ such that } x \in \bigcap_{\lambda \in \mathbb{Q}} \hat{F}_\lambda \text{; that is, } x \in \hat{F}_\lambda \text{ for each } \lambda \in \mathbb{Q} \text{. From the definition of } \hat{F}_\lambda, x \in \hat{F}_\lambda \text{ for each } \lambda \in \mathbb{Q} \text{ and } x \notin \hat{F}_\sigma \text{ for some } \sigma \in \mathbb{Q} \text{, which is a contradiction. Therefore } \bigcap_{\lambda \in \mathbb{Q}} \hat{F}_\lambda = \emptyset \text{. Moreover,}
\]

(iii)
\[
\bigcap_{\lambda < \mu} \hat{F}_\mu = \hat{F}_\lambda.
\]
Indeed, from the definition of $\bar{F}_\sigma$ we get
\[
\bigcap_{\lambda<\mu} \bar{F}_\mu = \bigcap_{\lambda<\mu} (\bigcap_{\mu<\rho} \bar{F}_\rho) = \bigcap_{\lambda<\rho} \bar{F}_\rho = \bar{F}_\lambda.
\]
Now
\[
\bigcap_{\lambda<\mu} \hat{F}_\mu = \bigcap_{\lambda<\mu} (\bar{F}_\mu \setminus \bigcap_{\sigma\in Q} \bar{F}_\sigma)
= (\bigcap_{\lambda<\mu} \bar{F}_\mu) \setminus \bigcap_{\sigma\in Q} \bar{F}_\sigma
= \bar{F}_\lambda \setminus \bigcap_{\sigma\in Q} \bar{F}_\sigma
= \hat{F}_\lambda.
\]

**Definition 3.30** Finally, define
\[
F_\lambda := \begin{cases} 
\hat{F}_\lambda, & \text{if } \lambda < 0, \lambda \in \mathbb{Q} \\
\hat{F}_\lambda \cup (X \setminus \bigcup_{\sigma\in Q} \hat{F}_\sigma), & \text{if } \lambda \geq 0, \lambda \in \mathbb{Q}.
\end{cases}
\]

**Theorem 3.31** [4] \(\{F_\lambda : \lambda \in \mathbb{Q}\}\) is a rational spectral resolution in \(\mathcal{M}\) and \(\eta(F_\lambda) = f_\lambda \ \forall \lambda \in \mathbb{Q}\). That is, we have lifted the rational spectral resolution \(\{f_\lambda : \lambda \in \mathbb{Q}\}\) in \(B\) through \(\eta\) to the rational spectral resolution \(\{F_\lambda : \lambda \in \mathbb{Q}\}\) in \(\mathcal{M}\).

**Proof.** We want to verify the conditions of Definition 1.17.

(i) If \(\lambda \leq \mu\), and they are nonnegative, then \(\hat{F}_\lambda \subseteq \hat{F}_\mu\) by Remark 3.29(ii), so we get \(\hat{F}_\lambda \cup (X \setminus \bigcup_{\sigma\in Q} \hat{F}_\sigma) \subseteq \hat{F}_\mu \cup (X \setminus \bigcup_{\sigma\in Q} \hat{F}_\sigma)\); that is, \(F_\lambda \subseteq F_\mu\). If \(\lambda, \mu\) are negative and \(\lambda \leq \mu\), then from Definition 3.30, \(F_\lambda = \hat{F}_\lambda \subseteq \hat{F}_\mu = F_\mu\). If \(\lambda < 0\) and \(\mu \geq 0\), then \(\lambda \leq \mu\) and \(F_\lambda = \hat{F}_\lambda \subseteq \hat{F}_\mu \subseteq \hat{F}_\mu \cup (X \setminus \bigcup_{\sigma\in Q} \hat{F}_\sigma) = F_\mu\).
(ii) 
\[
\bigcap_{\lambda \in \mathbb{Q}} F_{\lambda} = \left( \bigcap_{\lambda < 0} \hat{F}_{\lambda} \right) \cap \left( \bigcap_{\lambda \geq 0} \left( \hat{F}_{\lambda} \cup (X \setminus \bigcup_{\lambda \leq 0} \hat{F}_{\sigma}) \right) \right)
\]
\[
= \left( \bigcap_{\lambda < 0} \hat{F}_{\lambda} \right) \cap \left( \bigcap_{\lambda \geq 0} \hat{F}_{\lambda} \right) \cup \left( X \setminus \bigcup_{\lambda \leq 0} \hat{F}_{\sigma} \right)
\]
\[
= \bigcap_{\lambda \in \mathbb{Q}} \hat{F}_{\lambda} \cup \left( X \setminus \bigcup_{\lambda \leq 0} \hat{F}_{\sigma} \right)
\]
\[
= \emptyset,
\]
since \( \bigcap_{\lambda \in \mathbb{Q}} \hat{F}_{\lambda} = \emptyset \), and \( \left( \bigcap_{\lambda < 0} \hat{F}_{\lambda} \right) \cap \left( X \setminus \bigcup_{\lambda \leq 0} \hat{F}_{\sigma} \right) = \emptyset \).

(iii) 
\[
\bigcup_{\lambda \in \mathbb{Q}} F_{\lambda} = \left( \bigcup_{\lambda < 0} \hat{F}_{\lambda} \right) \cup \left( \bigcup_{\lambda \geq 0} \left( \hat{F}_{\lambda} \cup (X \setminus \bigcup_{\lambda \leq 0} \hat{F}_{\sigma}) \right) \right)
\]
\[
= \left( \bigcup_{\lambda < 0} \hat{F}_{\lambda} \right) \cup \left( \bigcup_{\lambda \geq 0} \hat{F}_{\lambda} \right) \cup \left( X \setminus \bigcup_{\lambda \leq 0} \hat{F}_{\sigma} \right)
\]
\[
= \bigcup_{\lambda \in \mathbb{Q}} \hat{F}_{\lambda} \cup \left( X \setminus \bigcup_{\lambda \leq 0} \hat{F}_{\sigma} \right)
\]
\[
= X.
\]

(iv) If \( \lambda \geq 0 \), then by Remark 3.29(iii) we get
\[
\bigcap_{\lambda \leq \mu} F_{\mu} = \bigcap_{\lambda \leq \mu} (\hat{F}_{\mu} \cup (X \setminus \bigcup_{\lambda \leq \mu} \hat{F}_{\sigma}))
\]
\[
= (\bigcap_{\lambda \leq \mu} \hat{F}_{\mu}) \cup (X \setminus \bigcup_{\lambda \leq \mu} \hat{F}_{\sigma})
\]
\[
= \hat{F}_{\lambda} \cup (X \setminus \bigcup_{\lambda \leq \mu} \hat{F}_{\sigma})
\]
\[
= F_{\lambda}.
\]
If \( \lambda < 0 \), then again, using Remark 3.29(iii),
\[
\bigcap_{\lambda < \mu} F_\mu = \left( \bigcap_{\lambda < \mu < 0} \hat{F}_\mu \right) \cap \left( \bigcap_{\lambda < 0 \leq \mu} (\hat{F}_\mu \cup \bigcup_{\sigma \in Q} \hat{F}_\sigma) \right) \\
= \left( \bigcap_{\lambda < \mu < 0} \hat{F}_\mu \right) \cap \left( \bigcap_{\lambda < 0 \leq \mu} (\hat{F}_\mu \cup \bigcup_{\sigma \in Q} \hat{F}_\sigma) \right) \\
= \left( \bigcap_{\lambda < \mu < 0} \hat{F}_\mu \right) \cup \left( \bigcap_{\lambda < \mu < 0} \hat{F}_\mu \right) \cap \left( \bigcap_{\lambda < 0 \leq \mu} \hat{F}_\sigma \right) \\
= \bigcap_{\lambda < \mu} \hat{F}_\mu \\
= \hat{F}_\lambda \\
= F_\lambda
\]

since \( \left( \bigcap_{\lambda < \mu < 0} \hat{F}_\mu \right) \cap \left( X \setminus \bigcup_{\sigma \in Q} \hat{F}_\sigma \right) = \emptyset \).

Finally, we will prove that \( \eta(F_\lambda) = f_\lambda \forall \lambda \in \mathbb{Q} \). If \( \lambda < 0 \), then \( \eta(F_\lambda) = \eta(\hat{F}_\lambda) = f_\lambda \) by Remark 3.29(ii). If \( \lambda \geq 0 \), then
\[
\eta(F_\lambda) = \eta(\hat{F}_\lambda \cup (X \setminus \bigcup_{\sigma \in Q} \hat{F}_\sigma)) \\
= \eta(\hat{F}_\lambda) \lor \eta(X \setminus \bigcup_{\sigma \in Q} \hat{F}_\sigma) \\
= f_\lambda \lor (\eta(X) \land (\eta(\bigcup_{\sigma \in Q} \hat{F}_\sigma))') \\
= f_\lambda \lor (\eta(X) \land (\bigvee_{\sigma \in Q} \eta(\hat{F}_\sigma))') \\
= f_\lambda \lor (1 \land (\bigvee_{\sigma \in Q} f_\sigma)') \\
= f_\lambda \lor (1 \land 1') \\
= f_\lambda. \quad \Box
\]

**Definition 3.32** [4] Extend the rational spectral resolution \( \{F_\lambda : \lambda \in \mathbb{Q}\} \) to a real spectral resolution \( \{E_\lambda : \lambda \in \mathbb{R}\} \) by defining \( E_\lambda = \bigcap_{\lambda \leq \rho} F_\rho, \rho \in \mathbb{Q}, \lambda \in \mathbb{R} \).

**Lemma 3.33** [4] \( \eta(E_\lambda) = e_\lambda \forall \lambda \in \mathbb{R} \).

Proof. \( \eta(E_\lambda) = \eta(\bigcap_{\lambda \leq \rho} F_\rho) = \bigwedge_{\lambda \leq \rho} \eta(F_\rho) = \bigwedge_{\lambda \leq \rho} f_\rho = e_\lambda. \quad \Box\)
Lemma 3.34 [4] Let \((X, \mathcal{M})\) be a measurable space and let \(\{E_\lambda : \lambda \in \mathbb{R}\}\) be any spectral resolution in \(\mathcal{M}\). Then there exists a unique measurable function \(f : X \to \mathbb{R}\) (i.e., \(f^{-1}(B) \in \mathcal{M}\) whenever \(B \in \mathcal{B}\)) such that \(E_\lambda = f^{-1}((\infty, \lambda]) \forall \lambda \in \mathbb{R}\).

**Proof.** Define \(f(x) := \inf \{\sigma \in \mathbb{R} : x \in E_\sigma\}\). Note that this infimum (and hence \(f\)) exists, since \(\bigcup_{\sigma \in \mathbb{R}} E_\sigma = X\). We will show that \(E_\lambda = f^{-1}((\infty, \lambda]), \forall \lambda \in \mathbb{R}\). Now fix \(\lambda \in \mathbb{R}\). Then

\[
f^{-1}((\infty, \lambda]) = \{x : f(x) \in (\infty, \lambda]\}
= \{x : f(x) \leq \lambda\}
= \{x : \inf \{\sigma \in \mathbb{R} : x \in E_\sigma\} \leq \lambda\},
\]
denote this by \(H\). We, next, show that \(H = E_\lambda\). If \(x \in H\), then \(x \in E_\mu, \forall \mu > \lambda\); otherwise there exists \(\mu > \lambda\) such that \(x \notin E_\mu\). Since \(\sigma \leq \mu\) implies \(E_\sigma \subseteq E_\mu\), it follows that \(x \notin E_\sigma \forall \sigma \leq \mu\). Hence, as \(\bigcup_{\sigma \in \mathbb{R}} E_\sigma = X\), we have \(x \in E_\theta\), for some \(\theta > \mu\); thus \(\inf \{\theta \in \mathbb{R} : x \in E_\theta\} \geq \mu > \lambda\), which is a contradiction. Now \(x \in \bigcap_{\lambda < \mu} E_\mu = E_\lambda\), so that \(H \subseteq E_\lambda\). On the other hand, let \(x \in E_\lambda\). Then \(\inf \{\sigma : x \in E_\sigma\} \leq \lambda\), so that \(E_\lambda \subseteq H\). Thus we have \(f^{-1}((\infty, \lambda]) = E_\lambda \in \mathcal{M} \forall \lambda \in \mathbb{R}\). Since the collection \(\mathcal{C} := \{(-\infty, \lambda] : \lambda \in \mathbb{R}\}\) generates the Borel \(\sigma\)-algebra \(\mathcal{B}\), it follows that \(f^{-1}(B) \in \mathcal{M} \forall B \in \mathcal{B}\) (see [27, page 71, problem 24]). Therefore, \(f\) is (Borel) measurable. Finally, the uniqueness of \(f\) follows from its definition. \(\square\)

**Corollary 3.35** [4] Let \((X, \mathcal{M})\) be any measurable space and let \(\{E_\lambda : \lambda \in \mathbb{R}\}\) be any spectral resolution in \(\mathcal{M}\). Then there exists a unique \(\mathcal{M}\)-valued measure \(A : \mathcal{B} \to \mathcal{M}\) such that \(E_\lambda = A((-\infty, \lambda])\).

**Proof.** Define \(A : \mathcal{B} \to \mathcal{M}\) by \(A(B) := f^{-1}(B) \forall B \in \mathcal{B}\), where \(f\) is the unique function given in Lemma 3.34. Then \(A(\mathbb{R}) = f^{-1}(\mathbb{R}) = X, A(\emptyset) = \)
\[ f^{-1}(\emptyset) = \emptyset \text{ and if } B_i \in \mathcal{B}, \ i \in \mathbb{N}, \text{then} \]
\[ A \left( \bigcup_{i=1} \bigcup_{i=1} B_i \right) = f^{-1}(\bigcup_{i=1} B_i) \]
\[ = \bigcup_{i=1} f^{-1}(B_i) \]
\[ = \bigcup_{i=1} A(B_i). \quad \Box \]

**Theorem 3.36** [4] Let \( B \) be a Boolean \( \sigma \)-algebra, and let \( e: \mathbb{R} \to B \) be a real spectral resolution in \( B \). Then there exists a unique \( B \)-valued measure \( A: \mathcal{B} \to B \) such that
\[ A((-\infty, \lambda]) = e_\lambda, \ \forall \lambda \in \mathbb{R}. \]

**Proof.** Use Lemma 3.33 to left \( e \) via the epimorphism \( \eta: \mathcal{M} \to B \) to a spectral resolution \( \{E_\lambda : \lambda \in \mathbb{R}\} \) in \( \mathcal{M} \) so that \( \eta(E_\lambda) = e_\lambda \ \forall \lambda \in \mathbb{R} \). By Corollary 3.35, there exists a unique \( \mathcal{M} \)-valued measure \( \tilde{A}: \mathcal{B} \to \mathcal{M} \) such that \( \tilde{A}((-\infty, \lambda]) = E_\lambda \ \forall \lambda \in \mathbb{R} \). Put \( A := \eta \circ \tilde{A} \). Then, we have, \( A((-\infty, \lambda]) = \eta(\tilde{A}((-\infty, \lambda])) = \eta(E_\lambda) = e_\lambda, \ \forall \lambda \in \mathbb{R} \). Now the fact that \( A: \mathcal{B} \to B \) is a \( B \)-valued measure follows from the facts that \( \tilde{A}: \mathcal{B} \to \mathcal{M} \) is an \( \mathcal{M} \)-valued measure and \( \eta: \mathcal{M} \to B \) is a \( \sigma \)-epimorphism. \( \Box \)

The following theorem is a generalization of Theorem 4.5 in [4] to \( \sigma \)-effect algebras.

**Theorem 3.37** Let \( L \) be a \( \sigma \)-effect algebra. If \( (e_\lambda)_{\lambda \in \mathbb{R}} \) is a spectral resolution in \( L \), and \( B \) is a Boolean \( \sigma \)-subeffect algebra of \( L \) containing \( (e_\lambda)_{\lambda \in \mathbb{R}} \), then there exists a unique observable \( A \) on \( L \) such that
\[ A((-\infty, \lambda]) = e_\lambda, \ \forall \lambda \in \mathbb{R}. \]

**Proof.** By Theorem 3.36, there exists a unique observable \( A \) on \( B \) such that \( A((-\infty, \lambda]) = e_\lambda, \ \forall \lambda \in \mathbb{R} \). It remains to prove that \( A \) is an observable on \( L \). Let \( \{E_i : i \in \mathbb{N}\} \subseteq \mathcal{B}(\mathbb{R}) \) where \( E_i \cap E_j = \emptyset \), for \( i \neq j \). Since \( A \) is an observable on \( B \), \( \{A(E_i) : i \in \mathbb{N}\} \) is a countable \( \oplus \)-orthogonal subset of \( B \); so, by Theorem 2.29, we have \( A \left( \bigcup_{i \in \mathbb{N}} E_i \right) = \bigvee_{i \in \mathbb{N}} A(E_i) = \bigoplus_{i \in \mathbb{N}} A(E_i). \) \( \Box \)
The converse of Theorem 3.37 need not be true; that is, if $A$ is an observable on $L$, then the spectral resolution defined by Theorem 3.27, may fail to exist in any Boolean $\sigma$-subeffect algebra of $L$.

Recall that if $(E_i)_{i\in\mathbb{N}}$ is a sequence of disjoint measurable sets in $\mathbb{R}$, and $E \subseteq \mathbb{R}$, then $m^*(E \cap \bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} m^*(E \cap E_i)$, where $m^*$ is Lebesgue’s outer measure on $\mathbb{R}$.

The following example shows that $(e_{\lambda})_{\lambda \in \mathbb{R}}$ in Theorem 3.27 may fail to lie in any Boolean $\sigma$-subeffect algebra.

**Example 3.38** Let $L$ be as in Example 2.3 and let $A : \mathcal{B}(\mathbb{R}) \to L$ be defined by $A(E) := m^*(E \cap [0, 1]) \forall E \in \mathcal{B}(\mathbb{R})$. Then $A$ is an observable on $L$. Indeed, if $(E_i)_{i\in\mathbb{N}} \subseteq \mathcal{B}(\mathbb{R})$ is pairwise disjoint, then

$$A(\bigcup_{i=1}^{\infty} E_i) = m^*(\bigcup_{i=1}^{\infty} E_i \cap [0, 1])$$

$$= \sum_{i=1}^{\infty} m^*([0, 1] \cap E_i)$$

$$= \sum_{i=1}^{\infty} A(E_i)$$

$$= \bigoplus_{i=1}^{\infty} A(E_i).$$

Now let $\lambda \in \mathbb{R}$. If $\lambda < 0$, then $A(-\infty, \lambda] = m^*([0, 1] \cap (-\infty, \lambda]) = m^*(\emptyset) = 0$ and if $\lambda = 0$, then $A(-\infty, 0] = m^*([0, 1] \cap (-\infty, 0]) = m^*(\{0\}) = 0$. If $0 < \lambda < 1$, then $A(-\infty, \lambda] = m^*([0, 1] \cap (-\infty, \lambda]) = m^*([0, \lambda]) = \lambda$. Finally if $\lambda > 1$, then $A(-\infty, \lambda] = m^*([0, 1]) = 1$. Therefore $(A(-\infty, \lambda])_{\lambda \in \mathbb{R}} = [0, 1]$. It remains to prove that $([0, 1], \oplus, 0, 1)$ is not a Boolean $\sigma$-effect algebra. Since $\frac{1}{2} + \frac{1}{4} \leq 1$, then $\frac{1}{2} \perp \frac{1}{4}$ but $\frac{3}{4} = \frac{1}{2} + \frac{1}{4} \neq \frac{1}{2} = \frac{1}{2} \lor \frac{1}{4}$; so, by Theorem 2.8, $([0, 1], \oplus, 0, 1)$ is not an OMP and so $([0, 1], \oplus, 0, 1)$ is not Boolean.
effect algebra by Definition 3.19.

The following theorem shows, however, that the converse of Theorem 3.37 would be true if the observable $A$ is sharp.

**Theorem 3.39** If $A$ is an observable on $L$ where $L$ is a $\sigma$-effect algebra, then $e_\lambda := A((-\infty, \lambda])$, $\lambda \in \mathbb{R}$, is a spectral resolution in $L$. Moreover, if $A$ is sharp, then $(e_\lambda)_{\lambda \in \mathbb{R}}$ is contained in a Boolean $\sigma$-subeffect algebra of $L$.

**Proof.** It follows immediately from Theorems 3.27 and 3.26. □

Since each orthoalgebra is a sharp effect algebra (see Definition 2.12 and 2.13), we get the following theorem, as a consequence of Theorems 3.37 and 3.39.

**Theorem 3.40** Let $L$ be a $\sigma$-orthoalgebra. If $(e_\lambda)_{\lambda \in \mathbb{R}}$ is a spectral resolution in $L$ contained in a Boolean $\sigma$-suborthoalgebra $B$ of $L$, then there exists a unique observable $A$ on $L$ such that

$$A((-\infty, \lambda]) = e_\lambda \forall \lambda \in \mathbb{R}.$$ 

Conversely, if $A$ is an observable on $L$, then $e_\lambda = A((-\infty, \lambda])$, $\lambda \in \mathbb{R}$, is a spectral resolution in $L$, which is contained in a Boolean $\sigma$-suborthoalgebra of $L$. 

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Chapter 4

Spectral Measures

In 2006, S. Pulmannova [23] introduced the notion of a spectral measure on an MV-algebra $M$. She showed that to every element $a$ in $M$ there can be associated a spectral measure $\Lambda_a : \mathcal{B}([0, 1]) \to \mathcal{B}(M)$, where $\mathcal{B}(M)$ denotes the Boolean $\sigma$-algebra of idempotent elements in $M$, and $\mathcal{B}([0, 1])$ denotes the $\sigma$-algebra of Borel subsets of $[0, 1]$. We will obtain the same result for a $\sigma$-complete lattice effect algebra in place of a $\sigma$-MV-algebra. This result is similar to the well-known spectral theorem for self-adjoint operators on a Hilbert space [22].

4.1 $\sigma$-MV-Algebras

Definition 4.1 [8] An $MV$-algebra is a nonempty set $M$ with two special elements 0 and 1 (0 $\neq$ 1), with a binary operation $\oplus : M \times M \to M$, and with a unary operation $^* : M \to M$ such that, for all $a, b, c \in M$, we have

- (MV1) $a \oplus b = b \oplus a$;
- (MV2) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$;
- (MV3) $a \oplus 0 = a$;
- (MV4) $a \oplus 1 = 1$;
- (MV5) $(a^*)^* = a$;
- (MV6) $a \oplus a^* = 1$;
- (MV7) $0^* = 1$;
- (MV8) $(a^* \oplus b)^* \oplus b = (a \oplus b^*)^* \oplus a$.

We define the following binary operations $\odot, \lor, \land$ on $M$ as follows:
\[ a \odot b := (a^* \oplus b^*)^* \]
\[ a \lor b := (a^* \oplus b^*)^* \oplus b, \quad a \land b = (a^* \lor b^*)^*, \quad a, b \in M \]

We write \( a \leq b \) in \( M \) iff \( a \lor b = b \) iff \( a \land b = a \). The relation \( \leq \) is a partial ordering on \( M \) and \( 0 \leq a \leq 1 \), for every \( a \in M \). Thus an MV-algebra is a distributive lattice with respect to \( \lor, \land \).

**Example 4.2** [8] Every Boolean algebra is an MV-algebra. Especially, let \( B \) be an algebra of subsets of a nonempty set \( X \). We put \( E \oplus F := E \cup F, E \odot F := E \cap F, E^* := X \setminus E \), for every \( E, F \in B \), \( 0 := \emptyset \) and \( 1 := X \). Then \( B \) forms an MV-algebra, where \( \leq \) is the inclusion relation.

**Example 4.3** [8] Let \( \mathcal{I} \) be a subset of the interval \([0, 1]\) of real numbers such that \( 0 \in \mathcal{I}, 1 \in \mathcal{I} \), and if \( a, b \in \mathcal{I} \), then \( a \oplus b := \min(1, a + b) \in \mathcal{I}, a \odot b := \max(0, a + b - 1) \in \mathcal{I}, a^* = 1 - a \in \mathcal{I} \), where + and - denote the usual sum and difference of real numbers. The system \( \mathcal{I} \) is an MV algebra. Moreover \( a \lor b = \max(a, b), a \land b = \min(a, b) \) and the relation \( \leq \) is the natural ordering of real numbers. It is not difficult to show that, for \( a, b \in \mathcal{I}, a \oplus b = a + b \) iff \( a \leq b^* = 1 - b \) and \( a \leq b \) implies \( b \odot a^* = b - a \)

**Lemma 4.4** Let \( M \) be an MV-algebra. If \( a, b \in M \), then

(i) \( a \lor b = b \lor a, a \land b = b \land a \);

(ii) \( a \lor b \leq a \oplus b \);

(iii) \( a = b \oplus c \) implies \( b \leq a \) and \( c \leq a \).

**Proof.** (i) The proof of this part follows from the definitions of \( a \lor b, a \land b \), and \( (MV8) \).

(ii) Using \( (MV8) \),

\[
(a \oplus b) \lor (a \lor b) = (a \oplus b) \lor ((a^* \oplus b^*)^* \oplus b)
\]
\[
= ( (a^* \oplus b^*)^* \oplus b) \oplus ((a^* \oplus b^*)^* \oplus (a^* \oplus b^*)^) \oplus b
\]
\[
= ( (a^* \oplus b^*)^* \oplus b) \oplus b^* \oplus (a^* \oplus b^*)^* \oplus b
\]
\[
= (a^* \oplus (a \oplus b))^* \oplus (a^* \oplus b)^* \oplus (b^* \oplus (a \oplus b))^* \oplus (a \oplus b)
\]
\[
= a \oplus b. \quad (MV6), (MV7)
\]
Therefore $a \lor b \leq a \oplus b$.

(iii)

\[
\begin{align*}
a \lor b &= (a \oplus b^*)^* \ominus a \\
&= ((b \ominus c) \ominus b^*)^* \ominus a \\
&= (c \ominus 1)^* \ominus a \\
&= 1^* \ominus a \\
&= 0 \ominus a \\
&= a.
\end{align*}
\]

Therefore, $b \leq a$. The proof of $c \leq a$ is similar. \qed

We define a binary operation $\setminus$ on the MV-algebra $A$ by the formula

$$b \setminus a := b \ominus a^*$$

for any $a, b \in A$.

It is evident that $1\setminus a = a^*, a\setminus 0 = a, a\setminus a = 0$ and $b\setminus a \leq b$, for every $a, b \in A$.

It is easy to check that

$$a \leq b \Rightarrow b = a \ominus (b \setminus a). \quad (4.1)$$

Indeed $a \ominus (b \setminus a) = a \ominus (b \ominus a^*) = a \ominus (b^* \ominus a)^* = (b^* \ominus a)^* \ominus a = b \lor a = b$. Also

$$\quad (a \ominus b) \setminus b = a \lor b^* \ominus a \ominus b,$$  

(4.2) is obvious since $a \ominus b) \setminus b = (a \ominus b) \ominus b^* = ((a \ominus b)^* \ominus b)^* = (a^* \lor b)^* = a \land b^* = a.$

**Definition 4.5** [23] A state on an MV-algebra $M$ is a mapping $m : M \rightarrow [0, 1]$ such that

(i) $m(1) = 1$;

(ii) $m(a \oplus b) = m(a) + m(b)$, whenever $a \leq b^*$.

A state $m$ on $M$ is said to be $\sigma$-additive if $a_n \not\rightarrow a$ (i.e., $a_n \leq a_{n+1} \forall n \in \mathbb{N}$ and $\text{Nand} \bigwedge_{n=1}^{\infty} a_n = a$) implies that $m(a_n) \rightarrow m(a)$.

If $a_1 \leq a_2 \leq \ldots \leq a_n \leq \ldots$, then $a_i = a_1 \ominus (a_2 \setminus a_1) \ominus \ldots \ominus (a_i \setminus a_{i-1}), i = 2, 3, \ldots$ Also $a_{k-1} \leq (a_k \setminus a_{k-1})^*, k = 2, 3, \ldots$ Indeed, $a_{k-1} \leq a_k^* \ominus a_{k-1} = \ldots$
\[(a_k \oplus a_{k-1}^*) = (a_k \setminus a_{k-1})^*.\] Hence, if \(m\) is a state on \(M\), then we have

\[m(a_i) = m(a_1) + \sum_{k=2}^{i} m(a_k \setminus a_{k-1}), \ i = 2, \ldots \quad (4.3)\]

We say that an MV-algebra \(M\) is a \(\sigma\)-MV algebra if \(M\) is a \(\sigma\)-complete lattice. It is clear that every Boolean \(\sigma\)-algebra is a \(\sigma\)-MV-algebra.

**Definition 4.6** [23] A mapping \(h : A \to A'\) between two MV-algebras is a homomorphism of MV algebras iff it preserves the operations \(\oplus, *\) and 1. An MV algebra homomorphism of two \(\sigma\)-MV-algebras is a \(\sigma\)-homomorphism if it preserves countable joins (and meets).

An element \(a \in M\) is idempotent iff \(a \oplus a = a\). Denote by \(B(M)\) the set of all idempotent elements of \(M\).

**Lemma 4.7** [5] For every MV-algebra \(M\), the set of all idempotent elements \(B(M)\) is a Boolean algebra. If \(M\) is a \(\sigma\)-MV-algebra, then \(B(M)\) is a Boolean \(\sigma\)-algebra.

**Lemma 4.8** If \(a, b \in B(M)\), then \(a \oplus b = a \lor b\) and \(a \land b = a \land b\).

**Proof.** By Lemma 4.4(i), \((a^* \oplus b)^* \oplus b = (b^* \oplus a)^* \oplus a\) implies that \((a^* \oplus b)^* \oplus b \oplus a = (b^* \oplus a)^* \oplus a \oplus a = (b^* \oplus a)^* \oplus a = b \lor a = a \lor b\). Hence \(a \lor b = (a^* \oplus b)^* \oplus b \oplus a\). By Lemma 4.4(iii), \(a \oplus b \leq a \lor b\), and, by Lemma 4.4(ii), \(a \lor b \leq a \oplus b\). Therefore \(a \lor b = a \oplus b\). It remains to prove that \(a \land b = a \land b\). Since \(a \land b = (a^* \lor b^*)^* = (a^* \oplus b^*)^* = a \oplus b\).

Recall that by a Borel probability measure \(\mu\) we mean a function \(\mu : \mathcal{B}(\mathbb{R}) \to [0, 1]\) such that

(i) \(\mu(\emptyset) = 0\),

(ii) \(\mu(\mathbb{R}) = 1\),

(iii) \(\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)\), whenever \(\{E_i \in \mathcal{B}(\mathbb{R}) : i \in \mathbb{N}\}\) is a countable family of pairwise disjoint Borel sets.
Lemma 4.9 Let \( M \) be a \( \sigma \)-MV-algebra and \( m \) be a state on \( M \). If \( \Lambda : \mathcal{B}([0, 1]) \to \mathcal{B}(M) \) is a homomorphism of Boolean \( \sigma \)-algebras, then \( m \circ \Lambda \) is a probability measure on \( \mathcal{B}([0, 1]) \), where \( \mathcal{B}([0, 1]) \) denotes the \( \sigma \)-algebra of all Borel subsets of \([0, 1] \).

**Proof.** Let \( (E_i)_{i \in \mathbb{N}} \subseteq \mathcal{B}([0, 1]) \) be a sequence of pairwise disjoint Borel subsets. Put \( A_n = \bigcup_{i=1}^{n} E_i \), \( n = 1, 2, \ldots \). The sequence \( (A_n)_{n \in \mathbb{N}} \) is monotonic and
\[
\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} E_n.
\]
Let us calculate
\[
m \circ \Lambda \left( \bigcup_{n=1}^{\infty} E_n \right) = m(\Lambda \left( \bigcup_{n=1}^{\infty} A_n \right))
\]
\[
= m(\bigvee_{n=1}^{\infty} \Lambda(A_n))
\]
\[
= \lim_{n \to \infty} m(\Lambda(A_n)).
\]
Now by (4.3) we have
\[
m \circ \Lambda \left( \bigcup_{n=1}^{\infty} E_n \right) = \lim_{n \to \infty} m(\Lambda(A_n))
\]
\[
= m(\Lambda(A_1))\ + \lim_{n \to \infty} \sum_{k=2}^{n} m(\Lambda(A_k) \setminus \Lambda(A_{k-1}))
\]
\[
= m(\Lambda(A_1))\ + \sum_{n=2}^{\infty} m(\Lambda(A_n) \setminus \Lambda(A_{n-1}))
\]
\[
= m(\Lambda(A_1))\ + \sum_{n=2}^{\infty} m(\Lambda(E_n)),
\]
since, if \( C \subseteq B \), then
\[
\Lambda(B - C) = \Lambda(B) \setminus \Lambda(C).
\]
Indeed, \( \Lambda(C) \), \( \Lambda(B - C) \in \mathcal{B}(M) \). By Lemma 4.8, we have \( B = C \cup (B - C) \) and hence
\[ \Lambda(B) = \Lambda(C) \lor \Lambda(B - C) = \Lambda(C) \oplus \Lambda(B - C). \]

Therefore, by (4.2),
\[ \Lambda(B) \setminus \Lambda(C) = (\Lambda(C) \oplus \Lambda(B - C)) \setminus \Lambda(C) = \Lambda(B - C). \]

\[ \square \]

**Theorem 4.10** [23] Let \( M \) be a \( \sigma \)-MV algebra. To every \( a \in M \) a \( \sigma \)-homomorphism \( \Lambda_a : \mathcal{B}[0, 1] \to \mathcal{B}(M) \) can be constructed such that the map \( a \to \Lambda_a \) is one-to-one and for every \( \sigma \)-additive state \( m \) on \( M \) we have
\[ m(a) = \int_0^1 \lambda d(m \circ \Lambda_a), \]
where \( \lambda \) is the identity function on \([0, 1] \).
Theorem 4.14 Let $L$ be a lattice effect algebra. Then each block $M$ of $L$ is

(i) a subeffect algebra of $L$, and is

(ii) a sublattice of $L$.

Proof. (i) If $x, y \in M$ and $x \oplus y$ exists in $L$, then $x \leftrightarrow z$ and $y \leftrightarrow z \forall z \in M$. Using Theorem 2.25(v), we have $x \oplus y \leftrightarrow z \forall z \in M$. If $x \oplus y \notin M$, then $M \cup \{x \oplus y\}$ is a set of mutually compatible elements where $M \subseteq M \cup \{x \oplus y\}$, and hence $M$ is not a maximal set of mutually compatible elements, which contradicts the assumption. Therefore $x \oplus y \in M$. Also, $x \in M$ implies $x' \leftrightarrow z \forall z \in M$. Using a similar argument, we get $x' \in M$. Also each element in $L$ is compatible with 0 and 1. Hence $0, 1 \in M$. Therefore $M$ is a subeffect algebra.

(ii) Using Theorem 2.25(i), (iv), and again mimicking the above argument, we get that $M$ is a sublattice of $L$. \qed

Remark 4.15 If $L$ is a $\sigma$-complete lattice effect algebra, then each block $M$ of $L$ is a $\sigma$-sublattice of $L$. Indeed, if $(a_i)_{i \in \mathbb{N}} \subseteq M$, then $a_i \leftrightarrow z \forall z \in M, i = 1, 2, \ldots$. By Theorem 2.26(i), $\bigvee_{i=1}^{\infty} a_i \leftrightarrow z \forall z \in M$. Therefore $\bigvee_{i=1}^{\infty} a_i \in M$.

The proof of the following theorem is different from the proof that appears in [7].

Theorem 4.16 Let $L$ be a lattice effect algebra. Then each block $M$ of $L$ is an MV-algebra.

Proof. We will prove that $M$ is an MV algebra under the operations $\hat{\oplus}$, where $a \hat{\oplus} b := a \oplus (a' \land b)$ and $a^* := a', a, b \in M$. Note, first, that $M$ is closed under the operation $\hat{\oplus}$, since, if $a, b \in M$, then, by Theorem 4.14, $a' \land b$ belongs to $M$ and $a \oplus (a' \land b)$ exists in $M$. Furthermore, from (†) in page 18 of this thesis, we have $a \hat{\oplus} b = (a' \ominus (a' \land b))'$. Now we want to verify the axioms
of Definition 4.1. It is clear that (MV5), (MV6), (MV7) hold. Let \(a, b \in M\),

(MV1) \(a \leftrightarrow b\), so by Theorem 2.23 \((a \lor b') \ominus a = b' \ominus (a \land b')\) and hence 
\(((a \lor b') \ominus a)' = (b' \ominus (a \land b'))'\). We conclude \(a \ominus (a' \land b) = (a \land b') \ominus b\); that is, \(a \oplus b = b \oplus a\).

(MV2) Using (MV1), Lemma 2.17(vi), Theorem 2.23, and Theorem 2.22(ii),

\[
(a \oplus b) \ominus c = (b \ominus a) \ominus c \\
= (b' \ominus (b' \land a))' \ominus c \\
= ((b' \ominus (b' \land a)) \ominus ((b' \ominus (b' \land a)) \land c))' \\
= (((b' \ominus ((b' \ominus (b' \land a)) \land c)) \ominus (b' \land a))')' \\
= (((b' \ominus ((b' \ominus (b' \land a)) \land (c \land b'))) \ominus (b' \land a))')' \\
= (((b' \ominus (b' \land c)) \ominus (b' \ominus (b' \land c) \land b' \land a))')' \\
= ((b' \ominus (b' \land c)) \ominus (b' \ominus (b' \land c) \land a))')' \\
= (b' \ominus (b' \land c))' \ominus a \\
= (b \ominus c) \ominus a \\
= a \ominus (b \ominus c).
\]

(MV8) Using Theorem 2.23(ii), Lemma 2.17(iv) and the fact that 
\((a \lor b) \ominus b \leq b' = 1 \ominus b\),

we have

\[
(a' \ominus b)' \ominus b = (a \ominus (a \land b)) \ominus b \\
= (b' \ominus (b' \land (a \ominus (a \land b))))' \\
= (b' \ominus (b' \land ((a \lor b) \ominus b)))' \\
= (b' \ominus ((a \lor b) \ominus b))' \\
= ((1 \ominus b) \ominus ((a \lor b) \ominus b))' \\
= a \lor b.
\]

Also, by same method, we have \((b' \ominus a)' \ominus a = a \lor b\). Therefore (MV8) holds.

(MV3) This part is trivial.
We have
\[ a \hat{\oplus} 1 = a \oplus (a' \land 1) = a \oplus a' = 1. \]

**Remark 4.17** The order \( \leq_{\hat{\ominus}} \) on \( M \) induced by \( \hat{\ominus} \) coincides with the order \( \leq \) on \( M \) induced by \( \ominus \). Indeed, if \( a, b \in M \), then by Theorem 4.16(MV8), we have \( a \lor \hat{\ominus} b = (a' \hat{\ominus} b)' \hat{\ominus} b = a \lor b \). Hence \( a \leq_{\hat{\ominus}} b \) iff \( b = a \lor b \) iff \( a \leq b \).

**Corollary 4.18** Let \( L \) be a \( \sigma \)-complete lattice effect algebra. If \( M \) is a block of \( L \), then \( M \) is a \( \sigma \)-MV algebra.

**Proof.** By Theorem 4.16, \( M \) is an MV-algebra. Let \((b_i)_{i \in \mathbb{N}} \subseteq M \). Then by Remark 4.15, \( \bigvee_{i=1}^{\infty} b_i \in M \). But, by Remark 4.17, the order of \( M \) as an MV-algebra coincides with the order of \( M \) as a subeffect-algebra. Hence \( \bigvee_{i \in \mathbb{N}} b_i = \bigvee_{i \in \mathbb{N}} b_i \in M \). That is, \( M \) is a \( \sigma \)-MV-algebra. \( \square \)

The following definition is equivalent to Definition 3.2.

**Definition 4.19** [22] A state on an effect algebra \( L \) is a mapping \( m : L \to [0, 1] \) such that \( m(a \oplus b) = m(a) + m(b) \) whenever \( a \perp b \), and \( m(1) = 1 \). A state is \( \sigma \)-additive if \( m(a_n) \to m(a) \) whenever \( a_n \not\rightarrow a \) in \( L \).

**Note:** If \( m \) is a state on a \( \sigma \)-complete lattice effect algebra \( L \) and \( a \leq b' \), then
\[ m(a \hat{\ominus} b) = m(a \oplus (a' \land b)) = m(a) + m(a' \land b) = m(a) + m(b); \]

hence by Corollary 4.18 and Definition 4.5, we conclude that each \( \sigma \)-additive state on a \( \sigma \)-complete lattice effect algebra \( L \) is a \( \sigma \)-additive state on \( M \subseteq L \), whenever \( M \) is a block of \( L \).

**Notation:** For any effect algebra \( L \),
\[ L_s = \{ a \in L : a \land a' = 0 \} \]

**Theorem 4.20** [22] Let \( L \) be a \( \sigma \)-complete lattice effect algebra. To every \( a \in L \), a \( \sigma \)-homomorphism \( \Lambda_a : \mathcal{B}[0, 1] \rightarrow L_s \) can be constructed such that the map \( a \mapsto \Lambda_a \) is one-to-one and for every \( \sigma \)-additive state \( m \) on \( L \) we have
\[
m(a) = \int_0^1 \lambda d(m \circ \Lambda_a) \]

**Proof.** By Theorem 4.13, every \( a \in L \) is contained in a block \( M \) of \( L \). By Corollary 4.18, \( M \) is a \( \sigma \)-MV-algebra. Hence, by Theorem 4.10, there exists a \( \sigma \)-homomorphism \( \Lambda_a : \mathcal{B}([0, 1]) \rightarrow \mathcal{B}(M) \) of Boolean \( \sigma \)-algebras such that for each \( \sigma \)-additive state \( m \) on \( M \), we have
\[
m(a) = \int_0^1 \lambda d(m \circ \Lambda_a). \quad (4.4)\]

Let \( m \) be a \( \sigma \)-additive state on \( L \). Then, by above note, \( m \) is a \( \sigma \)-additive state on \( M \). Hence each \( m \) on \( L \) satisfies (4.4). Finally \( \mathcal{B}(M) = L_s \cap M \). Indeed, if \( x \in \mathcal{B}(M) \), then \( x \oplus x = x \) and so \( x \oplus (x' \land x) = x \oplus 0 \), which implies \( x \land x' = 0 \); that is, \( x \in L_s \cap M \). Conversely, if \( x \in L_s \cap M \), then we have \( x \oplus x = x \oplus (x' \land x) = x \), so that \( x \in \mathcal{B}(M) \). Hence \( \Lambda_a : \mathcal{B}([0, 1]) \rightarrow L_s \cap M \subseteq L_s \). Therefore \( \Lambda_a : \mathcal{B}([0, 1]) \rightarrow L_s \). \( \square \)
CONCLUSION

The work presented in this thesis is concerned with some abstracting some results in spectral theory to the more general setting of an effect algebra without the (usual) aid of Hilbert space [4, 22]. These results form a generalization of the results that appear in [4]. Chapter 3 and 4 shed some light on this field.

Our proofs of some results in Chapter 3 are different from the proofs that appear in [4]. We use some general results on \( \sigma \)-effect algebras and we deduce our proofs from these results. Moreover, the general spectral results on an effect algebra enable us to derive a spectral Theorem on an orthoalgebra.

In Chapter 4, Theorem 4.20 depends on Theorem 4.16 and Theorem 4.10. The proof of Theorem 4.16 is different from the proof that appears in [7] and the references [22, 23] contain conclusive results concerning the Theorem 4.10 and the spectral measure. The important open questions that arise in this chapter and the previous chapter are the following:

(i) Is there other theorems in operator theory that can be abstracted to OMPs, orthoalgebras or effect algebras?

(ii) Is the spectral measure defined by (4.4) unique?
REFERENCES


