On Contra Continuous Functions

DECLARATION

The work provided in this thesis, unless otherwise referenced, is the researcher's own work, and has not been submitted elsewhere for any other degree or qualification.

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On Contra Continuous Functions In Topological Spaces

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On Contra Continuous Functions

After discussing the academic session held on the fourth of 24 Shaban 1435, the committee formed on the same day convened.

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And the committee decided to give this degree to her to work for the good of Allah and His servants and to serve her in

"Allah and His will."

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Abstract

In this thesis, we study contra continuous functions in topological spaces. We show that a function $f : X \rightarrow Y$ is contra continuous iff for each point $x \in X$ and each filter base $\Lambda$ in $X$ convergent to $x$, the filter base $f(\Lambda)$ is $c$-convergent to $f(x)$. Next, we introduce two independent classes of closed sets called $tgr$-closed sets and $t^*gr$-closed sets. We show that the class of $tgr$-closed sets contains all dense sets and regular closed sets and is contained in the class of $rg$-closed sets and $rwg$-closed sets. Also, we show that the class of $t^*gr$-closed sets contains all regular closed sets and is contained in the class of $gr$-closed sets and $swg$-closed sets. We show that these two classes are the same in a locally indiscrete space. Also, we study the topology generated by $tgr$-closed sets and show that the generated topology is contained in the space $\tau$ if and only if $(X, \tau)$ is $tgr$-locally indiscrete. Finally, we introduce and study new classes of contra continuous functions called contra $tgr$-continuous functions and contra $t^*gr$-closed sets.
Introduction

Continuous functions is of great importance in topological spaces. It has a fundamental role in compact spaces. One of the most important results in compact spaces is that the continuous image of a compact space is compact.

General topologists have introduced and investigated many different generalizations of continuous functions. One of the most significant of those notations is LC-continuity. Ganster and Reilly defined a function $f : (X, \tau) \rightarrow (Y, \sigma)$ to be LC-continuous if the preimage of every open set in $Y$ is locally closed in $X$.

In 1997, contra continuous functions were introduced by Dontchave as a dual notation of continuous functions. A function $f$ is called contra continuous if the preimage of every open set in the codomain is closed in the domain. Dontchave show that contra continuous images of spaces having a finite dense subset are compact, the spaces which have a finite dense subset is called strongly S-closed spaces. In 1999, Dontchave and Noiri show that the only contra continuous function on the real line is the constant one.

Generalized closed sets in topological spaces were first introduced by Levine in 1970. Later, new classes of closed sets were introduced, such as regular generalized, generalized regular, regular weakly, regular weakly generalized and $\pi$-generalized regular closed sets. This lead the topologists to define new classes of continuous and contra continuous functions.
The concept of contra $gr$-continuous functions was introduced by S.I Mahmood, also C.Janaki and Jeyanthi studied contra $\pi gr$-continuous functions. In 2014, M.Karpagadevi and A.Pushpalatha introduced a new class of functions called contra $rw$-continuous functions.

In this work, we introduce two independent classes of closed sets called $tgr$-closed sets and $t^*gr$-closed sets. We study the relations between these new classes and other different classes of closed sets. Also, we will study the properties of the topological spaces generated by these classes of closed sets.

Next, we introduce new classes of contra continuous functions by the means of $tgr$-closed sets, $t^*gr$-closed sets, which we called contra $tgr$-continuous and contra $t^*gr$-continuous functions.

This thesis consists of four chapters.

Chapter 1 contains two sections, in section 1 we give a basic informations for topological spaces which will be used in the remainder of the thesis. In chapter 2 , we will talk about contra continuous functions, this chapter is divided into 5 sections. In section 1 , we give a basic concepts about contra continuous functions, definitions, some remarks and examples. Section 2 is talking about the basic properties of contra continuous functions without considering any separation axioms on the spaces. In this section, we give a very important result which is a new characterization of contra continuous functions by means of convergence of a filter base. In section 3 , we study the properties that are preserved by contra continuous functions and some other properties. In section 4 , we talk about contra closed graphs and in section 5 , we define topologies called weak topology and strong topology. Chapter 3 is divided into three sections, section 1 is talking about $tgr$-closed sets, and section 2 is talking about $t^*gr$-closed sets, then in section 3 , we define and study the topology generated by these classes.
In chapter 4, we introduce new classes of contra continuous functions called contra $tgr$-continuous functions, contra $t^*gr$-continuous functions. This chapter is divided into two sections, in section 1 we will talk about contra $tgr$-continuous functions. In section 2, we will talk about contra $t^*gr$-continuous functions. We will study basic properties of each class and discuss it’s relations with other classes of contra continuous functions.
Chapter 1

Preliminaries

In this chapter, we give a basic informations which will be used in the remainder of this thesis.

1.1 Introduction to topological spaces

Definition 1.1.1. [20] A topology on a nonempty set $X$ is a collection $\tau$ of subsets of $X$ called the open sets, satisfying:

1. $X$ and $\phi$ belong to $\tau$
2. Any union of elements of $\tau$ belongs to $\tau$.
3. Any finite intersection of elements of $\tau$ belongs to $\tau$.

We say $(X, \tau)$ is a topological space, sometimes abbreviated $X$ is a topological space when no confusion can result about $\tau$.

Definition 1.1.2. [20] If $X$ is a topological space and $E \subseteq X$, we say that $E$ is closed iff $X - E$ is open. The set that is both open and closed is called clopen set.

Example 1.1.3. [20] For any set $X$ the collection $\tau = \{\phi, X\}$ forms a topology called
the trivial topology. The collection $\tau = P(X)$ the power set of $X$ forms a topology called the discrete topology.

**Definition 1.1.4.** [20] Let $X$ be any nonempty set. A topology $\tau$ on $X$ is called the cofinite topology if the closed subsets of $X$ are $X$ and all finite subsets of $X$, that is the open sets are $\phi$ and all subsets of $X$ which have finite complements.

**Definition 1.1.5.** [20] A topological space $(X,\tau)$ is called alexandroff space iff the arbitrary intersection of open sets is open, or equivalently, if the arbitrary union of closed sets is closed.

**Definition 1.1.6.** [20] If $X$ is a topological space, and $E \subseteq X$, then:

1. The closure of $E$ in $X$ is $cl(E) = \bigcap \{A \subseteq X : A$ is closed in $X$ and $E \subseteq A\}$
2. The interior of $E$ in $X$ is $int(E) = \bigcup \{U \subseteq X : U$ is open and $U \subseteq E\}$.

**Definition 1.1.7.** [20] If $X$ is a topological space and $x \in X$, a neighborhood (abbreviated nhood) of $x$ is a set $U$ which contains an open set $V$ containing $x$, or equivalently $U$ is a nhood of $x$ iff $x \in int(U)$. The collection $U_x$ of all nhoods of $x$ is called the nhood system at $x$.

**Definition 1.1.8.** [20] A nhood base at $x$ in $X$ is a subcollection $B_x$ taken from the nhood system $U_x$ having the property that each $U \in U_x$ contains some $V \in B_x$. That is $U_x$ is completely determined by $B_x$ as follows: $U_x = \{U \subseteq X : V \subseteq U$ for some $V \in B_x\}$.

**Example 1.1.9.** [20] The sets $(a,b)$ for $z > x$ form a nhood base at $x$ for a topology on the real line, this topology is called the Sorgenfrey line $E$.

**Definition 1.1.10.** [20] A topological space $X$ is called first countable iff every $x \in X$ has a countable nhood base.
**Definition 1.1.11.** [20] If \((X, \tau)\) is a topological space, a *base* for \(\tau\) (some times we call it a base for \(X\) when no confusion can result) is a collection \(\beta \subseteq \tau\) such that whenever \(G\) is open set in \(X\) and \(p \in G\), there is some \(B \in \beta\) such that \(p \in B \subseteq G\).

**Example 1.1.12.** [20] In \(\mathbb{R}\), the collection \(\beta\) of all open intervals is a base for the usual topology.

**Definition 1.1.13.** [20] If \((X, \tau)\) is a topological space, a *subbase* for \(\tau\) is a collection \(\mathcal{S} \subseteq \tau\) such that the collection of all finite intersections of elements from \(\mathcal{S}\) forms a base for \(\tau\).

**Example 1.1.14.** [20] In \(\mathbb{R}\), the family of sets of the form \((-\infty, a)\) together with those of the form \((b, \infty)\) is a subbase for the usual topology.

**Definition 1.1.15.** [20] A space \(X\) is called *connected* if there is no decomposition \(X = A \cup B\) such that \(A \cap B = \emptyset\) and \(A, B\) are both nonempty open sets in \(X\). It is easy to see that \(X\) is connected iff the only clopen subsets of \(X\) are \(\emptyset\) and \(X\).

**Definition 1.1.16.** [20] A subset \(A\) of a topological space \(X\) is called *dense* if \(\text{cl}(A) = X\).

**Definition 1.1.17.** [20], [3] If \(X, Y\) are topological spaces and \(f : X \to Y\) is a function, then:

1. \(f\) is *continuous* iff \(f^{-1}(V)\) is open in \(X\) for any open set \(V \subseteq Y\).
2. \(f\) is *open* (respectively *closed*) iff \(f(U)\) is open (closed) in \(Y\) for any open (closed) set \(U \subseteq X\).
3. \(f\) is *contra open* (respectively *contra closed*) iff \(f(U)\) is closed (open) in \(Y\) for any open (closed) set \(U \subseteq X\).

**Definition 1.1.18.** [20] If \((X, \tau)\) is a topological space and \(A \subseteq X\), the collection \(\tau_A = \{G \cap A : G \in \tau\}\) is a topology on \(A\), called the *relative topology of \(A\)*. This topological space is denoted by \((A, \tau_A)\).
Definition 1.1.19. [18], [5], [2], [19], [7] Let \( X \) be a topological space, a subset \( A \) of \( X \) is called:

1. **Regular open** iff \( A = \text{int} (\text{cl}(A)) \)

2. **Regular closed** iff \( A = \text{cl} (\text{int}(A)) \).

3. **Locally closed** if it can be represented as the intersection of an open set and a closed set.

4. **Semi open** iff \( A \subseteq \text{cl}(\text{int}(A)) \).

5. **Semi closed** iff \( \text{int}(\text{cl}(A)) \subseteq A \).

6. **\( \pi \)-open** iff \( A \) is a finite union of regular open sets.

7. **Regular semi open** if there is a regular open set \( U \) such that \( U \subseteq A \subseteq \text{cl}(U) \).

Definition 1.1.20. [20] If \((X, \tau)\) is a topological space, then \( RC(X) \) is the family of all regular closed sets in \((X, \tau)\).

Remark 1.1.21. [20] Every regular closed set is closed and every regular open set is open set.

Lemma 1.1.22. [20] (a) The union of two regular closed sets is regular closed.

(b) The intersection of two regular open sets is regular open.

Definition 1.1.23. [21] Let \((X, \tau)\) be a topological space, let \( A \subseteq X \) then:

(a) \( A \) is called \( t \)-set iff \( \text{int}(A) = \text{int}(\text{cl}(A)) \).

(b) \( A \) is called \( t^{*} \)-set iff \( \text{cl}(\text{int}(A)) = \text{cl}(\text{int}(A)) \).

Proposition 1.1.24. [21] Let \((X, \tau)\) be a topological space, let \( A \subseteq X \), then:

1. \( A \) is a \( t \)-set iff \( A \) is semi closed set.

2. \( A \) is a \( t^{*} \)-set iff \( A \) is semi open set.
Proposition 1.1.25. [4] Let $(X, \tau)$ be a topological space, let $A \subseteq X$ then:
(a) If $A$ is open, then $A$ is $t^*$-set.
(b) If $A$ is closed, then $A$ is $t$-set.

Proposition 1.1.26. [4] Let $(X, \tau)$ be a topological space, let $A \subseteq X$ then:
(a) Every regular open set is a $t$-set.
(b) Every regular closed set is a $t^*$-set.

Proposition 1.1.27. [20] If $X$ and $Y$ are topological spaces, then a nonempty subset $U \times V \subseteq X \times Y$ is regular open iff both $U$, $V$ are regular open.

Proposition 1.1.28. [20] If $X$ and $Y$ are topological spaces, then a nonempty subset $U \times V \subseteq X \times Y$ is regular closed iff both $U$, $V$ are regular closed.

Definition 1.1.29. [20] A family of subsets of a topological space is called *locally finite* iff each point of the space has a neighborhood meeting only finitely many elements of the family.

Lemma 1.1.30. [10] If $\{A_\alpha : \alpha \in \Delta\}$ is a locally finite family of closed sets, then $\bigcup_{\alpha \in \Delta} A_\alpha$ is closed.

Definition 1.1.31. [13] A topological space $(X, \tau)$ is called extremally disconnected iff every regular closed set in $X$ is open.

Definition 1.1.32. [5] A topological space $(X, \tau)$ is called locally indiscrete if every open set is closed.

Definition 1.1.33. [20], [2] A topological space $X$ is called:
(1) $T_\circ$-space iff whenever $x$ and $y$ are distinct points in $X$, there is an open set containing one and not the other.
(2) $T_1$-space iff whenever $x$ and $y$ are distinct points in $X$, there is a neighborhood of each not
containing the other.

(3) **$T_2$-space** (Hausdorff space) iff whenever $x$ and $y$ are distinct points of $X$, there are disjoint open sets $U$ and $V$ in $X$ with $x \in U$ and $y \in V$.

(4) **Regular space** iff whenever $A$ is closed in $X$ and $x \notin A$, then there are disjoint open sets $U$ and $V$ with $x \in U$ and $A \subseteq V$.

(5) **Normal space** iff whenever $A$ and $B$ are disjoint closed sets in $X$, there are disjoint open sets $U$ and $V$ with $A \subseteq U$ and $B \subseteq V$.

(6) **Urysohn** space iff whenever $x$ and $y$ are distinct points in $X$, there are nhoods $U$ of $x$ and $V$ of $y$ with $\text{cl}(U) \cap \text{cl}(V) = \emptyset$.

(7) **Ultra Hausdorff** iff whenever $x$ and $y$ are distinct points, there exist disjoint closed sets $U$ and $V$ containing $x$ and $y$ respectively.

(8) **Weakly Hausdorff** iff whenever $x$ and $y$ are distinct points, there exist regular closed sets $U$ and $V$ such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$.

(9) **Ultra Normal** iff whenever $A$ and $B$ are disjoint closed sets in $X$, there are disjoint clopen sets $U$ and $V$ with $A \subseteq U$ and $B \subseteq V$.

(10) **Hyperconnected** iff each nonempty open set is dense, or equivalently the only nonempty regular closed set is $X$.

(11) **Ultra connected** iff every two nonvoid closed subsets of $X$ intersects.

**Remark 1.1.34.** Every $T_2$-space is $T_1$-space and every $T_1$-space is $T_\circ$.

**Definition 1.1.35.** [20], [5], [13] A topological space $(X, \tau)$ is called:

(1) **Compact** iff every open cover for $X$ has a finite subcover.

(2) **Lindeloff** iff every open cover for $X$ has a countable subcover.

(3) **Almost compact** iff every open cover has a finite proximate subcover (subfamily the closures of whose members cover $X$).
(4) **S-closed** iff every regular closed cover of $X$ has a finite subcover.

(5) **Strongly S-closed** iff every closed cover of $X$ has a finite subcover.

(6) **$S$-Lindelöf** iff every regular closed cover of $X$ has a countable subcover.

(7) **Countably compact** iff every countable open cover of $X$ has a finite subcover.

(8) **Countably S-closed** iff every countable regular closed cover of $X$ has a finite subcover.

**Remark 1.1.36.** Strongly S-closed spaces are S-closed, since regular closed sets are closed.

**Theorem 1.1.37.** [20] The following are equivalent for a topological space $X$ :

(a) $X$ is regular.

(b) If $U$ is open in $X$ and $x \in U$, there is an open set $V$ containing $x$ such that $\text{cl}(V) \subseteq U$.

(c) Each $x \in X$ has a nhood base consisting of closed sets.

**Definition 1.1.38.** [20] A filter $F$ on a set $S$ is a nonempty collection of nonempty subsets of $S$ with the properties:

(a) if $F_1, F_2 \in F$, then $F_1 \cap F_2 \in F$,

(b) if $F \in F$ and $F \subseteq F^*$, then $F^* \in F$.

**Definition 1.1.39.** [20] A subcollection $F_0$ of $F$ is a filter base for $F$ iff each element of $F$ contains some element of $F_0$.

**Definition 1.1.40.** [20] A nonempty collection $\zeta$ of nonempty subsets of $S$ is a filter base for some filter on $S$ iff :

(a) if $C_1, C_2 \in \zeta$, then $C_3 \subset C_1 \cap C_2$ for some $C_3 \in \zeta$.

**Example 1.1.41.** [20] Let $X$ be any set and $A \subseteq X$, then $\{F \subseteq X : A \subseteq F\}$ is a filter on $X$.

**Example 1.1.42.** [20] Let $(X, \tau)$ be any topological space, let $x \in X$, then:

$\Lambda = \{U \subseteq X : U \in \tau \text{ and } x \in U\}$ is a filter base for some filter on $X$. 

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Remark 1.1.43. If $\Lambda$ is a filter base for a filter $F$ on $X$ and $f : X \to Y$, then $f(\Lambda)$ is a filter base for $f(F)$.

Definition 1.1.44. [20] Let $(X, \tau)$ be a topological space. A filter base $\Lambda$ converges to $x \in X$ iff each open set $U$ contains $x$ must contain some $B \in \Lambda$.

Example 1.1.45. [20] The filter base $\Lambda = \{U \subseteq X : U \in \tau$ and $x \in U\}$ converges to $x$ since any open set $U$ contains $x$ belongs to $\Lambda$.

Definition 1.1.46. [15] Let $(X, \tau)$ be a topological space. A filter base $\Lambda$ is called c-convergent to a point $x \in X$ if for each closed set $U$ contains $x$, there exists $B \in \Lambda$ such that $B \subseteq U$.

Definition 1.1.47. [20] Let $X, Y$ be topological spaces and $f : X \to Y$ be a function, then the graph function $g : X \to X \times Y$ is defined by $g(x) = (x, f(x))$.

### 1.2 Generalized closed sets

In this section we give a brief summary about different classes of closed sets, different classes of continuous functions and then we summarize the relations between them.

Definition 1.2.1. [18] The regular closure of a set $A \subseteq X$ is the set :

$$rcl(A) = \bigcap \{F \subseteq X : F \text{ is regular closed and } A \subseteq F\}.$$ 

Theorem 1.2.2. [18] Let $(X, \tau)$ be a topological space, then for any set $A \subseteq X$, $cl(A) \subseteq rcl(A)$.

Theorem 1.2.3. [2] Let $(X, \tau)$ be any topological space, and $A, B$ two subsets of $X$, then $rcl(A \cup B) = rcl(A) \cup rcl(B)$. 

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Definition 1.2.4. [18], [6] Let $X$ be a topological space, a subset $A$ of $X$ is called:

(1) A generalized closed set (briefly g-closed) if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$.

(2) A regular generalized closed set (rg-closed) if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is regular open in $X$.

(3) A generalized regular closed set (gr-closed) if $\text{rcl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$.

(4) $\text{Pre}$-closed if $\text{cl}(\text{int}(A)) \subseteq A$.

(5) $\alpha$-closed if $\text{cl}(\text{int}(\text{cl}(A))) \subseteq A$.

(6) Semi $\text{pre}$-closed set if $\text{int}(\text{cl}(\text{int}(A))) \subseteq A$.

(7) Weakly-closed (w-closed) if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is semi open.

(8) Weakly generalized closed (wg-closed) if $\text{cl}(\text{int}(A)) \subseteq U$ whenever $A \subseteq U$ and $U$ is open.

(9) Semi weakly generalized (swg-closed) if $\text{cl}(\text{int}(A)) \subseteq U$ whenever $A \subseteq U$ and $U$ is semi open.

(10) Regular weakly generalized (rwg-closed) iff $\text{cl}(\text{int}(A)) \subseteq U$ whenever $A \subseteq U$ and $U$ is regular open.

(11) Regular weakly closed (rw-closed) if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is regular semi open.

(12) $\pi gr$-closed iff $\text{rcl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\pi$-open.

Theorem 1.2.5. [19], [17], [6] In any topological space $X$, we have:

(1) Every $\text{gr}$-closed set is $\text{g}$-closed set.

(2) Every $\text{g}$-closed set is $\text{rg}$-closed set.

(3) Every $\text{w}$-closed set is $\text{rw}$-closed set.

(4) Every $\text{w}$-closed set is $\text{wg}$-closed set.
(5) Every $w$-closed set is $g$-closed set.

(6) Every $rw$-closed set is $rwg$-closed set.

(7) Every $gr$-closed set is $\pi gr$-closed set.

**Definition 1.2.6.** [2], [5], [18] A function $f : X \to Y$ is called:

1. *$g$-continuous* iff $f^{-1}(V)$ is $g$-closed set in $X$ for any open set $V \subseteq Y$.
2. *$gr$-continuous* iff $f^{-1}(V)$ is $gr$-closed set in $X$ for any open set $V \subseteq Y$.
3. *$rg$-continuous* iff $f^{-1}(V)$ is $rg$-closed set in $X$ for any open set $V \subseteq Y$.
4. *RC-continuous* iff $f^{-1}(V)$ is regular-closed set in $X$ for any open set $V \subseteq Y$.
5. *Perfectly continuous* iff $f^{-1}(V)$ is clopen set in $X$ for any open set $V \subseteq Y$.
6. *LC-continuous* iff $f^{-1}(V)$ is locally closed set in $X$ for any open set $V \subseteq Y$.
7. *Regular set connected* iff $f^{-1}(V)$ is clopen set in $X$ for any regular open set $V \subseteq Y$.

**Remark 1.2.7.** [18], [16], [12], [2] (1) Every $gr$-continuous function is $g$-continuous, since every $gr$-closed set is $g$-closed.

(2) Every $g$-continuous function is $rg$-continuous, since every $g$-closed set is $rg$-closed.

(3) Every continuous function is $LC$-continuous, since every open set is locally closed.

(4) Every perfectly continuous function is continuous, since every clopen set is open.

(5) Every $RC$-continuous function is gr-continuous, since every regular closed set is gr-closed.
Chapter 2

Contra Continuous Functions

2.1 Basic Concepts

In this section we study some examples of contra continuous functions and almost contra continuous functions and some remarks.

Definition 2.1.1. [5] A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called contra continuous if $f^{-1}(U)$ is closed in $(X, \tau)$ for each open set $U$ in $(Y, \sigma)$.

Theorem 2.1.2. [5] For a function $f : (X, \tau) \rightarrow (Y, \sigma)$, the following conditions are equivalent:

1) $f$ is contra continuous.

2) For each $x \in X$ and each closed set $V$ in $Y$ with $f(x) \in V$, there exists an open set $U$ in $X$ such that $x \in U$ and $f(U) \subseteq V$.

3) The inverse image of each closed set in $Y$ is open in $X$.

Proof. (1 $\rightarrow$ 2) Suppose that $f$ is contra continuous and let $V$ be closed set in $Y$ with $f(x) \in V$. Since $V$ is closed, $(Y - V)$ is open, so $f^{-1}(Y - V) = X - f^{-1}(V)$ is closed in $X$, ie $f^{-1}(V)$ is open in $X$, let $U = f^{-1}(V)$, then (2) follows.
(2 → 3) Let \( V \) be any closed set in \( Y \), let \( x \in f^{-1}(V) \), then \( f(x) \in V \), from (2) \( \exists \) an open set \( U_x \) such that \( x \in U_x \) and \( f(U_x) \subseteq V \), i.e, \( x \in U_x \subseteq f^{-1}(V) \), therefor, \( f^{-1}(V) \) is open.

(3 → 1) Let \( U \) be open set in \( Y \), then \( Y - U \) is closed set, hence \( f^{-1}(Y - U) = X - f^{-1}(U) \) is open set in \( X \). Therefor, \( f^{-1}(U) \) is closed in \( X \) and \( f \) is contra continuous.

Example 2.1.3.

(a) Every constant function is contra continuous.

(b) If \((X, \tau)\) is discrete and \((Y, \sigma)\) is an arbitrary topological space, then any function \( f : X \to Y \) is contra continuous. Again, if \((X, \tau)\) is any arbitrary topological space, and \((Y, \sigma)\) is trivial, then any mapping \( g : X \to Y \) is contra continuous.

Example 2.1.4. Let \( X = \{a, b\} \) be the Sierpinski space by setting \( \tau = \{\phi, \{a\}, X\} \) and \( \sigma = \{\phi, \{b\}, X\} \). The identity function \( f : (X, \tau) \to (X, \sigma) \) is contra continuous.

Example 2.1.5. The characteristic function of a subset \( A \) of \( X \) defined by:

\[
    f(x) = \begin{cases} 
        1, & \text{if } x \in A \\
        0, & \text{if } x \notin A 
    \end{cases}
\]

\( f : X \to \mathbb{R} \) is contra continuous iff \( A \) is both open and closed in \( X \).

Proof. Suppose that \( A \) is both open and closed, let \( U \subseteq \mathbb{R} \) be any arbitrary open set,

\[
    f^{-1}(U) = \begin{cases} 
        X, & \text{if } \{0, 1\} \subseteq U \\
        A, & \text{if } 1 \in U \text{ and } 0 \notin U \\
        X - A, & \text{if } 0 \in U \text{ and } 1 \notin U \\
        \phi, & \text{if } 0 \notin U \text{ and } 1 \notin U 
    \end{cases}
\]

In all cases, \( f^{-1}(U) \) is closed set in \( X \), hence \( f \) is contra continuous.

Conversely, suppose that \( f \) is contra continuous, then \( f^{-1}(\{1\}) = A \) is open set in \( X \) (since \( \{1\} \) is closed set in \( \mathbb{R} \)), also \( f^{-1}(\{0\}) = X - A \) is open, hence \( A \) is closed in \( X \). Therefor, \( A \) is both open and closed in \( X \). \( \Box \)
Example 2.1.6. Let \((X, \tau)\) be the Sorgenfrey line topology defined in example (1.1.9), let \((Y, \sigma)\) be the integers with the cofinite topology, let \(f : (X, \tau) \to (Y, \sigma)\) defined by \(f(x) = \lfloor x \rfloor\), then \(f\) is contra continuous function.

Proof. To prove this, let \(A\) be any closed set in \((Y, \sigma)\). If \(A = Y\) then \(f^{-1}(A) = X\) which is open. Now suppose that \(A\) is any proper nonempty closed set, then \(A\) is finite, say, \(A = \{a_1, a_2, \ldots, a_n\}\) then \(f^{-1}(A) = \bigcup_{i=1}^{n} [a_i, a_i + 1)\) which is open in \(X\). Hence, \(f\) is contra continuous.

Example 2.1.7. [5] Contra continuity and continuity are independent. The identity function on the real line with the usual topology is continuous but not contra continuous since \((0, 1)\) is open in \(\mathbb{R}\) but \(f^{-1}(0, 1) = (0, 1)\) is not closed in \(\mathbb{R}\). However, a contra continuous function need not be continuous. For example, let \(X = \{a, b\}\) be the Sierpinski space by setting \(\tau = \{\emptyset, \{a\}, X\}\) and \(\sigma = \{\emptyset, \{b\}, X\}\). The identity function \(f : (X, \tau) \to (X, \sigma)\) is contra continuous but not continuous since \(\{b\}\) is open in \((X, \sigma)\) but \(f^{-1}(\{b\}) = \{b\}\) which is not open in \((X, \tau)\).

Definition 2.1.8. [5] A function \(f : X \to Y\) is said to be almost contra continuous if \(f^{-1}(V)\) is closed in \(X\) for every regular open set \(V\) of \(Y\).

Theorem 2.1.9. For a function \(f : (X, \tau) \to (Y, \sigma)\) the following conditions are equivalent:

1. \(f\) is almost contra continuous.

2. The inverse image of each regular closed set in \(Y\) is open in \(X\).

Proof. Follows directly from definition (2.1.8). □

Remark 2.1.10. Every contra continuous function is almost contra continuous, since every regular open set is open set, but almost contra continuous functions need not be contra continuous as shown in the following example.
**Example 2.1.11.** Consider the identity function $f : \mathbb{R} \to (\mathbb{R}, \tau)$ where $\tau$ is the cofinite topology, then $f$ is almost contra continuous function but not contra continuous. To prove this, notice that the only regular open sets in the cofinite topology are $\mathbb{R}$ and $\phi$, $f^{-1}(\phi) = \phi$ and $f^{-1}(\mathbb{R}) = \mathbb{R}$, in any case the inverse image of regular open set is closed, so $f$ is almost contra continuous. On the other hand, $\{0\}$ is closed in the cofinite topology but $f^{-1}\{0\} = \{0\}$ which is not open in the usual topology, hence $f$ is not contra continuous.

**Remark 2.1.12.** [5] The composition of two contra continuous functions need not be contra continuous as shown in the following example.

**Example 2.1.13.** [5] Let $X = \{a, b\}$ be the Sierpinski space and set $\tau = \{\phi, \{a\}, X\}$ and $\sigma = \{\phi, \{b\}, X\}$. The identity functions $f : (X, \tau) \to (X, \sigma)$ and $g : (X, \sigma) \to (X, \tau)$ are both contra continuous but their composition is the identity function $g \circ f : (X, \tau) \to (X, \tau)$ is not contra continuous, since $\{a\}$ is open but $(g \circ f)^{-1}\{a\} = \{a\}$ which is not closed in $(X, \tau)$.

**Remark 2.1.14.** General topologists have introduced different classes of contra continuous functions by the means of different classes of closed sets.

**Definition 2.1.15.** [18], [12], [2] A function $f : (X, \tau) \to (Y, \sigma)$ is called:

1. Contra $gr$-continuous iff $f^{-1}(V)$ is $gr$-closed in $(X, \tau)$ whenever $V$ is open in $(Y, \sigma)$.
2. Contra $rg$-continuous iff $f^{-1}(V)$ is $rg$-closed in $(X, \tau)$ whenever $V$ is open in $(Y, \sigma)$.
3. Contra $RW$-continuous iff $f^{-1}(V)$ is $RW$-closed in $(X, \tau)$ whenever $V$ is open in $(Y, \sigma)$.
4. Contra $\pi gr$-continuous iff $f^{-1}(V)$ is $\pi gr$-closed in $(X, \tau)$ whenever $V$ is open in $(Y, \sigma)$.
5. Contra $rwg$-continuous iff $f^{-1}(V)$ is $rwg$-closed in $(X, \tau)$ whenever $V$ is open in $(Y, \sigma)$.
6. Contra $wg$-continuous iff $f^{-1}(V)$ is $wg$-closed in $(X, \tau)$ whenever $V$ is open in $(Y, \sigma)$.
2.2 Basic properties of contra continuous functions

In this section we study the relations between contra continuous functions and some classes of functions. Also, we study some basic properties of contra continuous functions. Finally, we give a new characterization of contra continuity by means of convergence.

**Theorem 2.2.1.** [5] Every contra continuous function is LC continuous.

*Proof.* The result follows from the fact that every closed set is locally closed. □

**Remark 2.2.2.** [5] An LC continuous function need not be contra continuous. For a counter example let $f$ be the identity function on the real line with the usual topology. If $U$ is any open set, then $f^{-1}(U) = U$ which is open and hence locally closed, so $f$ is LC continuous. However, $(0, 1)$ is open in the usual topology, but $f^{-1}(0, 1) = (0, 1)$ which is not closed in the usual topology, hence $f$ is not contra continuous.

**Theorem 2.2.3.** Every RC-continuous function is contra continuous function.

*Proof.* Follows from the fact that every regular closed set is a closed set. □

**Remark 2.2.4.** Contra continuous functions need not be RC-continuous. For a counter example, let $X = \{a, b\}$ and $\tau = \{X, \phi, \{a\}\}$, define $f : (X, \tau) \rightarrow (X, \tau)$ by $f(a) = b, f(b) = a$, then $f$ is contra continuous function but it is not RC-continuous since $\{a\}$ is open but $f^{-1}\{a\} = \{b\}$ is not regular closed.

**Theorem 2.2.5.** [5] Every perfectly-continuous function is contra continuous function.

*Proof.* Follows from the fact that every clopen set is regular closed set and every regular closed set is closed. □

**Remark 2.2.6.** Contra continuous functions need not be perfectly-continuous. For a counter example, let $X = \{a, b\}$ and $\tau = \{X, \phi, \{a\}\}$, define $f : (X, \tau) \rightarrow (X, \tau)$ by
f(a) = b, f(b) = a, then f is contra continuous function but it is not perfectly-continuous since \{a\} is open but \(f^{-1}\{a\} = \{b\}\) is not clopen.

**Theorem 2.2.7.** Let \((X, \tau)\) be an alexandroff space and let \(\beta\) be a base for a topological space \((Y, \sigma)\), then \(f : (X, \tau) \to (Y, \sigma)\) is contra continuous function if and only if \(f^{-1}(B)\) is closed in \(X\) for any \(B \in \beta\).

*Proof. (⇒) It is clear since any basic open set is open, ie \(\beta \subseteq \sigma\).

(⇐) Suppose that \(f^{-1}(B)\) is closed in \(X\) for any \(B \in \beta\). Let \(V\) be any open set in \(Y\), then \(V = \bigcup_{\alpha \in \Delta} B_\alpha\), where \(B_\alpha \in \beta\) for any \(\alpha \in \Delta\), so \(f^{-1}(B_\alpha)\) is closed in \(X\) for each \(\alpha \in \Delta\), but \(X\) is an alexandroff space, so \(\bigcup_{\alpha \in \Delta} f^{-1}(B_\alpha) = f^{-1}(\bigcup_{\alpha \in \Delta} B_\alpha) = f^{-1}(V)\) is closed in \(X\). Hence, \(f\) is contra continuous. \(\square\)

**Proposition 2.2.8.** \(X\) has the discrete topology iff whenever \(Y\) is a topological space and \(f : X \to Y\) is a function, then \(f\) is contra continuous.

*Proof. Suppose that \(X\) has the discrete topology, let \(Y\) be any topological space and \(f : X \to Y\) any function. Let \(V \subseteq Y\) be any open set, then \(f^{-1}(V)\) is closed set in \(X\) since \(X\) has the discrete topology. Conversely, suppose that any function \(f : X \to Y\) is contra continuous for any topological space \(Y\). Let \(Y\) be the space \(X\) with the discrete topology, and \(f\) is the identity function, let \(A\) be arbitrary subset of \(X\), then \(A\) is closed in the discrete topology and hence \(f^{-1}(A) = A\) is open set in \(X\), so \(X\) has the discrete topology. \(\square\)

**Proposition 2.2.9.** \(X\) has the trivial topology iff whenever \(Y\) is a topological space and \(f : Y \to X\), then \(f\) is contra continuous.

*Proof. Suppose that \(X\) has the trivial topology and \(f : Y \to X\) is any function where \(Y\) is any topological space. Let \(A \subseteq X\) be any open set, then \(A = X\) or \(A = \emptyset\) and
$f^{-1}(A) = Y$ or $\phi$. In any case, $f^{-1}(A)$ is closed in $Y$, hence $f$ is contra continuous. Conversely, suppose that for any topological space $Y$, and any $f : Y \to X$ we have that $f$ is contra continuous. Let $Y$ be $X$ with the trivial topology and $f$ be the identity function, let $A$ be any open subset of $X$, then $f^{-1}(A) = A$ is closed in the trivial topology, this implies that $A = \phi$ or $A = X$, hence $X$ has the trivial topology.

Proposition 2.2.10. Let $(X, \tau)$ be a topological space, then the identity function $I : (X, \tau) \to (X, \tau)$ is contra continuous if and only if $(X, \tau)$ is locally indiscrete.

Proof. Suppose that $I : (X, \tau) \to (X, \tau)$ is contra continuous. Let $V$ be any open set in $(X, \tau)$, then $I^{-1}(V) = V$ is closed (since $I$ is contra continuous), hence $(X, \tau)$ is locally indiscrete. Conversely, suppose that $(X, \tau)$ is locally indiscrete. Let $V$ be any open set in $(X, \tau)$, so $V$ is closed (since $(X, \tau)$ is locally indiscrete), then $I^{-1}(V) = V$ is closed, hence $I$ is contra continuous.

Theorem 2.2.11. [5] A contra continuous function $f : X \to Y$ is continuous when $X$ is locally indiscrete.

Proof. Suppose that $(X, \tau)$ is locally indiscrete and $f$ is contra continuous function, let $V \subseteq Y$ be any open set, then $f^{-1}(V)$ is closed (since $f$ is contra continuous), hence $f^{-1}(V)$ is open since $(X, \tau)$ is locally indiscrete, hence $f$ is continuous.

Theorem 2.2.12. [5] Let $f : X \to Y$ be a surjective closed contra continuous function, then $Y$ is locally indiscrete.

Proof. Let $V$ be an open set of $Y$. Since $f$ is contra continuous, $f^{-1}(V)$ is closed in $X$. Let $f^{-1}(V) = U$. Then, $V = f(U)$ is closed since $f$ is closed.

In section (2.1) we said that the composite of two contra continuous functions need not be contra continuous, but in the following theorem we show that their composite
is continuous. Also, we show that the composite of a contra continuous function and a continuous function is contra continuous.

**Theorem 2.2.13.** [5] Let \( f : (X, \tau) \to (Y, \sigma) \) and \( g : (Y, \sigma) \to (Z, \nu) \) be two functions, then:

(i) \( g \circ f \) is contra continuous, if \( g \) is continuous and \( f \) is contra continuous.

(ii) \( g \circ f \) is contra continuous, if \( g \) is contra continuous and \( f \) is continuous.

(iii) \( g \circ f \) is contra continuous, if \( f \) and \( g \) are continuous and \( Y \) is locally indiscrete.

(iv) \( g \circ f \) is continuous, if both \( g \) and \( f \) are contra continuous.

**Proof.** (i) Let \( V \) be an open set in \( (Z, \nu) \), then \( g^{-1}(V) \) is open set in \( (Y, \sigma) \) since \( g \) is continuous, and so \( f^{-1}(g^{-1}(V)) \) is closed set in \( (X, \tau) \) since \( f \) is contra continuous, hence \( (g \circ f)^{-1}(V) \) is closed set in \( (X, \tau) \), therefor \( g \circ f \) is contra continuous.

(iii) Let \( V \) be an open set in \( (Z, \nu) \), then \( g^{-1}(V) \) is open set in \( (Y, \sigma) \) since \( g \) is continuous, and hence \( g^{-1}(V) \) is closed in \( Y \) since \( Y \) is locally indiscrete, therefor, \( f^{-1}(g^{-1}(V)) \) is closed set in \( (X, \tau) \) since \( f \) is continuous, hence \( (g \circ f)^{-1}(V) \) is closed set in \( (X, \tau) \), therefor \( g \circ f \) is contra continuous.

(iv) Let \( V \subseteq Z \) be open set, then \( g^{-1}(V) \) is closed in \( (Y, \sigma) \) since \( g \) is contra continuous, so \( f^{-1}(g^{-1}(V)) \) is open in \( (X, \tau) \) since \( f \) is contra continuous, ie ,\( (g \circ f)^{-1}(V) \) is open in \( (X, \tau) \). Hence, \( g \circ f \) is continuous.

**Theorem 2.2.14.** Let \( f : X \to Y \) and \( g : Y \to Z \) be two functions. If \( f \) is a closed surjective function and \( g \circ f \) is contra continuous, then \( g \) is contra continuous.

**Proof.** Let \( V \) be any open set in \( Z \), then \( (g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V)) \) is closed in \( X \). But \( f \) is a closed surjective function, so \( f(f^{-1}(g^{-1}(V))) = g^{-1}(V) \) is closed in \( Y \). Hence, \( g \) is contra continuous.
In the following three theorems we study the restriction and extension properties of contra continuous functions, also we give some related examples.

**Theorem 2.2.15.** If \( f : (X, \tau) \to (Y, \sigma) \) is contra continuous, and \( A \subseteq X \), then \((f|_A) : (A, \tau_A) \to (Y, \sigma)\) is contra continuous.

*Proof.* Let \( V \subseteq Y \) be an open set, then \((f|_A)^{-1}(V) = f^{-1}(V) \cap A\), but \( f^{-1}(V) \) is closed set in \( X \), so \((f|_A)^{-1}(V)\) is closed in \((A, \tau_A)\). Hence, \((f|_A)\) is contra continuous. \(\square\)

**Theorem 2.2.16.** *(Pasting Lemma for Contra continuous functions)*

Let \( X = A \cup B \) be a topological space with a topology \( \tau \) and \( Y \) be a topological space with a topology \( \sigma \). Let \( f : (A, \tau_A) \to (Y, \sigma) \) and \( g : (B, \tau_B) \to (Y, \sigma) \) be contra continuous maps such that \( f(x) = g(x) \) \( \forall x \in A \cap B \). Suppose \( A \) and \( B \) are both open (or both closed) sets in \( X \), then the function \( h : (X, \tau) \to (Y, \sigma) \) defined by \( h(x) = f(x) \) for \( x \in A \) and \( h(x) = g(x) \) for \( x \in B \) is contra continuous.

*Proof.* Let \( V \subseteq Y \) be a closed set, then \( h^{-1}(V) = f^{-1}(V) \cup g^{-1}(V) \), since \( f \) and \( g \) are contra continuous on \((A, \tau_A)\) and \((B, \tau_B)\) respectively, then both \( f^{-1}(V) \) and \( g^{-1}(V) \) are open sets in \((A, \tau_A)\) and \((B, \tau_B)\) respectively. But both \( A \) and \( B \) are open subspaces of \((X, \tau)\), therefor both \( f^{-1}(V) \) and \( g^{-1}(V) \) are open sets in \((X, \tau)\) and so \( h^{-1}(V) \) is open in \((X, \tau)\). Hence, \( h \) is contra continuous. \(\square\)

**Remark 2.2.17.** The openness condition in the above theorem cannot be dropped as shown in the following example.

**Example 2.2.18.** Consider \( \mathbb{R} \) with the usual topology, \( \mathbb{R} = \mathbb{Z} \cup H \) where \( H = \bigcup_{n \in \mathbb{Z}} (n, n+1) \). Define \( f_Z : \mathbb{Z} \to \mathbb{R}^1 \) by \( f(n) = 0 \) \( \forall n \in \mathbb{Z} \) and \( f_H : H \to \mathbb{R}^1 \) by \( f(h) = 1 \) \( \forall h \in H \), then both \( f_H \) and \( f_Z \) are contra continuous functions (since any constant function is contra
continuous, but if we define \( f : \mathbb{R} \rightarrow \mathbb{R} \) by
\[
f(x) = \begin{cases} 
0, & \text{if } x \in \mathbb{Z} \\
1, & \text{if } x \in H
\end{cases}
\]
then \( f \) is not contra continuous since \( \{0\} \) is closed in \( \mathbb{R} \) but \( f^{-1}\{0\} = \mathbb{Z} \) is not open, this does not contradict the above theorem since \( H \) is open but \( \mathbb{Z} \) is not. Also, \( \mathbb{Z} \) is closed but \( H \) is not.

**Example 2.2.19.** Consider \( \mathbb{R} \) with the usual topology, \( \mathbb{R} = \mathbb{Q} \cup \mathbb{P} \), define \( f_\mathbb{Q} : \mathbb{Q} \rightarrow \mathbb{R} \) by \( f(q) = 0 \) \( \forall q \in \mathbb{Q} \), and \( f_\mathbb{P} : \mathbb{P} \rightarrow \mathbb{R} \) by \( f(p) = 1 \) \( \forall p \in \mathbb{P} \), then both \( f_\mathbb{Q} \) and \( f_\mathbb{P} \) are contra continuous but if we defined \( f : \mathbb{R} \rightarrow \mathbb{R} \) by
\[
f(x) = \begin{cases} 
0, & \text{if } x \in \mathbb{Q} \\
1, & \text{if } x \in \mathbb{P}
\end{cases}
\]
then \( f \) is not contra continuous since \( \{0\} \) is closed but \( f^{-1}\{0\} = \mathbb{Q} \) which is not open. This does not contradict the above theorem since \( \mathbb{P} \) and \( \mathbb{Q} \) are not open sets, also not closed.

**Theorem 2.2.20.** Let \( X = \bigcup_{\alpha \in \Delta} A_\alpha \) and \( \{A_\alpha : \alpha \in \Delta\} \) be a locally finite covering of closed sets. Let \( f_\alpha : A_\alpha \rightarrow Y \) be contra continuous for all \( \alpha \in \Delta \) such that \( f_\alpha(x) = f_\beta(x) \) for all \( x \in A_\alpha \cap A_\beta \). Define \( f(x) = f_\alpha(x) \) for \( x \in A_\alpha \), then \( f \) is contra continuous.

**Proof.** Let \( F \) be an open set in \( Y \), then \( f^{-1}(F) = \bigcup_{\alpha \in \Delta} f_\alpha^{-1}(F) \). Since \( f_\alpha \) is contra continuous in \( A_\alpha \), so \( f_\alpha^{-1}(F) \) is closed in \( A_\alpha \) for all \( \alpha \in \Delta \), therefor, \( f_\alpha^{-1}(F) \) is closed in \( X \) for all \( \alpha \in \Delta \), now since \( f_\alpha^{-1}(F) \subseteq A_\alpha \) for all \( \alpha \in \Delta \) and \( \{A_\alpha : \alpha \in \Delta\} \) is locally finite, then \( \{f_\alpha^{-1}(F) : \alpha \in \Delta\} \) is also a locally finite family, using lemma (1.1.30) we get, \( f^{-1}(F) = \bigcup_{\alpha \in \Delta} f_\alpha^{-1}(F) \) is closed in \( X \) and \( f \) is contra continuous.

**Remark 2.2.21.** In the above theorem, the condition that the family is locally finite cannot be dropped as shown in the following example.
**Example 2.2.22.** Consider \( \mathbb{R} \) with the usual topology, consider the family of closed subsets of \( \mathbb{R} \) \( \{ \{r\} : r \in \mathbb{R} \} \). Let \( f_r : \{r\} \to \mathbb{R} \) be the constant function, each function \( f_r \) is contra continuous, but if we defined the function \( f : \mathbb{R} \to \mathbb{R} \) to be the identity function, then \( f \) is not contra continuous since \((0, 1)\) is open in \( \mathbb{R} \) but \( f^{-1}(0, 1) = (0, 1) \) which is not closed in \( \mathbb{R} \). The family \( \{ \{r\} : r \in \mathbb{R} \} \) is not locally finite, notice that any nhhood about 0 meets infinitely many elements of the family since each interval contains infinite number of real numbers.

The following theorems are about almost contra continuous functions on a product of two spaces.

**Theorem 2.2.23.** Let \( f : X_1 \times X_2 \to Y_1 \times Y_2 \) defined by \( f(x_1, x_2) = (f_1(x_1), f_2(x_2)) \), where \( f_1 : X_1 \to Y_1 \) and \( f_2 : X_2 \to Y_2 \) and \( X_1, X_2, Y_1, Y_2 \) are topological spaces. Then, \( f \) is almost contra continuous iff both \( f_1 \) and \( f_2 \) are almost contra continuous.

*Proof.* Suppose that both \( f_1 \) and \( f_2 \) are almost contra continuous. Let \( U_1 \times U_2 \subseteq Y_1 \times Y_2 \) be a nonempty regular open set, then by proposition (1.1.27) both \( U_1 \) and \( U_2 \) are nonempty regular open sets in \( Y_1, Y_2 \) respectively. Now, \( f^{-1}(U_1 \times U_2) = f_1^{-1}(U_1) \times f_2^{-1}(U_2) \), but \( f_1^{-1}(U_1) \) and \( f_2^{-1}(U_2) \) are both closed sets in \( X_1, X_2 \) respectively, and thus \( f_1^{-1}(U_1) \times f_2^{-1}(U_2) \) is closed in \( X_1 \times X_2 \). Hence, \( f \) is almost contra continuous.

Conversely, suppose that \( f \) is almost contra continuous. Let \( V_1 \subseteq Y_1 \) be a regular open set, then \( V_1 \times Y_2 \) is a regular open set in \( Y_1 \times Y_2 \), so \( f^{-1}(V_1 \times Y_2) \) is closed in \( X_1 \times X_2 \).

\[
f^{-1}(V_1 \times Y_2) = \{(x_1, x_2) \in X_1 \times X_2 : (f_1(x_1), f_2(x_2)) \in V_1 \times Y_2\}
= \{(x_1, x_2) \in X_1 \times X_2 : x_1 \in f_1^{-1}(V_1), x_2 \in f_2^{-1}(Y_2)\}
= f_1^{-1}(V_1) \times f_2^{-1}(Y_2)
= f_1^{-1}(V_1) \times X_2
\] is closed,

so \( f_1^{-1}(V_1) \) is closed, hence \( f_1 \) is almost contra continuous. Similarly, we can prove the
result for $f_2$. 

**Theorem 2.2.24.** Let $f : X_1 \times X_2 \to Y_1 \times Y_2$ defined by $f(x_1, x_2) = (f_1(x_1), f_2(x_2))$, where $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ and $X_1$, $X_2$, $Y_1$, $Y_2$ are topological spaces. Then, $f$ is contra continuous iff both $f_1$ and $f_2$ are contra continuous.

**Proof.** The proof is similar to the previous theorem. 

**Theorem 2.2.25.** Let $h : X \to X_1 \times X_2$ defined by $h(x) = (h_1(x), h_2(x))$, then for $i=1,2$:

$h_i : X \to X_i$ is almost contra continuous iff $h$ is almost contra continuous.

**Proof.** Suppose that $h$ is almost contra continuous, let $U_1$ be a regular open set in $X_1$, then $U_1 \times X_2$ is regular open in $X_1 \times X_2$ and so $h^{-1}(U_1 \times X_2)$ is closed in $X$, but:

$h^{-1}(U_1 \times X_2) = \{x \in X : h(x) = (h_1(x), h_2(x)) \in U_1 \times X_2\}$

$= \{x \in X : h_1(x) \in U_1, h_2(x) \in X_2\}$

$= h_1^{-1}(U_1)$

so, $h_1^{-1}(U_1)$ is closed in $X$ and $h_1$ is almost contra continuous. Similarly, we can prove the result for $h_2$.

Conversely, suppose that both $h_1$ and $h_2$ are almost contra continuous. Let $U_1 \times U_2 \subseteq X_1 \times X_2$ be a nonempty regular open set, then by proposition (1.1.27) both $U_1$ and $U_2$ are nonempty regular open sets in $X_1$, $X_2$ respectively. Now, we get that $h_1^{-1}(U_1)$, $h_2^{-1}(U_2)$ are both closed sets in $X$, then:

$h^{-1}(U_1 \times U_2) = h_1^{-1}(U_1) \cap h_2^{-1}(U_2)$ which is closed in $X$, so $h$ is almost contra continuous. 

**Theorem 2.2.26.** Let $h : X \to X_1 \times X_2$ defined by $h(x) = (h_1(x), h_2(x))$, let $h_i : X \to X_i$, $i = 1, 2$, then:

$h_i : X \to X_i$ is contra continuous iff $h$ is contra continuous.
Proof. The proof is similar to the previous theorem.

We close this section with a very important theorem which is a new characterization of contra continuity in topological spaces.

**Theorem 2.2.27.** A function \( f : X \to Y \) is contra continuous iff for each point \( x \in X \) and each filter base \( \Lambda \) in \( X \) convergent to \( x \), the filter base \( f(\Lambda) \) is c-convergent to \( f(x) \).

*Proof.* Suppose that \( f : X \to Y \) is contra continuous. Let \( x \in X \) be arbitrary, \( \Lambda \) be any filter base in \( X \) converges to \( x \). Let \( V \subseteq Y \) be any closed set such that \( f(x) \in V \), then \( \exists U \in \tau \) such that \( x \in U \) and \( f(U) \subseteq V \) (since \( f \) is contra continuous), but \( \Lambda \) converges to \( x \), so \( \exists B \in \Lambda \) such that \( B \subseteq U \), therefore, \( f(B) \subseteq f(U) \subseteq V \), but \( f(B) \in f(\Lambda) \). Hence, \( f(\Lambda) \) is c-convergent to \( f(x) \).

Conversely, suppose that the above condition holds, let \( x \in X \) be arbitrary and \( V \subseteq Y \) be any closed set such that \( f(x) \in X \). Now, consider the filter base \( \Lambda = \{ U \subseteq X : U \in \tau \text{ and } x \in U \} \), then \( \Lambda \) converges to \( x \) (see example 1.1.45), thus it follows that \( f(\Lambda) \) must c-converges to \( f(x) \), so \( \exists U \in \Lambda \) such that \( f(U) \subseteq V \). Hence, for any closed set \( V \) contains \( f(x) \), \( \exists \) an open set \( U \) contains \( x \) and \( f(U) \subseteq V \), this means that \( f \) is contra continuous. \( \Box \)

### 2.3 The preservation theorems and some other properties

**Theorem 2.3.1.** If a function \( f : X \to Y \) is contra continuous and \( Y \) is regular, then \( f \) is continuous.

*Proof.* Let \( x \in X \) and \( V \) be an open set of \( Y \) containing \( f(x) \). Since \( Y \) is regular, there exist an open set \( G \) in \( Y \) containing \( f(x) \) such that \( Cl(G) \subseteq V \). Again, since \( f \)
is contra continuous, so there exists $U$ in $X$ such that $x \in U$ and $f(U) \subseteq Cl(G)$. Then $f(U) \subseteq Cl(G) \subseteq V$. Hence, $f$ is continuous.

**Corollary 2.3.2.** Let $f : X \to Y$ be a contra continuous surjection function, and $Y$ be regular. If $X$ is compact, then $Y$ is compact.

**Proof.** Follows from the previous theorem and the fact that continuous image of a compact space is compact.

**Theorem 2.3.3.** If $f : X \to Y$ is contra continuous and open injection and $Y$ is first countable, then $X$ is regular.

**Proof.** In order to get the regularity of $X$, we will show that each $x \in X$ has a nhood base consisting of closed sets. Now, let $x \in X$ be arbitrary, then $y = f(x) \in Y$ and $Y$ is first countable, so $y$ has a countable nhood base say $\beta = \{V_i : i \in \mathbb{N}\}$, we will show that the family $\{f^{-1}(V_i) : i \in \mathbb{N}\}$ is a nhood base of closed sets about $x$. First, we have $f^{-1}(V_i)$ is closed set in $X$ (since $f$ is contra continuous), and $x \in f^{-1}(V_i)$ for each $i \in \mathbb{N}$.

Now, Let $U$ be any nhood about $x$, then $x \in int(U)$ and $f(int(U))$ is open set in $Y$ contains $f(x) = y$ (since $f$ is open), but $\beta$ is a nhood base about $y$, so there exist $V_i \in \beta$ such that $f(x) \in V_i \subseteq f(int(U))$, hence $x \in f^{-1}(V_i) \subseteq int(U) \subseteq U$, therefor, the family $\{f^{-1}(V_i) : i \in \mathbb{N}\}$ is a nhood base of closed sets about $x$ and $X$ is regular.

**Theorem 2.3.4.** If $f : X \to Y$ is contra continuous, closed injection and $Y$ is ultranormal, then $X$ is normal.

**Proof.** Let $V$ and $W$ be disjoint closed subsets of $X$. Since $f$ is closed injection, $f(V)$ and $f(W)$ are disjoint closed subsets of $Y$. Again, since $Y$ is ultranormal $f(V)$ and $f(W)$ are separated by disjoint clopen sets $P$ and $Q$ respectively. Therefor, $f(V) \subseteq P$ and $f(W) \subseteq Q$ i.e. $V \subseteq f^{-1}(P)$ and $W \subseteq f^{-1}(Q)$, where $f^{-1}(P)$ and $f^{-1}(Q)$ are disjoint open sets of $X$ ( since $f$ is contra continuous ). This shows that $X$ is normal.
Theorem 2.3.5. If \( f : X \to Y \) is contra continuous surjection, where \( X \) is connected and \( Y \) is any topological space, then \( Y \) is not a discrete space.

Proof. If possible, suppose that \( Y \) is a discrete space. Let \( P \) be a proper nonempty open and closed subset of \( Y \). Then \( f^{-1}(P) \) is a proper nonempty open and closed subset of \( X \), which contradicts the fact that \( X \) is connected. Hence the result follows. \( \Box \)

Theorem 2.3.6. If \( f : X \to Y \) is contra continuous surjection and \( X \) is connected, then \( Y \) is connected.

Proof. If possible, suppose that \( Y \) is not connected. Then there exist nonempty disjoint open sets \( P \) and \( Q \) such that \( Y = P \cup Q \). So \( P \) and \( Q \) are clopen sets of \( Y \). Since \( f \) is contra continuous function, \( f^{-1}(P) \) and \( f^{-1}(Q) \) are open sets of \( X \). Also \( f^{-1}(P) \) and \( f^{-1}(Q) \) are nonempty disjoint open sets of \( X \) and \( X = f^{-1}(P) \cup f^{-1}(Q) \), which contradicts the fact that \( X \) is connected. Hence \( Y \) is connected. \( \Box \)

Theorem 2.3.7. If \( X \) is ultraconnected and \( f : X \to Y \) is contra continuous surjection, then \( Y \) is hyperconnected.

Proof. Suppose that \( Y \) is not hyperconnected, so there exist a nonempty open set \( V \) such that \( V \) is not dense in \( Y \). So, there exist nonempty open subsets \( B_1 = Int(cl(V)) \) and \( B_2 = Y - cl(V) \). Now, we have \( B_1, B_2 \) are two disjoint open sets, so \( f^{-1}(B_1) \) and \( f^{-1}(B_2) \) are two disjoint closed sets in \( X \), but this contradicts that \( X \) is ultraconnected. \( \Box \)

Theorem 2.3.8. Contra continuous images of hyperconnected spaces are connected.

Proof. Suppose that \( f : X \to Y \) is contra continuous surjection and \( X \) is hyperconnected, let \( A \) be a proper nonempty clopen set in \( Y \), then \( B = f^{-1}(A) \) is both open and closed (since \( f \) is contra continuous), so \( B \) is both dense and closed, this implies that \( f(B) = \)
\( A = cl(A) = Y \), hence \( A = Y \). This contradicts that \( A \) is proper subset of \( Y \). Therefor, there is no proper nonempty clopen subset in \( Y \) and so \( Y \) is connected.

**Theorem 2.3.9.** [7] A space \( X \) is connected if and only if every contra continuous function from \( X \) into any \( T_1 \) space \( Y \) is constant.

*Proof.* Let \( X \) be connected. Now, since \( Y \) is a \( T_1 \) space, \( \Omega = \{ f^{-1}(y) : y \in Y \} \) is disjoint open partition of \( X \) (since any singeltone set is closed in \( T_1 \) space). If \( |\Omega| \geq 2 \) (where \( |\Omega| \) denotes the cardinality of \( \Omega \)), then \( X \) is the union of two nonempty disjoint open sets. Since \( X \) is connected, we get \( |\Omega| = 1 \). Hence, \( f \) is constant.

Conversely, suppose that \( X \) is not connected and every contra continuous function from \( X \) into any \( T_1 \) space \( Y \) is constant. Since \( X \) is not connected, there exist a nonempty proper open as well as closed set \( V \) in \( X \). We consider the space \( Y = \{0, 1\} \) with the discrete topology \( \sigma \). The function \( f : X \rightarrow Y \) defined by \( f(V) = \{0\} \) and \( f(X-V) = \{1\} \) is contra continuous which is not constant, a contradiction. Hence, \( X \) is connected.

**Corollary 2.3.10.** [7] The only contra continuous function defined on the real line \( \mathbb{R} \) is the constant one.

*Proof.* Follows from theorem (2.3.9) and the fact that real line \( \mathbb{R} \) is both connected and \( T_1 \) space.

**Theorem 2.3.11.** [5] If \( f : X \rightarrow Y \) is contra continuous surjection, then the following statements hold:

1. If \( X \) is compact, then \( Y \) is almost compact.
2. If \( X \) is compact, then \( Y \) is strongly \( S \)-closed.

*Proof.* (1) Let \( \{V_\alpha : \alpha \in \Delta\} \) be an open cover for \( Y \), then \( \{Cl(V_\alpha) : \alpha \in \Delta\} \) is a closed cover of \( Y \), ie, \( Y = \bigcup_{\alpha \in \Delta} Cl(V_\alpha) \), therefor, \( X = \bigcup_{\alpha \in \Delta} f^{-1}(Cl(V_\alpha)) \). Hence, \( \{f^{-1}(Cl(V_\alpha)) : \alpha \in \Delta\} \)
is an open cover for $X$ (since $f$ is contra continuous), but $X$ is compact, so there exist a finite subset $\Omega \subseteq \Delta$ such that $X = \bigcup_{\alpha \in \Omega} f^{-1}(\text{Cl}(V_\alpha))$, this implies that $Y = \bigcup_{\alpha \in \Omega} \text{Cl}(V_\alpha)$. Hence, $Y$ has a finite proximate subcover, ie $Y$ is almost compact.

(2) Let $\{V_\alpha : \alpha \in \Delta\}$ be a closed cover for $Y$, then $\{f^{-1}(V_\alpha) : \alpha \in \Delta\}$ is an open cover for $X$ (since $f$ is contra continuous), but $X$ is compact, so $\exists$ a finite set $\Omega \subseteq \Delta$ such that $\{f^{-1}(V_\alpha) : \alpha \in \Omega\}$ covers $X$ this implies that $\{V_\alpha : \alpha \in \Omega\}$ is a finite subcover for $Y$. Therefor, $Y$ is strongly S-closed and hence S-closed.

\textbf{Theorem 2.3.12.} If $f : X \to Y$ is almost contra continuous surjection, then the following results hold:

(1) If $X$ is compact, then $Y$ is S-closed space.

(2) If $X$ is Lindeloff, then $Y$ is S-Lindeloff.

(3) If $X$ is countably compact, then $Y$ is countably S-closed.

\textit{Proof.} (1) Suppose that $\{U_\alpha : \alpha \in \Delta\}$ is a regular closed cover for $Y$, then $f^{-1}(U_\alpha)$ is open for each $\alpha \in \Delta$ (since $f$ is almost contra continuous), therefor $\{f^{-1}(U_\alpha) : \alpha \in \Delta\}$ is an open cover for $X$ and $X$ is compact, so $\exists$ $\Omega \subseteq \Delta$ such that $\Omega$ is finite and $\{f^{-1}(U_\alpha) : \alpha \in \Omega\}$ is a cover for $X$, this implies that $\{U_\alpha : \alpha \in \Omega\}$ is a cover for $Y$ and hence, $Y$ is S-closed.

(2) Let $\{V_\alpha : \alpha \in \Delta\}$ be a regular closed cover for $Y$, then $f^{-1}(V_\alpha)$ is an open set in $X$ for each $\alpha \in \Delta$, therefor $\{f^{-1}(V_\alpha) : \alpha \in \Delta\}$ is an open cover for $X$, but $X$ is Lindeloff, so $\exists$ a countable set $\Omega \subseteq \Delta$ such that $\{f^{-1}(V_\alpha) : \alpha \in \Omega\}$ covers $X$, therefor $\{V_\alpha : \alpha \in \Omega\}$ is a countable subcover for $Y$. Hence, $Y$ is S-Lindeloff.

(3) Let $\{V_\alpha : \alpha \in \Delta\}$ be a countable regular closed cover for $Y$, since $f$ is almost contra continuous, so $\{f^{-1}(V_\alpha) : \alpha \in \Delta\}$ is a countable open cover for $X$, but $X$ is countably compact, so there exist a finite subset $\Omega \subseteq \Delta$ such that $X = \bigcup_{\alpha \in \Omega} f^{-1}(\text{Cl}(V_\alpha))$, this implies that $Y = \bigcup_{\alpha \in \Omega} \text{Cl}(V_\alpha)$. Therefore, $Y$ is strongly S-closed and hence S-closed.
compact, so \( \exists \) a finite subset \( \Omega \subseteq \Delta \) such that \( \{ f^{-1}(V_{\alpha}) : \alpha \in \Omega \} \) covers \( X \), therefor \( \{ V_{\alpha} : \alpha \in \Omega \} \) is a finite subcover for \( Y \). Hence, \( Y \) is countably S-closed. \( \square \)

**Corollary 2.3.13.** If \( f : X \to Y \) is contra continuous surjection, then the following results hold:

1. If \( X \) is compact, then \( Y \) is S-closed space.
2. If \( X \) is Lindeloff, then \( Y \) is S-Lindeloff.
3. If \( X \) is countably compact, then \( Y \) is countably S-closed.

*Proof.* Follows directly from theorem (2.3.12), since every contra continuous function is almost contra continuous. \( \square \)

**Theorem 2.3.14.** If \( f : X \to Y \) is contra continuous injection where \( Y \) is a Urysohn space, then \( X \) is \( T_2 \).

*Proof.* Let \( x, y \in X \) and \( x \neq y \). Since \( f \) is injection, so \( f(x) \neq f(y) \). Now, since \( Y \) is Urysohn, there exist open sets \( U \) and \( V \) of \( Y \) containing \( f(x) \) and \( f(y) \) respectively, such that \( Cl(U) \cap Cl(V) = \phi \). Also, \( f \) being contra continuous there exist open sets \( P \) and \( Q \) containing \( x \) and \( y \) respectively such that \( f(P) \subseteq Cl(U) \) and \( f(Q) \subseteq Cl(V) \). Then \( f(P) \cap f(Q) = \phi \) and so \( P \cap Q = \phi \). Therefor \( X \) is \( T_2 \). \( \square \)

**Theorem 2.3.15.** If \( f : X \to Y \) is contra continuous injection where \( Y \) is ultra Hausdorff, then \( X \) is \( T_2 \).

*Proof.* Let \( x, y \in X \) where \( x \neq y \). Then, since \( f \) is an injection and \( Y \) is ultra Hausdorff, \( f(x) \neq f(y) \) and there exist disjoint closed sets \( U \) and \( V \) containing \( f(x) \) and \( f(y) \) respectively. Again, since \( f \) is contra continuous, \( f^{-1}(U), f^{-1}(V) \) are disjoint open sets in \( X \) containing \( x \) and \( y \) respectively. This shows that \( X \) is \( T_2 \). \( \square \)
Theorem 2.3.16. If \( f : X \to Y \) is an almost contra continuous injection and \( Y \) is weakly Hausdorff, then \( X \) is \( T_1 \).

Proof. Let \( x, y \in X \) such that \( x \neq y \), since \( Y \) is weakly Hausdorff, there exist regular closed sets \( U \) and \( V \) such that \( f(x) \in U \), \( f(y) \notin U \) and \( f(y) \in V \), \( f(x) \notin V \). Now, since \( f \) is almost contra continuous \( f^{-1}(U) \) and \( f^{-1}(V) \) are open subsets of \( X \) such that \( x \in f^{-1}(U) \), \( y \notin f^{-1}(U) \) and \( y \in f^{-1}(V) \), \( x \notin f^{-1}(V) \). This shows that \( X \) is \( T_1 \). \( \square \)

Corollary 2.3.17. If \( f : X \to Y \) is contra continuous injection and \( Y \) is weakly Hausdorff, then \( X \) is \( T_1 \).

Theorem 2.3.18. If \( f : X \to Y \) is an almost contra continuous surjection and \( X \) is connected, then \( Y \) is connected.

Proof. If possible, suppose that \( Y \) is not connected. Then there exist disjoint nonempty open sets \( U \) and \( V \) of \( Y \) such that \( Y = U \cup V \). Since \( U \) and \( V \) are clopen sets in \( Y \), they are regular closed sets of \( Y \). Again, since \( f \) is almost contra continuous surjection \( f^{-1}(U) \) and \( f^{-1}(V) \) are open sets of \( X \) and \( X = f^{-1}(U) \cup f^{-1}(V) \). This shows that \( X \) is not connected which is a contradiction. Hence \( Y \) is connected. \( \square \)

2.4 Contra closed graphs

In this section we define contra closed graph and study some results.

Definition 2.4.1. For a function \( f : X \to Y \) the subset \( \{(x, f(x)) : x \in X\} \subseteq X \times Y \) is called the graph of \( f \) and is denoted by \( G(f) \).

Definition 2.4.2. [15] The graph \( G(f) \) of a function \( f : X \to Y \) is said to be contra closed if for each \( (x, y) \in (X \times Y) - G(f) \), there exist an open set \( U \) in \( X \) contains \( x \) and a closed set \( V \) in \( Y \) containing \( y \) such that \( (U \times V) \cap G(f) = \emptyset \).
**Lemma 2.4.3.** The graph $G(f)$ of a function $f : X \to Y$ is contra closed in $X \times Y$ iff for each $(x, y) \in (X \times Y) - G(f)$, there exist an open set $U$ in $X$ contains $x$ and a closed set $V$ in $Y$ containing $y$ such that $f(U) \cap V = \phi$.

**Proof.** We shall prove that $f(U) \cap V = \phi$ iff $(U \times V) \cap G(f) = \phi$. Let $(U \times V) \cap G(f) \neq \phi$, then there exist $(x, y) \in (U \times V)$ and $(x, y) \in G(f)$, so $x \in U$, $y \in V$ and $y = f(x) \in V$, hence $f(U) \cap V \neq \phi$. Conversely, if $f(U) \cap V \neq \phi$, so there exist $y \in f(U)$ and $y \in V$, so $y = f(x)$ for some $x \in U$ and $y \in V$. Therefore, $(x, y) \in U \times V$ and $y = f(x)$ implies that $(x, y) \in G(f)$, then $(U \times V) \cap G(f) \neq \phi$. Hence, the result follows.\[\square\]

**Theorem 2.4.4.** If a function $f : X \to Y$ is contra continuous and $Y$ is Urysohn, then $G(f)$ is contra closed in $X \times Y$.

**Proof.** Let $(x, y) \in (X \times Y) - G(f)$, then $y \neq f(x)$ and since $Y$ is Urysohn, there exist open sets $P$ and $Q$ in $Y$ such that $f(x) \in P$ and $y \in Q$ and $Cl(P) \cap Cl(Q) = \phi$. Now, since $f$ is contra continuous, there exist an open set $U$ in $X$ contains $x$ such that $f(U) \subseteq Cl(P)$ which implies that $f(U) \cap Cl(Q) = \phi$. Hence, by lemma (2.4.3) $G(f)$ is contra closed in $X \times Y$.\[\square\]

**Theorem 2.4.5.** If a function $f : X \to Y$ and $g : X \to Y$ are contra continuous functions, where $Y$ is Urysohn, then $D = \{x \in X : f(x) = g(x)\}$ is closed in $X$.

**Proof.** Let $x \in (X - D)$, then $f(x) \neq g(x)$. Since $Y$ is Urysohn, there exist open sets $U$ and $V$ such that $f(x) \in U$ and $g(x) \in V$ with $cl(U) \cap cl(V) = \phi$. Again, since $f$ and $g$ are contra continuous functions, then $f^{-1}(cl(U))$ and $f^{-1}(cl(V))$ are open sets in $X$. Let $P = f^{-1}(cl(U))$ and $Q = f^{-1}(cl(V))$, then $P$ and $Q$ are open sets of $X$ containing $x$. Let $H = P \cap Q$, then $H$ is open in $X$. Hence, $f(H) \cap g(H) = f(P \cap Q) \cap g(P \cap Q) \subseteq f(P) \cap g(Q) \subseteq cl(U) \cap cl(V) = \phi$. Therefore, $D \cap H = \phi$, thus $D$ is closed in $X$.\[\square\]
Theorem 2.4.6. Let $f : X \to Y$ and $g : X \to Y$ be any two functions. If $Y$ is Urysohn, $f, g$ are contra continuous and $f = g$ on a dense set $A \subseteq X$, then $f = g$ on $X$.

Proof. Since $f, g$ are contra continuous and $Y$ is Urysohn, using theorem (2.4.5), $D = \{ x \in X : f(x) = g(x) \}$ is closed in $X$. Also, we have $f = g$ on a dense set $A \subseteq X$. Now, since $A \subseteq D$ and $A$ is dense in $X$, we have $X = \text{cl}(A) \subseteq \text{cl}(D) = D$. Hence, $f = g$ on $X$. \qed

Theorem 2.4.7. Let $f : X \to Y$ be a function and $g : X \to X \times Y$ be the graph function of $f$. Then, $f$ is contra continuous if $g$ is contra continuous.

Proof. Suppose that $g$ is contra continuous. Let $U$ be an open set of $Y$, then $X \times U$ is an open set of $X \times Y$. Since $g$ is contra continuous, we get that $g^{-1}(X \times U)$ is a closed set in $X$, but $g^{-1}(X \times U) = \{ x \in X : (x, f(x)) \in X \times U \} = \{ x \in X : f(x) \in U \} = f^{-1}(U)$, therefor $f^{-1}(U)$ is closed in $X$. Hence, $f$ is contra continuous. \qed

2.5 Weak topology and Strong topology

If we consider a collection of functions $\{ f_{\alpha} : \alpha \in \Delta \}$ where $f_{\alpha} : X \to X_{\alpha}$, $X$ is any set and $X_{\alpha}$ is a topological space for each $\alpha$. If $X$ is equipped with the discrete topology, then each $f_{\alpha}$ is contra continuous. The discrete topology is the largest topology which have this property. In this section we try to describe the smallest topology on $X$ with this property.

Definition 2.5.1. If $X$ is a set and $X_{\alpha}$ are topological spaces with $f_{\alpha} : X \to X_{\alpha}$ for each $\alpha \in \Delta$. The weak topology induced on $X$ by the collection $\{ f_{\alpha} : \alpha \in \Delta \}$ of functions is the smallest topology on $X$ making each $f_{\alpha}$ contra continuous.
**Theorem 2.5.2.** If $X$ is a set and $X_\alpha$ are topological spaces with $f_\alpha : X \rightarrow X_\alpha$ for each $\alpha \in \Delta$, then the weak topology is the topology $\tau$ on $X$ for which the sets $f_\alpha^{-1}(V_\alpha)$, for $\alpha \in \Delta$ and $V_\alpha$ is closed in $X_\alpha$ form a subbase.

**Proof.** If we consider $X$ with the topology $\tau$, then we will show that each $f_\alpha$ is contra continuous. Let $V_\alpha$ be closed set in $X_\alpha$, then $f_\alpha^{-1}(V_\alpha)$ is a subbasic open set, so $f_\alpha^{-1}(V_\alpha) \in \tau$, hence $f_\alpha$ is contra continuous.

Now, we show that $\tau$ is the smallest topology on $X$ making each $f_\alpha$ contra continuous, suppose that $\sigma$ is a topology on $X$ such that each $f_\alpha$ is contra continuous, let $U \in \tau$, then $U = \bigcup \bigcap_{i=1}^n B_{\alpha_i} = \bigcup \bigcap_{i=1}^n f_\alpha^{-1}(V_{\alpha_i})$

for some $V_{\alpha_i} \subseteq X_{\alpha_i}$ a closed set, since $f_{\alpha_i}$ is contra continuous and $V_{\alpha_i}$ is closed, so $f_{\alpha_i}^{-1}(V_{\alpha_i})$ is open in $(X, \sigma)$, and $U$ is open in $(X, \sigma)$, therefor $\tau \subseteq \sigma$ and hence $\tau$ is the weak topology on $X$. $\square$

**Theorem 2.5.3.** If $X$ has the weak topology induced by a collection $\{f_\alpha : \alpha \in \Delta\}$ of functions where $f_\alpha : X \rightarrow X_\alpha$ and $Y$ is a topological space, then $f : Y \rightarrow X$ is continuous iff $f_\alpha \circ f : Y \rightarrow X_\alpha$ is contra continuous.

**Proof.** Suppose that $f$ is continuous, then $f_\alpha \circ f$ is contra continuous (since $f_\alpha$ is contra continuous as $X$ has the weak topology).

Conversely, suppose that $X$ has the weak topology and $f_\alpha \circ f$ is contra continuous, we will show that the inverse image of each subbasic open set in $X$ is open in $Y$. Let $U_\alpha$ be a subbasic open set in $X$, then $U_\alpha = f_\alpha^{-1}(V_\alpha)$ where $V_\alpha$ is closed set in $X_\alpha$, so $f^{-1}(U_\alpha) = f^{-1}(f_\alpha^{-1}(V_\alpha)) = (f_\alpha \circ f)^{-1}(V_\alpha)$ is open in $Y$ (by the contra continuity of $f_\alpha \circ f$), therefor $f : Y \rightarrow X$ is continuous. $\square$

If $X$ is a topological space and $Y$ is any set, if we consider $Y$ with the trivial topology,
then any function $f : X \to Y$ is contra continuous, the trivial topology is the smallest topology on $Y$ with this property, so we look for the largest topology on $Y$ making $f$ contra continuous, such topology is called the strong topology on $Y$ induced by $f$.

**Theorem 2.5.4.** Consider $f : X \to Y$, where $X$ is Alexandroff space and $Y$ is any set, then the strong topology on $Y$ induced by $f$ is the topology $\tau_f$ which have the collection $\beta = \{V \subseteq Y : f^{-1}(V) \text{ is closed in } X\}$ as a subbase.

**Proof.** First, we show that $\tau_f$ makes $f$ contra continuous, let $W \subseteq Y$ be open set in $\tau_f$ then, $W = \bigcup_{\alpha} \bigcap_{i=1}^{n} V_{\alpha i}$ where $f^{-1}(V_{\alpha i})$ is closed in $X$ for each $\alpha_i$, $f^{-1}(W) = \bigcup_{\alpha} \bigcap_{i=1}^{n} f^{-1}(V_{\alpha i})$ is closed in $X$ since $X$ is Alexandroff space, hence $f$ is contra continuous.

Now, we show that $\tau_f$ is the largest topology on $Y$ that makes $f$ contra continuous, suppose that $\sigma$ is another topology on $Y$ such that $f : X \to Y$ is contra continuous, let $U \in \sigma$, then $f^{-1}(U)$ is closed in $X$ (since $f : X \to (Y, \sigma)$ is contra continuous), so $U$ is a basic open set in $\tau_f$. Hence $\sigma \subseteq \tau_f$ and $\tau_f$ is the strong topology induced on $Y$ by $f$. \(\square\)

An important question we must deal with is: under what conditions on $f$ will a preassigned topology $\tau$ on $Y$ be identical to the strong topology $\tau_f$ induced by $f$ ? It is clear that contra continuity of $f$ is an important condition to make $\tau \subseteq \tau_f$. Thus, we search for additional conditions to force $\tau \supseteq \tau_f$. The following theorem states the fundamental result about this.

**Theorem 2.5.5.** If $X$ is Alexandroff space, $Y$ is any topological space and $f : X \to Y$ is contra continuous, contra closed, then the topology $\tau$ on $Y$ is the strong topology $\tau_f$.

**Proof.** Suppose that $f$ is contra continuous and contra closed, since $\tau_f$ is the largest topology making $f$ contra continuous, so $\tau \subseteq \tau_f$. But if $U \in \tau_f$, so $U = \bigcup_{\alpha} \bigcap_{i=1}^{n} V_{\alpha i}$ where
$f^{-1}(V_{\alpha i})$ is closed.

Therefor, $f^{-1}(U) = \bigcup_{\alpha} \bigcap_{i=1}^{n} f^{-1}(V_{\alpha i})$ is closed in $X$ since $X$ is Alexandroff, and $f(f^{-1}(U)) = U$ is open in $(Y, \tau)$ (since $f$ is contra closed) and $\tau_f \subseteq \tau$. Hence, $\tau \subseteq \tau_f$ and this establishes equality. 

**Theorem 2.5.6.** Let $Y$ have the strong topology induced by a map $f : X \to Y$, where $X$ ia an Alexandroff space. Then, an arbitrary map $g : Y \to Z$ is continuous iff $g \circ f : X \to Z$ is contra continuous.

**Proof.** Necessity is trivial, since the composition of continuous map and a contra continuous map is contra continuous.

To prove sufficiency, suppose that $g \circ f$ is contra continuous and let $U$ be open set in $Z$, then $(g \circ f)^{-1}(U)$ is closed in $X$ (by the contra continuity of $g \circ f$), but $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ is closed in $X$. So, $g^{-1}(U)$ is a basic open set in $\tau_f$ and $g : Y \to Z$ is continuous.
Chapter 3

New classes of generalized closed sets in topological spaces

3.1 On $tgr$-closed sets

In this section a new class of closed sets called $tgr$-closed sets is introduced. We will study some examples and some results about it. Then we will study their relations with other classes of closed sets.

Definition 3.1.1. Let $(X, \tau)$ be a topological space. A subset $A \subseteq X$ is called $tgr$-closed iff $rcl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $t$-set.

Example 3.1.2. Let $(X, \tau)$ be an infinite set with the cofinite topology. The $t$-sets are $X$, $\phi$ and the finite subsets, while the $tgr$-closed sets are $X$, $\phi$ and the infinite sets. To prove this, notice that the only regular closed sets in $(X, \tau)$ are $X$, $\phi$, so $rcl(A) = X$ for any nonempty set in $X$. Now, if $A$ is nonempty finite set, then $A$ is a $t$-set that contains itself but does not contain $rcl(A) = X$, so $A$ is not $tgr$-closed. If $A$ is infinite, then the only $t$-set that contains $A$ is $X$ and so it contains $rcl(A)$, hence $A$ is $tgr$-closed set.
Example 3.1.3. Let \( X = \{a, b\} \) be the Sierpinski space by setting \( \tau = \{ \emptyset, \{a\}, X \} \). The only regular closed sets are \( X, \emptyset \), so \( rcl \{a\} = rcl \{b\} = rcl(X) = X \), and the \( t \)-sets are \( X, \emptyset \) and \( \{b\} \). The set \( \{a\} \) is \( tgr \)-closed set since the only \( t \)-set that contains \( \{a\} \) is \( X \), however, the set \( \{b\} \) is not \( tgr \)-closed set since it is a \( t \)-set contains itself but does not contain \( rcl \{b\} = X \).

Example 3.1.4. Consider the real line with the usual topology, we will show that the only intervals that are \( tgr \)-closed sets are the closed intervals.

**case(1) :** If \( I=[a,b] \). Note that \( [a,b] \) is a regular closed set, so \( rcl[a,b] = [a,b] \), therefore if \( U \) is any \( t \)-set that contains \( [a,b] \), it contains \( rcl[a,b] = [a,b] \) and \( [a,b] \) is \( tgr \)-closed set in \( \mathbb{R} \).

**case(2) :** If \( I \) is an infinite closed interval, ie \( I = (-\infty, a] \) or \( I = [a, \infty) \). Note that \( I \) is regular closed set, so it is \( tgr \)-closed set similar to case 1.

**case(3) :** If \( I = (a,b) \). Note that \( [a,b] \) is a regular closed set that contains \( (a,b) \), then we have: \( cl(a,b) = [a,b] \subseteq rcl(a,b) \subseteq [a,b] \), hence \( rcl(a,b) = [a,b] \). However, \( (a,b) \) is a \( t \)-set that contains itself but it does not contain \( rcl(a,b) = [a,b] \), hence \( (a,b) \) is not \( tgr \)-closed set.

**case(4) :** If \( I=(a,\infty) \). Note that \( [a,\infty) \) is a regular closed set that contains \( (a,\infty) \), then we have: \( cl(a,\infty) = [a,\infty) \subseteq rcl(a,\infty) \subseteq [a,\infty) \), hence \( rcl(a,\infty) = [a,\infty) \). However, \( (a,\infty) \) is a \( t \)-set that contains itself but it does not contain \( rcl(a,\infty) = [a,\infty) \), hence \( (a,\infty) \) is not \( tgr \)-closed set. Similarly if \( I = (-\infty, a) \).

**case(5) :** If \( I = (a,b] \). Note that \( [a,b] \) is a regular closed set that contains \( (a,b] \), then we have: \( cl(a,b] = [a,b] \subseteq rcl(a,b] \subseteq [a,b] \), hence \( rcl(a,b] = [a,b] \). However, \( (a,b] \) is a \( t \)-set that contains itself but it does not contain \( rcl(a,b] = [a,b] \), hence \( (a,b] \) is not \( tgr \)-closed set. Similarly if \( I = [a,b) \).
Theorem 3.1.5. Let \((X, \tau)\) be a topological space:

1) If \((X, \tau)\) is the discrete topology, then any subset \(A \subseteq X\) is tgr-closed.

2) If \((X, \tau)\) is the trivial topology, then any subset \(A \subseteq X\) is tgr-closed.

Proof. (1) Suppose that \((X, \tau)\) is the discrete topology, let \(A \subseteq X\), then \(A\) is clopen so it is regular closed, hence \(rcl(A) = A\), it follows that \(A\) is tgr-closed.

(2) Suppose that \((X, \tau)\) is the trivial topology. Now for any nonempty subset \(A \subseteq X\), \(rcl(A) = X\) since the only regular closed sets in the trivial topology are \(X\), \(\phi\), also the only \(t\)-sets in the trivial topology are \(X\), \(\phi\). Hence for any nonempty set \(A\), the only \(t\)-set that contains \(A\) is \(X\), this means that \(A\) is tgr-closed set. \(\square\)

Example 3.1.6. Consider \(\mathbb{R}\) with the usual topology. The set of rationals \(\mathbb{Q}\) is tgr-closed.

We will prove that the only \(t\)-set that contains \(\mathbb{Q}\) is \(\mathbb{R}\). Suppose that \(U\) is a \(t\)-set such that \(\mathbb{Q} \subseteq U\), then we have:

\[
\text{int(cl(Q))} \subseteq \text{int(cl(U))}, \text{ so } X \subseteq \text{int(cl(U))} = \text{int(U)}, \text{ then } \text{int(U)} = X \text{ therefor, } U = X.
\]

Hence, \(\mathbb{Q}\) is tgr-closed set.

In fact, any dense set in a topological space is tgr-closed set. We can prove this as in example (3.1.6). See the following theorem.

Theorem 3.1.7. Let \((X, \tau)\) be any topological space. If \(A\) is dense in \(X\), then \(A\) is a tgr-closed set.

Proof. Suppose that \(A\) is dense set in \(X\), let \(U\) be any \(t\)-set such that \(A \subseteq U\), then \(\text{int(cl(A))} \subseteq \text{int(cl(U))}\), this implies that \(X \subseteq \text{int(U)}\) and therefor \(U = X\) and \(rcl(A) \subseteq U = X\). Hence, \(A\) is a tgr-closed set. \(\square\)

Remark 3.1.8. If \(A\) is a tgr-closed set, \(A\) need not be dense. As a counterexample, the empty set is a tgr-closed set but not dense.
Theorem 3.1.9. Let \((X, \tau)\) be a topological space. If \(A \subseteq X\) is tgr-closed set, then \(rcl(A) = cl(A)\).

Proof. Suppose that \(A\) is tgr-closed set, then any \(t\)-set that contains \(A\) must contain \(rcl(A)\), but \(cl(A)\) is a closed set that contains \(A\), so \(cl(A)\) is a \(t\)-set that contains \(A\) (from proposition 1.1.25), hence \(rcl(A) \subseteq cl(A)\) and we have \(cl(A) \subseteq rcl(A)\), therefor \(cl(A) = rcl(A)\). \(\square\)

Theorem 3.1.10. Let \(A \subseteq B \subseteq rcl(A)\), and \(A\) is tgr-closed subset in a topological space \((X, \tau)\), then \(B\) is also tgr-closed.

Proof. If \(A \subseteq B \subseteq rcl(A)\), then \(rcl(A) \subseteq rcl(B) \subseteq rcl(A)\) therefor, \(rcl(A) = rcl(B)\). Now, if \(V\) is any \(t\)-set such that \(B \subseteq V\), then \(A \subseteq V\) this implies that \(rcl(B) = rcl(A) \subseteq V\), hence \(B\) is tgr-closed set. \(\square\)

Theorem 3.1.11. The union of two tgr-closed sets is tgr-closed.

Proof. Suppose that both \(A\) and \(B\) are tgr-closed sets. Let \(U\) be any \(t\)-set such that \(A \cup B \subseteq U\), then \(A \subseteq U\) and \(B \subseteq U\), so \(rcl(A) \subseteq U\) and \(rcl(B) \subseteq U\) (since both \(A\) and \(B\) are tgr-closed sets), but by theorem (1.2.3) we get, \(rcl(A \cup B) = rcl(A) \cup rcl(B) \subseteq U\), hence \(A \cup B\) is tgr-closed. \(\square\)

Theorem 3.1.12. Let \((X, \tau)\) be a hyperconnected space. If \(\{A_\alpha : \alpha \in \Delta\}\) is a family of subsets of \(X\) such that \(A_\beta\) is a nonempty tgr-closed set for some \(\beta \in \Delta\), then \(\bigcup_{\alpha \in \Delta} A_\alpha\) is tgr-closed set.

Proof. Let \(U\) be any \(t\)-set such that \(\bigcup_{\alpha \in \Delta} A_\alpha \subseteq U\), then we have \(A_\beta \subseteq U\), but \(A_\beta\) is tgr-closed set, hence \(rcl(A_\beta) \subseteq U\). However, \(X\) is hyperconnected space, so the only nonempty regular closed set is \(X\), hence \(rcl(A_\beta) = X \subseteq U\), this implies that \(U = X\), and \(X\) is the only \(t\)-set that contains \(\bigcup_{\alpha \in \Delta} A_\alpha\). Hence, \(\bigcup_{\alpha \in \Delta} A_\alpha\) is tgr-closed. \(\square\)
Corollary 3.1.13. Arbitrary union of tgr-closed sets in a hyperconnected space is tgr-closed.

Proof. Follows directly from theorem (3.1.12).

Corollary 3.1.14. Let \((X, \tau)\) be a hyperconnected space. If \(A \subseteq X\) is a nonempty tgr-closed set, then any superset of \(A\) is also tgr-closed.

Proof. First proof, suppose that \(A\) is a nonempty tgr-closed set in a hyperconnected space \(X\), let \(B \subseteq X\) such that \(A \subseteq B\), then using theorem (3.1.12), \(B = A \cup (B - A)\) is tgr-closed set.

Second proof, since \(X\) is hyperconnected, then \(rcl(A) = X\), hence we get that: \(A \subseteq B \subseteq rcl(A) = X\), and \(A\) is tgr-closed, so \(B\) is also tgr-closed by Theorem (3.1.10).

Remark 3.1.15. The intersection of two tgr-closed sets need not be a tgr-closed set. Consider the real line with the cofinite topology, then from example (3.1.2) we have: for any \(a \in \mathbb{R}\), \((-\infty, a]\) and \([a, \infty)\) are both tgr-closed sets but \((-\infty, a] \cap [a, \infty) = \{a\}\) which is not tgr-closed set in the cofinite space.

In the following theorems and examples we discuss the relations between tgr-closed sets and other classes of closed sets.

Theorem 3.1.16. Let \((X, \tau)\) be a topological space, if \(A \subseteq X\) is a tgr-closed set, then it is rg-closed set.

Proof. Suppose that \(A\) is a tgr-closed set and \(U\) is a regular open set such that \(A \subseteq U\), then by proposition (1.1.26) \(U\) is a t-set, so \(rcl(A) \subseteq U\) (since \(A\) is a tgr-closed set), but \(cl(A) \subseteq rcl(A)\), hence \(cl(A) \subseteq U\) and \(A\) is rg-closed set.

Remark 3.1.17. rg-closed sets need not be tgr-closed sets as shown in the following example.
Example 3.1.18. If $X$ is infinite space with the cofinite topology. Let $A$ be any nonempty finite set, then $A$ is closed and $cl(A) = A$, hence whenever $A \subseteq U$ and $U$ is regular open in $X$, then $cl(A) = A \subseteq U$, so $A$ is $rg$-closed set but from example (3.1.2) it is not $tgr$-closed set.

Theorem 3.1.19. If $A$ is regular closed set in any topological space, then $A$ is a $tgr$-closed set.

Proof. Suppose that $A$ is regular closed set in a topological space $(X, \tau)$, so $rcl(A) = A$, hence whenever $U$ is a $t$-set contains $A$, it must contain $rcl(A) = A$, hence $A$ is $tgr$-closed set.

Remark 3.1.20. $tgr$-closed sets need not be regular closed set as shown in the following example.

Example 3.1.21. Consider the set of rational numbers $\mathbb{Q}$ in the usual topology, then $\mathbb{Q}$ is $tgr$-closed set from example (3.1.6), but it is not regular closed, since $cl(int(\mathbb{Q})) = cl(\phi) = \phi \neq \mathbb{Q}$.

Theorem 3.1.22. Every $tgr$-closed set is $rwg$-closed.

Proof. Suppose that $A$ is $tgr$-closed. Let $U$ be any regular open set such that $A \subseteq U$, so $U$ is $t$-set that contains $A$ (from proposition 1.1.26), hence $rcl(A) \subseteq U$, but $cl(int(A)) \subseteq cl(A) \subseteq rcl(A) \subseteq U$. Hence, $A$ is $rwg$-closed.

Remark 3.1.23. $rwg$-closed sets need not be $tgr$-closed sets. Consider the real line with the cofinite topology, then $\{1\}$ is $rwg$-closed set since $cl(int(\{1\})) = \phi$, but from example (3.1.2) it is not $tgr$-closed.

Theorem 3.1.24. If $A$ is $tgr$-closed set and $t$-set, then $A$ is closed set.
Proof. Suppose that $A$ is both $tgr$-closed and $t$-set, then $A$ is a $t$-set that contains itself, so it must contain $rcl(A)$, therefor we have $A \subseteq rcl(A) \subseteq A$, hence $A = rcl(A)$, but $rcl(A)$ is a closed set, so the result follows. \qed

Remark 3.1.25. The following examples show that $tgr$-closed sets are independent of closed sets, pre-closed sets, semi closed sets, $\alpha$-closed sets, semi-pre closed sets, $w$-closed sets, $g$-closed sets, $wg$-closed sets, $swg$-closed sets.

Example 3.1.26. $tgr$-closed sets and closed sets are independent. In $\mathbb{R}$ with the usual topology, the set of rationals $\mathbb{Q}$ is a $tgr$-closed set but not closed. On the other hand, in any infinite space $X$ with the cofinite topology, any nonempty finite set is closed set but not $tgr$-closed set. See example (3.1.2).

Example 3.1.27. $tgr$-closed sets and pre-closed sets are independent. Consider $\mathbb{R}$ with the cofinite topology, then $\mathbb{R} - \{1\}$ is $tgr$-closed set, but $cl(int(\mathbb{R} - \{1\})) = \mathbb{R}$, so $\mathbb{R} - \{1\}$ is not pre-closed set. On the other hand, consider $X = \{a, b\}$ with the Serpinski space, let $\tau = \{X, \phi, \{a\}\}$, then $\{b\}$ is pre-closed since $int\{b\} = \phi$ and $cl(int\{b\}) = \phi \subseteq \{b\}$, but $\{b\}$ is not $tgr$-closed. See examples (3.1.2) and (3.1.3).

Example 3.1.28. $tgr$-closed sets and semi closed sets are independent. Consider $\mathbb{R}$ with the cofinite topology, $\mathbb{Z}$ is $tgr$-closed set but $int(cl(\mathbb{Z})) = \mathbb{R} \neq int(\mathbb{Z})$, hence $\mathbb{Z}$ is not semi closed set. On the other hand, $\{1\}$ is not a $tgr$-closed set, but $int(cl\{1\}) = int\{1\} = \phi$, hence $\{1\}$ is semi closed set but not $tgr$-closed set.

Example 3.1.29. $tgr$-closed sets and $\alpha$-closed sets are independent. Consider $\mathbb{R}$ with the cofinite topology, then $\mathbb{R} - \mathbb{Z}$ is $tgr$-closed set but $cl(int(cl(\mathbb{R} - \mathbb{Z}))) = \mathbb{R}$, hence it is not $\alpha$-closed set. On the other hand, $\{1\}$ is $\alpha$-closed set since $cl(int(cl(\{1\}))) = \phi$, but from example (3.1.2) it is not $tgr$-closed set.
Example 3.1.30. \( tgr \)-closed sets and semi pre closed sets are independent. Consider \( \mathbb{R} \) with the cofinite topology. The set \( \mathbb{R} - \{1\} \) is \( tgr \)-closed but not semi pre closed since \( \text{int}(\text{cl}(\text{int}(\mathbb{R} - \{1\}))) = \mathbb{R} \) is not contained in \( \mathbb{R} - \{1\} \). On the other hand, the interval \([0,1)\) is semi pre closed set in the usual topology since \( \text{int}(\text{cl}(\text{int}[0,1))) = (0,1) \subseteq [0,1) \), but from example (3.1.4) it is not \( tgr \)-closed set.

Example 3.1.31. \( tgr \)-closed sets and \( g \)-closed sets are independent. Consider \( \mathbb{R} \) with the cofinite topology. \( \mathbb{R} - \{1\} \) is \( tgr \)-closed but not \( g \)-closed since \( \mathbb{R} - \{1\} \) is open set contains itself but does not contain \( \text{cl}(\mathbb{R} - \{1\}) = \mathbb{R} \). On the other hand, any nonempty finite set in the cofinite topology is closed, so it is \( g \)-closed but it is not \( tgr \)-closed set. See example (3.1.2).

Example 3.1.32. \( tgr \)-closed sets and \( wg \)-closed sets are independent. Consider \( \mathbb{R} \) with the cofinite topology. \( \mathbb{R} - \{1\} \) is \( tgr \)-closed but not \( wg \)-closed since \( \mathbb{R} - \{1\} \) is open set but it does not contain \( \text{cl}(\text{int}(\mathbb{R} - \{1\})) = \mathbb{R} \). On the other hand, any nonempty finite set \( A \) in the cofinite topology is not \( tgr \)-closed but it is \( wg \)-closed since \( \text{cl}(\text{int}(A)) = \phi \).

Example 3.1.33. \( tgr \)-closed sets and \( swg \)-closed sets are independent. Consider \( \mathbb{R} \) with the cofinite topology. If \( A \) is any nonempty finite set, then \( \text{int}(A) = \phi \), hence \( \text{cl}(\text{int}(A)) = \phi \subseteq U \) whenever \( A \subseteq U \) and \( U \) is semiopen in \( X \), therefore \( A \) is \( swg \)-closed but it is not \( tgr \)-closed. On the other hand, let \( A = \mathbb{R} - \{1\} \), then \( A \) is \( tgr \)-closed set but it is not \( swg \)-closed since \( A \) is semi open set contains itself but does not contain \( \text{cl}(\text{int}(A)) = \text{cl}(A) = \mathbb{R} \).

3.2 On \( t^*gr \)-closed sets

In this section a new class of closed sets called \( t^*gr \)-closed sets is introduced. We will study some examples and their relations with other classes of closed sets.
**Definition 3.2.1.** Let \((X, \tau)\) be a topological space. A subset \(A \subseteq X\) is called \(t^*gr\)-closed iff \(rcl(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is \(t^*\)-set.

**Example 3.2.2.** Let \(X = \{a, b\}\) be the Sierpinski space by setting \(\tau = \{\emptyset, \{a\}, X\}\). The only regular closed sets are \(X, \emptyset\), so \(rcl\{a\} = rcl\{b\} = rcl(X) = X\), the \(t^*\)-sets are \(X, \emptyset\) and \(\{a\}\). The set \(\{b\}\) is \(t^*gr\)-closed set but \(\{a\}\) is not.

**Example 3.2.3.** Let \((X, \tau)\) be an infinite set with the cofinite topology. The open sets are the only \(t^*\)-sets. Now suppose that \(A\) is not open, then \(X - \{a\}\), where \(a \notin A\) is a \(t^*\) set that contains \(A\) but does not contain \(rcl(A) = X\), so any non-open set is not \(t^*gr\)-closed set. However, if \(A\) is nonempty proper open set, then \(A\) is a \(t^*\)-set that contains itself but it does not contain \(rcl(A) = X\), so it is not a \(t^*gr\)-closed set. Hence, the only \(t^*gr\)-closed sets are \(\emptyset, X\).

**Example 3.2.4.** Consider the real line with the usual topology. As in example (3.1.4), we can show that the only intervals that are \(t^*gr\)-closed sets are the closed intervals.

**Theorem 3.2.5.** Let \((X, \tau)\) be a topological space:

1. If \((X, \tau)\) is the discrete topology, then any subset \(A \subseteq X\) is \(t^*gr\)-closed.
2. If \((X, \tau)\) is the trivial topology, then any subset \(A \subseteq X\) is \(t^*gr\)-closed.

**Proof.**

1. Suppose that \((X, \tau)\) is the discrete topology, let \(A \subseteq X\), then \(A\) is clopen so it is regular closed, hence \(rcl(A) = A\), it follows that \(A\) is \(t^*gr\)-closed.

2. Suppose that \((X, \tau)\) is the trivial topology. Now for any nonempty subset \(A\) of \(X\), \(rcl(A) = X\) since the only regular closed sets in the trivial topology are \(X, \emptyset\). Also the only \(t^*\)-sets in the trivial topology are \(X, \emptyset\). To show this, let \(A\) be a \(t^*\)-set such that \(A \notin \{\emptyset, X\}\), hence \(cl(int(A)) = cl(A)\). Now \(int(A) = \emptyset\) (since the only open set that contained in \(A\) is \(\emptyset\)), so \(cl(int(A)) = \emptyset\) but \(cl(A) = X\) (since the only closed set that
contains \( A \) is \( X \), therefor \( \text{cl} (\text{int} (A)) \neq \text{cl} (A) \) this is a contradiction, hence the only \( t^* \)-sets in the trivial topology are \( \phi \), \( X \). Hence for any nonempty set \( A \), the only \( t^* \)-set that contains \( A \) is \( X \), this means that \( A \) is \( t^* \)-gr-closed set.

\[ \square \]

**Theorem 3.2.6.** Let \( A \subseteq B \subseteq \text{rcl} (A) \), and \( A \) is \( t^* \)-gr-closed subset in a topological space \((X, \tau)\), then \( B \) is also \( t^* \)-gr-closed.

**Proof.** If \( A \subseteq B \subseteq \text{rcl} (A) \), then \( \text{rcl} (A) \subseteq \text{rcl} (B) \subseteq \text{rcl} (A) \) therefor, \( \text{rcl} (A) = \text{rcl} (B) \).

Now, if \( V \) is any \( t^* \)-set such that \( B \subseteq V \), then \( A \subseteq V \) this implies that \( \text{rcl} (B) = \text{rcl} (A) \subseteq V \), hence \( B \) is \( t^* \)-gr-closed set.

\[ \square \]

**Theorem 3.2.7.** The union of two \( t^* \)-gr-closed sets is \( t^* \)-gr-closed.

**Proof.** Suppose that both \( A \) and \( B \) are \( t^* \)-gr-closed sets. Let \( U \) be any \( t^* \)-set such that \( A \cup B \subseteq U \), then \( A \subseteq U \) and \( B \subseteq U \), so \( \text{rcl} (A) \subseteq U \) and \( \text{rcl} (B) \subseteq U \) (since both \( A \) and \( B \) are \( t^* \)-gr-closed sets), but from theorem (1.2.3), \( \text{rcl} (A \cup B) = \text{rcl} (A) \cup \text{rcl} (B) \subseteq U \), hence \( A \cup B \) is \( t^* \)-gr-closed.

\[ \square \]

**Theorem 3.2.8.** Let \((X, \tau)\) be a hyperconnected space. If \( \{A_\alpha : \alpha \in \Delta\} \) is a family of subsets of \( X \) such that \( A_\beta \) is a nonempty \( t^* \)-gr-closed set for some \( \beta \in \Delta \), then \( \bigcup_{\alpha \in \Delta} A_\alpha \) is \( t^* \)-gr-closed set.

**Proof.** Let \( U \) be any \( t^* \)-set such that \( \bigcup_{\alpha \in \Delta} A_\alpha \subseteq U \), then \( \forall \beta \in \Delta \) we have \( A_\beta \subseteq U \), but \( A_\beta \) is \( t^* \)-gr-closed set, hence \( \text{rcl} (A_\beta) \subseteq U \). However, \( X \) is hyperconnected space, so the only nonempty regular closed set is \( X \), hence \( \text{rcl} (A_\beta) = X \subseteq U \), this implies that \( U = X \), and \( X \) is the only \( t^* \)-set that contains \( \bigcup_{\alpha \in \Delta} A_\alpha \). Hence, \( \bigcup_{\alpha \in \Delta} A_\alpha \) is \( t^* \)-gr-closed.

\[ \square \]

**Corollary 3.2.9.** Arbitrary union of \( t^* \)-gr-closed sets in a hyperconnected space is \( t^* \)-gr-closed.
Proof. Follows directly from theorem (3.2.8). □

Corollary 3.2.10. Let \((X, \tau)\) be a hyperconnected space. If \(A \subseteq X\) is a nonempty \(t^*gr\)-closed set, then any superset of \(A\) is also \(t^*gr\)-closed.

Proof. First proof, suppose that \(A\) is a nonempty \(t^*gr\)-closed set in a hyperconnected space \(X\), let \(B \subseteq X\) such that \(A \subseteq B\), then using theorem (3.2.8), \(B = A \cup (B - A)\) is \(t^*gr\)-closed set.

Second proof, since \(X\) is hyperconnected, then \(rcl(A) = X\), hence we get that: \(A \subseteq B \subseteq rcl(A) = X\), and \(A\) is \(t^*gr\)-closed, so \(B\) is also \(t^*gr\)-closed by Theorem (3.2.6). □

Remark 3.2.11. In section (3.1), we define the class of \(tgr\)-closed sets, in fact \(tgr\)-closed sets and \(t^*gr\)-closed sets are independent, see the following example.

Example 3.2.12. Consider the real line with the cofinite topology, then \(\mathbb{R} - \{1\}\) is \(tgr\)-closed but it is not \(t^*gr\)-closed (see examples 3.1.2 and 3.2.3). On the other hand, consider the Sierpinski space where \(X = \{a, b\}\) and \(\tau = \{\emptyset, \{a\}, X\}\), then \(\{b\}\) is \(t^*gr\)-closed set but not \(tgr\)-closed set (see examples 3.1.3 and 3.2.2).

In the following theorems and examples we discuss the relations between \(t^*gr\)-closed sets and other classes of closed sets.

Theorem 3.2.13. Every \(t^*gr\)-closed set is \(gr\)-closed set.

Proof. Suppose that \(A\) is \(t^*gr\)-closed set, let \(U\) be an open set such that \(A \subseteq U\), by proposition (1.1.25) \(U\) is \(t^*\)-set so, \(rcl(A) \subseteq U\) (since \(A\) is \(t^*gr\)-closed set). Hence, \(A\) is \(gr\)-closed set. □

Remark 3.2.14. \(gr\)-closed sets need not be \(t^*gr\)-closed sets as shown in the following example.
Example 3.2.15. Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}, \{b, c\}\}$, $A = \{b\}$ is $gr$-closed but not $t^* gr$-closed. To show this, notice that the regular closed sets that contains $\{b\}$ are $X$ and $\{b, c\}$, so $rcl \{b\} = \{b, c\}$ and any open set that contains $\{b\}$ also contains $\{b, c\}$, therefor $A = \{b\}$ is $gr$-closed. On the other hand, the $t^*$-sets that contains $\{b\}$ are $X$ and $\{a, b\}$, so $\{a, b\}$ is $t^*$-set contains $\{b\}$ but it does not contain $rcl \{b\} = \{b, c\}$. Hence, $\{b\}$ is $gr$-closed set but it is not $t^* gr$-closed set.

Corollary 3.2.16. Every $t^* gr$-closed set is $\pi gr$-closed set.

Proof. The result follows from theorem (3.2.14) and the fact that every $gr$-closed set is $\pi gr$-closed set. See theorem (1.2.5). □

Theorem 3.2.17. Every $t^* gr$-closed set is $w$-closed.

Proof. Suppose that $A$ is a $t^* gr$-closed set in a topological space $(X, \tau)$. Let $U$ be a semi open set such that $A \subseteq U$, then by proposition (1.1.24) $U$ is $t^*$-set. since $A$ is a $t^* gr$-closed, we have $rcl(A) \subseteq U$, but $cl(A) \subseteq rcl(A)$, this implies that $cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is semi open set, therefor $A$ is $w$-closed set. □

Remark 3.2.18. The converse of theorem (3.2.18) need not be true. Consider $X = \{a, b, c, d\}$ and $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. The set $\{d\}$ is $w$-closed since it is closed, but it is not $t^* gr$-closed since $rcl \{d\} = \{c, d\}$ and $\{a, b, d\}$ is a $t^*$-set that contains $\{d\}$ but it does not contains $rcl \{d\}$.

Corollary 3.2.19. Every $t^* gr$-closed set is $g$-closed set.

Proof. The result follows from theorem (3.2.18) and the fact that every $w$-closed set is $g$-closed set. See theorem (1.2.5). □

Corollary 3.2.20. Every $t^* gr$-closed set is $rg$-closed set.
Proof. The result follows from theorem (3.2.18) and the fact that every $w$-closed set is $rg$-closed set. See theorem (1.2.5).

Corollary 3.2.21. Every $t^*gr$-closed set is $wg$-closed set.

Proof. The result follows from theorem (3.2.18) and the fact that every $w$-closed set is $wg$-closed set. See theorem (1.2.5).

Corollary 3.2.22. Every $t^*gr$-closed set is $rw$-closed set.

Proof. The result follows from theorem (3.2.18) and the fact that every $w$-closed set is $rw$-closed set. See theorem (1.2.5).

Corollary 3.2.23. Every $t^*gr$-closed set is $rwg$-closed set.

Proof. The result follows from theorem (3.2.18) and the fact that every $w$-closed set is $rwg$-closed set. See theorem (1.2.5).

Theorem 3.2.24. Every $t^*gr$-closed set is $swg$-closed set.

Proof. Suppose that $A$ is a $t^*gr$-closed set in a topological space $X$. Let $U$ be a semi open set such that $A \subseteq U$, then by proposition (1.1.24) $U$ is $t^*$-set. Since $A$ is a $t^*gr$-closed, we have $rcl(A) \subseteq U$, but $cl(int(A)) \subseteq cl(A) \subseteq rcl(A)$, this implies that $cl(int(A)) \subseteq U$ whenever $A \subseteq U$ and $U$ is a semi open set, therefor A is $swg$-closed set.

Remark 3.2.25. The converse of this theorem need not be true. Consider $X = \{a, b, c, d\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. The set $\{d\}$ is $swg$-closed set since $cl(int(\{d\})) = \phi$, but it is not $t^*gr$-closed since $rcl \{d\} = \{c, d\}$ and $\{a, b, d\}$ is a $t^*$-set that contains $\{d\}$ but it does not contains $rcl \{d\}$.

Theorem 3.2.26. Every regular closed set is $t^*gr$-closed set.
Proof. Suppose that $A$ is a regular closed set in a topological space $(X, \tau)$, then $rcl(A) = A$, hence it follows that $A$ is $t^*gr$-closed. \hfill \square

Remark 3.2.27. $t^*gr$-closed sets need not be regular closed set. Consider the Sierpinski space where $X = \{a, b\}$ and $\tau = \{\emptyset, \{a\}, X\}$, then $\{b\}$ is $t^*gr$-closed set but not regular closed set since $cl(int \{b\}) = \emptyset \neq \{b\}$.

The following diagram summarizes the relations of $t^*gr$-closed sets with other classes of closed sets.

![Diagram of closed sets](image)

Figure 3.1: $t^*gr$-closed sets

Theorem 3.2.28. Let $(X, \tau)$ be a locally indiscrete space, then a subset $A \subseteq X$ is $tgr$-closed set iff $A$ is $t^*gr$-closed.

Proof. Suppose that $(X, \tau)$ is a locally indiscrete, we will show that the $t$-sets and the $t^*$-sets are the same. First, suppose that $A$ is a $t$-set, so $int(cl(A)) = int(A)$, but since
$(X, \tau)$ is locally indiscrete, any closed set is open, so $cl(A)$ is open set and $int(cl(A)) = cl(A) = int(A)$, hence $cl(cl(A)) = cl(A) = cl(int(A))$ and $A$ is a $t^*$-set.

Conversely, suppose that $A$ is a $t^*$-set, so $int(cl(A)) = cl(A)$, but $int(A)$ is open set, so it is closed (since $(X, \tau)$ is locally indiscrete), hence $cl(int(A)) = int(A) = cl(A)$ and this implies that $int(int(A)) = int(A) = int(cl(A))$, therefor $A$ is a $t$-set. So, we proved that in a locally indicrete space any $t$-set is $t^*$-set which implies that any $t^*gr$-closed set is $tgr$-closed set, also we proved that any $t^*$-set is $t$-set which implies that any $tgr$-closed set is $t^*gr$-closed set, hence the result follows.

3.3 On the topology generated by $tgr$-closed sets

In section (3.1) we have introduced the notation of $tgr$-closed sets, the family of all $tgr$-closed sets is not a topology, however it can be considered as a subbase for some topology. In this section, we will prove some results about this topology.

Definition 3.3.1. Let $(X, \tau)$ be a topological space, then the topology generated by the $tgr$-closed sets is the topology which have the collection of $tgr$-closed sets as a subbase and it is denoted by $\tau_{tgr}$.

Example 3.3.2. Consider the real line with the cofinite topology. In example (3.1.2) we get that the nonempty $tgr$-closed sets are the infinite sets, then the topology generated by the $tgr$-closed sets is the discrete topology. To prove this, we will show that each sigeltone set belongs to $\tau_{tgr}$. Let $a \in \mathbb{R}$ be arbitrary, then $(-\infty, a]$ and $[a, \infty)$ are $tgr$-closed sets and belong to $\tau_{tgr}$, hence $(-\infty, a] \cap [a, \infty) = \{a\} \in \tau_{tgr}$ since $\tau_{tgr}$ is a topology. Therefor, $\tau_{tgr}$ is the discrete topology.

Theorem 3.3.3. If $(X, \tau)$ is locally indiscrete topological space, then $\tau \subseteq \tau_{tgr}$. 

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Proof. Let \( U \in \tau \), then \( U \) is closed in \( \tau \) since \((X,\tau)\) is locally indiscrete. Then, \( U \) is a clopen set in \((X,\tau)\), so \( U \) is regular closed and hence by theorem (3.1.19) it is \( tgr \)-closed set in \((X,\tau)\), therefore \( U \in \tau_{tgr} \). \( \Box \)

**Theorem 3.3.4.** Let \((X,\tau)\) be a topological space. If \((X,\tau)\) is hyperconnected, then \( \tau \subseteq \tau_{tgr} \)

*Proof.* Suppose that \((X,\tau)\) is a hyperconnected space, if \( U \) is a nonempty open set in \((X,\tau)\), then \( U \) is dense in \((X,\tau)\), hence by theorem (3.1.7), \( U \) is a \( tgr \)-closed set in \((X,\tau)\), this implies that \( U \) is a subbasic open set in \((X,\tau_{tgr})\), therefore \( \tau \subseteq \tau_{tgr} \). \( \Box \)

**Theorem 3.3.5.** Let \((X,\tau)\) be a topological space, then \( RC(X) \subseteq \tau_{tgr} \)

*Proof.* The result follows from the fact that every regular closed set is \( tgr \)-closed set. \( \Box \)

**Definition 3.3.6.** A topological space \((X,\tau)\) is called \( tgr \)-locally indiscrete if every \( tgr \)-open set is closed, or equivalently every \( tgr \)-closed set is open.

**Theorem 3.3.7.** The following are equivalent for a topological space \((X,\tau)\):

1. \( \tau_{tgr} \subseteq \tau \).
2. \((X,\tau)\) is \( tgr \)-locally indiscrete.

*Proof.* Suppose that \( \tau_{tgr} \subseteq \tau \). Let \( V \) be a \( tgr \)-closed set in \((X,\tau)\), then it is a subbasic open set in \( \tau_{tgr} \) and \( V \in \tau_{tgr} \subseteq \tau \), hence \( V \) is open set in \((X,\tau)\) and \((X,\tau)\) is \( tgr \)-locally indiscrete.

Conversely, suppose that \((X,\tau)\) is \( tgr \)-locally indiscrete space. Let \( V \in \tau_{tgr} \), then \( V = \bigcup_{\alpha} \bigcap_{i=1}^{n} V_{\alpha} \) where each \( V_{\alpha} \) is \( tgr \)-closed set in \((X,\tau)\), but \((X,\tau)\) is \( tgr \)-locally indiscrete , so each \( V_{\alpha} \) is open in \((X,\tau)\). Hence \( V = \bigcup_{\alpha} \bigcap_{i=1}^{n} V_{\alpha} \) is open set in \((X,\tau)\) and \( \tau_{tgr} \subseteq \tau \). \( \Box \)

**Proposition 3.3.8.** Let \((X,\tau)\) be a topological space. If \( \tau_{tgr} \subseteq \tau \), then \((X,\tau)\) is extremally disconnected.
Proof. Suppose that $\tau_{tgr} \subseteq \tau$, then by theorem (3.3.7) we get that $(X, \tau)$ is $tgr$-locally indiscrete. To prove that $(X, \tau)$ is extremally disconnected, let $U$ be a regular closed set, then $U$ is $tgr$-closed set in $(X, \tau)$, so $U \in \tau_{tgr} \subseteq \tau$, therefor, $U$ is open set in $(X, \tau)$. Hence, $(X, \tau)$ is extremally disconnected. \hfill \Box

Remark 3.3.9. The converse of the above theorem need not be true. Consider the real line with the cofinite topology, this space is extremally disconnected but the cofinite topology does not contain the generated topology which is the discrete topology. See example (3.3.2).

**Theorem 3.3.10.** Let $(X, \tau)$ be a topological space. If $(X, \tau_{tgr})$ is compact space, then $(X, \tau)$ is S-closed space.

Proof. Let $\{V_\alpha : \alpha \in \Delta\}$ be a cover for $(X, \tau)$ of regular closed sets, then by theorem (3.1.19) each $V_\alpha$ is $tgr$-closed set in $(X, \tau)$, and $V_\alpha \in \tau_{tgr}$ for each $\alpha \in \Delta$, so $\{V_\alpha : \alpha \in \Delta\}$ is an open cover for $(X, \tau_{tgr})$. Since $(X, \tau_{tgr})$ is compact, there exist a finite set $\Omega \subseteq \Delta$ such that $\{V_\alpha : \alpha \in \Omega\}$ is a cover for $(X, \tau_{tgr})$, hence $(X, \tau)$ has a finite subcover of regular closed sets and it is S-closed space. \hfill \Box

Remark 3.3.11. The converse of the above theorem need not be true in general, consider $\mathbb{R}$ with the cofinite topology, this space is S-closed since the only regular closed sets are $\mathbb{R}$, $\emptyset$. However, the generated topology is the discrete topology which is not compact.

**Theorem 3.3.12.** Let $(X, \tau)$ be a topological space. If $(X, \tau_{tgr})$ is Lindeloff space, then $(X, \tau)$ is S-Lindeloff.

Proof. Let $\{V_\alpha : \alpha \in \Delta\}$ be a cover for $(X, \tau)$ of regular closed sets, then by theorem (3.1.19) each $V_\alpha$ is $tgr$-closed set in $(X, \tau)$, and $V_\alpha \in \tau_{tgr}$ for each $\alpha \in \Delta$, so $\{V_\alpha : \alpha \in \Delta\}$ is an open cover for $(X, \tau_{tgr})$. Since, $(X, \tau_{tgr})$ is Lindeloff, there exist a countable set $\Omega \subseteq \Delta$
such that \( \{ V_\alpha : \alpha \in \Omega \} \) is a cover for \((X, \tau_{tgr})\), hence \((X, \tau)\) has a countable subcover of regular closed sets and it is S-Lindeloff space.

\[ \square \]

**Theorem 3.3.13.** Let \((X, \tau)\) be a topological space. If \((X, \tau)\) is weakly Hausdorff, then \((X, \tau_{tgr})\) is \( T_1 \)-space.

**Proof.** Suppose that \((X, \tau)\) is weakly Hausdorff. Let \(x, y\) be two distinct points in \(X\), so there exist \(U, V\) two regular closed sets in \((X, \tau)\) such that \(x \in U, y \notin U\) and \(y \in V, x \notin V\) (since \((X, \tau)\) is weakly Hausdorff). Therefore, \(U, V\) are two \(tgr\)-closed sets in \((X, \tau)\), so \(U, V\) are basic open sets in \((X, \tau_{tgr})\) such that \(x \in U, y \notin U\) and \(y \in V, x \notin V\), hence \((X, \tau_{tgr})\) is \( T_1 \)-space. \( \square \)

**Remark 3.3.14.** The converse of the above theorem need not be true. Consider the real line with the cofinite topology, then the generated topology is the discrete topology which is \( T_1 \)-space, but the cofinite topology is not weakly Hausdorff since the only nonempty regular closed set is \(X\).

**Theorem 3.3.15.** If \((X, \tau)\) is weakly Hausdorff topological space, then the following are equivalent:

1. \((X, \tau_{tgr})\) is strongly \(S\)-closed.
2. \(X\) is finite.

**Proof.** It is clear that if \(X\) is finite, then \((X, \tau_{tgr})\) is strongly \(S\)-closed. Conversely, suppose that \((X, \tau_{tgr})\) is strongly \(S\)-closed space and \((X, \tau)\) is weakly Hausdorff, then by theorem (3.3.13) \((X, \tau_{tgr})\) is \( T_1 \) space. Hence we get a closed cover \(\beta\) for \((X, \tau_{tgr})\) where \(\beta = \{ \{ x \} : x \in X \}\), and since \((X, \tau_{tgr})\) is strongly \(S\)-closed, \(\beta\) must have a finite subcover say \(\{ x_1, x_2, x_3, \ldots\} \), then \(X = \bigcup_{i=1}^{n} x_i\). Hence, \(X\) is finite. \( \square \)

**Theorem 3.3.16.** If \((X, \tau)\) is a discrete topological space, then \((X, \tau_{tgr})\) is also discrete.
Proof. Let \( A \) be arbitrary subset of a discrete space \((X, \tau)\), then \( A \) is a tgr-closed set by theorem (3.1.5), therfor \( A \in \tau_{tgr} \) and \( \tau_{tgr} \) is the discrete space. \( \square \)

Remark 3.3.17. The converse of the above theorem need not be true. Consider the real line with the cofinite topology, in example (3.3.2) we have shown that \((\mathbb{R}, \tau_{tgr})\) is the discrete topology although the cofinite topology is not discrete.

Theorem 3.3.18. If \((X, \tau)\) is the trivial topology, then \((X, \tau_{tgr})\) is the discrete topology.

Proof. Let \( A \) be arbitrary subset of the trivial space \((X, \tau)\), then \( A \) is a tgr-closed set by theorem (3.1.5), therfor \( A \in \tau_{tgr} \) and \( \tau_{tgr} \) is the discrete space. \( \square \)

Remark 3.3.19. If \((X, \tau), (X, \sigma)\) are two topological spaces such that \( \tau \subseteq \sigma \), it is need not be true that \( \tau_{tgr} \subseteq \sigma_{tgr} \). Consider \( X = \{a, b\} \), let \( \tau \) be the trivial topology and \( \sigma \) be the Sierpinski space \( \sigma = \{X, \{a\}, \emptyset\} \), we have \( \tau \subseteq \sigma \) and \( \tau_{tgr} \) is the discrete space but \( \sigma_{tgr} = \sigma \) (since the only tgr-closed sets in the Sierpinski space are \( X, \{a\} \) and \( \phi \)). Therefor, we have \( \sigma \) contains \( \tau \) but \( \sigma_{tgr} \) does not contain \( \tau_{tgr} \).

Theorem 3.3.20. Let \((X, \tau) \), \((Y, \sigma)\) be topological spaces and \( f : (X, \tau) \rightarrow (Y, \sigma) \) be a continuous open surjection function, then the following are equivalent:

(1) \((Y, \sigma)\) is tgr-locally indiscrete.

(2) \( f : (X, \tau) \rightarrow (Y, \sigma_{tgr}) \) is continuous.

Proof. Suppose (1) holds, then we get that \( \sigma_{tgr} \subseteq \sigma \) (from theorem 3.3.7). Now, let \( U \in \sigma_{tgr} \subseteq \sigma \), so \( U \in \sigma \) and we have \( f : (X, \tau) \rightarrow (Y, \sigma) \) is continuous, we get that \( f^{-1}(U) \) is open in \((X, \tau)\), hence \( f : (X, \tau) \rightarrow (Y, \sigma_{tgr}) \) is continuous.

Conversely, suppose that (2) holds, and let \( V \) be a tgr-closed set \((Y, \sigma)\), so \( V \in \sigma_{tgr} \), hence \( f^{-1}(V) \) is open in \((X, \tau)\) (since \( f : (X, \tau) \rightarrow (Y, \sigma_{tgr}) \) is continuous ), but since \( f \)
is open surjection, we have \( f(f^{-1}(V)) = V \) is open in \((X, \tau)\). Hence, \((Y, \sigma)\) is tgr-locally indiscrete.

**Corollary 3.3.21.** Let \((X, \tau)\) be a topological space, then the identity function \( I : (X, \tau) \to (X, \tau_{tgr}) \) is continuous if and only if \((X, \tau)\) is tgr-locally indiscrete.

*Proof.* Consider the identity function \( I : (X, \tau) \to (X, \tau) \), then \( I \) is a continuous open surjection function, and the result follows directly from theorem (3.3.20).

**Theorem 3.3.22.** If \( f : (X, \tau) \to (Y, \sigma) \) is continuous and \((X, \tau)\) is locally indiscrete, then \( f : (X, \tau_{tgr}) \to (Y, \sigma) \) is continuous.

*Proof.* Let \( V \) be open set in \((Y, \sigma)\), then \( f^{-1}(V) \) is open in \((X, \tau)\) (since \( f : (X, \tau) \to (Y, \sigma) \) is continuous), but \((X, \tau)\) is locally indiscrete, so \( \tau \subseteq \tau_{tgr} \) (from theorem 3.3.3), hence \( f^{-1}(V) \) is open in \((X, \tau_{tgr})\) and this means that \( f : (X, \tau_{tgr}) \to (Y, \sigma) \) is continuous.

**Theorem 3.3.23.** If \( f : (X, \tau) \to (Y, \sigma) \) is continuous and \((X, \tau)\) is hyperconnected, then \( f : (X, \tau_{tgr}) \to (Y, \sigma) \) is continuous.

*Proof.* Suppose that \((X, \tau)\) is hyperconnected, then by theorem (3.3.4) \( \tau \subseteq \tau_{tgr} \) and the proof is similar to the proof of theorem (3.3.22).

**Theorem 3.3.24.** Let \((X, \tau)\), \((Y, \sigma)\) be topological spaces and \( f : (X, \tau) \to (Y, \sigma) \) be a contra continuous, contra closed surjection function, then the following are equivalent:

1. \((Y, \sigma)\) is tgr-locally indiscrete.
2. \( f : (X, \tau) \to (Y, \sigma_{tgr}) \) is contra continuous.

*Proof.* Suppose (1) holds, then we get that \( \sigma_{tgr} \subseteq \sigma \) (from theorem 3.3.7). Now, let \( U \in \sigma_{tgr} \subseteq \sigma \), so \( U \in \sigma \) and we have \( f : (X, \tau) \to (Y, \sigma) \) is contra continuous, so we get that \( f^{-1}(U) \) is closed in \((X, \tau)\), hence \( f : (X, \tau) \to (Y, \sigma_{tgr}) \) is contra continuous.

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Conversely, suppose that (2) holds, and let $V$ be a $tgr$-closed set $(Y, \sigma)$, so $V \in \sigma_{tgr}$, hence $f^{-1}(V)$ is closed set in $(X, \tau)$ (since $f : (X, \tau) \rightarrow (Y, \sigma_{tgr})$ is contra continuous), but since $f$ is contra closed surjection, we have $f(f^{-1}(V)) = V$ is open in $(X, \tau)$. Hence, $(Y, \sigma)$ is $tgr$-locally indiscrete.

**Theorem 3.3.25.** If $f : (X, \tau) \rightarrow (Y, \sigma)$ is contra continuous and $(X, \tau)$ is locally indiscrete, then $f : (X, \tau_{tgr}) \rightarrow (Y, \sigma)$ is contra continuous.

**Proof.** Let $V$ be any closed set in $(Y, \sigma)$, then $f^{-1}(V)$ is open in $(X, \tau)$ since $f : (X, \tau) \rightarrow (Y, \sigma)$ is contra continuous, but since $(X, \tau)$ is locally indiscrete, then by theorem (3.3.3), we get that $\tau \subseteq \tau_{tgr}$, hence $f^{-1}(V)$ is open in $(X, \tau_{tgr})$ and this means that $f : (X, \tau_{tgr}) \rightarrow (Y, \sigma)$ is contra continuous.

**Theorem 3.3.26.** If $f : (X, \tau) \rightarrow (Y, \sigma)$ is contra continuous and $(X, \tau)$ is hyperconnected, then $f : (X, \tau_{tgr}) \rightarrow (Y, \sigma)$ is contra continuous.

**Proof.** The proof is similar to the previous theorem.

**Theorem 3.3.27.** Let $(X, \tau)$, $(Y, \sigma)$ be topological spaces, if $f : (X, \tau) \rightarrow (Y, \sigma)$ is $RC$-continuous, then $f : (X, \tau_{tgr}) \rightarrow (Y, \sigma)$ is continuous.

**Proof.** Suppose that $f : (X, \tau) \rightarrow (Y, \sigma)$ is RC-continuous. Let $V \subseteq Y$ be open set, then $f^{-1}(V)$ is regular closed set in $(X, \tau)$, so $f^{-1}(V)$ is $tgr$-closed set in $(X, \tau)$, so $f^{-1}(V)$ is a basic open set in $(X, \tau_{tgr})$. Hence, $f : (X, \tau_{tgr}) \rightarrow (Y, \sigma)$ is continuous.

In the same way that we have defined $\tau_{tgr}$ we define $\tau^{*}_{tgr}$ to be the topology which have the collection of $t^{*}gr$-closed sets as a subbase.

**Example 3.3.28.** Consider the real line with the cofinite topology. In example (3.2.3) we get that the only $t^{*}gr$-closed sets in this space are $X$, $\phi$, hence $\tau^{*}_{tgr}$ is the trivial topology.
Theorem 3.3.29. If \((X, \tau)\) is a discrete topological space, then \((X, \tau_{t^*gr})\) is also discrete.

Proof. Let \(A\) be arbitrary subset of \(X\), then \(A\) is a \(t^*gr\)-closed set from theorem (3.2.5), therefor \(A \in \tau_{t^*gr}\) and \(\tau_{t^*gr}\) is the discrete space. \(\square\)

Theorem 3.3.30. If \((X, \tau)\) is the trivial topology, then \((X, \tau_{t^*gr})\) is the discrete topology.

Proof. Let \(A\) be arbitrary subset of \(X\) with the trivial topology, then \(A\) is \(t^*gr\)-closed set by theorem (3.2.5), hence \(A \in \tau_{t^*gr}\) and \(\tau_{t^*gr}\) is the discrete topology. \(\square\)

Example 3.3.31. Let \((X, \tau)\) be the Serpinski space where \(X = \{a, b\}\) and \(\tau = \{X, \phi, \{a\}\}\). From example (3.2.2), the \(t^*gr\)-closed sets are \(X\), \(\phi\), \(\{b\}\) and \(\tau_{t^*gr} = \{X, \phi, \{b\}\}\).

In the Serpinski space where \(X = \{a, b\}\) and \(\tau = \{X, \phi, \{a\}\}\) we have: \(\tau_{tgr} = \{X, \phi, \{a\}\}\), so in this example \(\tau_{tgr} \neq \tau_{t^*gr}\), hence the question that we must deal with is: under what conditions on \((X, \tau)\) we get the equality \(\tau_{tgr} = \tau_{t^*gr}\). The following theorem states a fundamental result about this.

Theorem 3.3.32. If \((X, \tau)\) is a locally indiscrete space, then \(\tau_{tgr} = \tau_{t^*gr}\).

Proof. Since \((X, \tau)\) is locally indiscrete, then from theorem (3.2.29), \(tgr\)-closed sets and \(t^*gr\)-closed sets are the same, hence \(\tau_{tgr} = \tau_{t^*gr}\). \(\square\)
Chapter 4

New classes of contra continuous functions

In this chapter we introduce new classes of contra continuous functions, namely, contra \( tgr \)-continuous and contra \( t^*gr \)-continuous functions. We get some of their properties and discuss the relationship with other classes of contra continuous functions. In this chapter, the notation \( C(Y, f(x)) \) denotes the class of all closed subsets of \( Y \) that contain \( f(x) \), the notation \( tgrO(X, \tau) \) denotes the class of all \( tgr \)-open sets in the space \((X, \tau)\), and the notation \( tgrO(X, x) \) denotes the class of all \( tgr \)-open sets in \( X \) that contain \( x \).

4.1 Contra \( tgr \)-continuous functions

In this section we introduce the notation of contra \( tgr \)-continuous functions in topological spaces, then we discuss some examples and some characterizations. Also, we study the relations between contra \( tgr \)-continuous functions and other classes of functions.

**Definition 4.1.1.** A function \( f : (X, \tau) \to (Y, \sigma) \) is called contra \( tgr \)-continuous iff
$f^{-1}(V)$ is tgr-closed in $X$ whenever $V$ is open in $Y$.

**Example 4.1.2.** Define $f : (\mathbb{R}, \tau) \to \mathbb{R}^1$ where $\tau$ is the cofinite topology, such that $f(x) = \lfloor x \rfloor$ for any real number $x$. Then, $f$ is contra tgr-continuous. To prove this, let $V$ be an open subset of $\mathbb{R}^1$, then $f^{-1}(V) = \bigcup_{n \in V \cap \mathbb{Z}} [n, n+1)$ is empty or it is an infinite subset of $\mathbb{R}$, hence $f^{-1}(V)$ is tgr-closed set in the cofinite topology. Therefore, $f$ is contra tgr-continuous function.

**Example 4.1.3.** Let $(X, \tau)$ be the Sierpinski space where $X = \{a, b\}$ and $\tau = \{X, \phi, \{a\}\}$ and let $\sigma = \{X, \phi, \{b\}\}$. Define $f : (X, \tau) \to (X, \sigma)$ such that $f(a) = b$ and $f(b) = a$, then $f$ is contra tgr-continuous. However, the identity function, $I$ is not contra tgr-continuous since $\{b\} \in \sigma$ but $I^{-1}\{b\} = \{b\}$ is not tgr-closed in $(X, \tau)$. See example (3.1.3).

**Theorem 4.1.4.** Let $f : X \to Y$ be any function, then the following are equivalent:

(i) $f$ is contra tgr-continuous.

(ii) The inverse image of each closed set in $Y$ is tgr-open in $X$.

*Proof.* Follows directly from definition (4.1.1). \hfill \Box

**Theorem 4.1.5.** If $f : X \to Y$ is contra tgr-continuous function, then for every $x \in X$, each $F \in C(Y, f(x))$ there exists $U \in tgrO(X, x)$ such that $f(U) \subseteq F$ (ie), For each $x \in X$ and each closed subset $F$ of $Y$ with $f(x) \in F$, there exists a tgr-open set $U$ of $Y$ such that $x \in U$ and $f(U) \subseteq F$.

*Proof.* Let $f : X \to Y$ be a contra tgr-continuous function, let $F$ be any closed set of $Y$ and $f(x) \in F$ where $x \in X$. Then, $f^{-1}(F)$ is tgr-open in $X$, also $x \in f^{-1}(F)$. Take $U = f^{-1}(F)$, then $U$ is a tgr-open set containing $x$ and $f(U) \subseteq F$. \hfill \Box

**Theorem 4.1.6.** Let $(X, \tau)$ be a hyperconnected space, and let $\beta$ be a base for a topological space $(Y, \sigma)$. Then, $f : (X, \tau) \to (Y, \sigma)$ is contra tgr-continuous iff $f^{-1}(B)$ is tgr-closed set in $(X, \tau)$ for any $B \in \beta$. 

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Proof. ($\Rightarrow$) It is clear since any basic open set is open, ie $\beta \subseteq \sigma$.

($\Leftarrow$) Suppose that $f^{-1}(B)$ is $tgr$-closed set in $X$ for any $B \in \beta$. Let $V$ be any open set in $Y$, then $V = \bigcup_{\alpha \in \Delta} B_{\alpha}$, where $B_{\alpha} \in \beta$ for any $\alpha \in \Delta$, so $f^{-1}(B_{\alpha})$ is $tgr$-closed in $X$ for each $\alpha \in \Delta$, but $X$ is hyperconnected space, so from corollary (3.1.13), $\bigcup_{\alpha \in \Delta} f^{-1}(B_{\alpha}) = f^{-1}(\bigcup_{\alpha \in \Delta} B_{\alpha}) = f^{-1}(V)$ is $tgr$-closed in $X$. Hence, $f$ is contra $tgr$-continuous. \qed

Theorem 4.1.7. Suppose that $tgrO(X, \tau)$ is closed under arbitrary unions. Then the following are equivalent for a function $f : (X, \tau) \to (Y, \sigma)$:

1. $f$ is contra $tgr$-continuous.
2. For each $x \in X$ and each $F \in C(Y, f(x))$, there exists a set $U \in tgrO(X, x)$ such that $f(U) \subseteq F$.

Proof. (1 $\rightarrow$ 2) follows from theorem (4.1.5).

(2 $\rightarrow$ 1) Let $F$ be any closed set in $Y$, we will show that $f^{-1}(F)$ is a $tgr$-open set in $(X, \tau)$. Let $x \in f^{-1}(F)$, then $f(x) \in F$, so there exist a set $U_x \in tgrO(X, \tau)$ such that $f(U_x) \subseteq F$. Hence, $f^{-1}(F) = \bigcup_{x \in f^{-1}(F)} U_x$, then it follows that $f^{-1}(F)$ is a $tgr$-open set in $(X, \tau)$. \qed

Proposition 4.1.8. Let $f : (X, \tau) \to (Y, \sigma)$ be a function, then :

(a) If $\tau$ is the discrete topology, then $f$ is contra $tgr$-continuous.

(b) If $\tau$ is the trivial topology, then $f$ is contra $tgr$-continuous.

(c) If $\sigma$ is the trivial topology, then $f$ is contra $tgr$-continuous.

Proof. (a) follows from the fact that in the discrete topology any set is $tgr$-closed set. Similarly in the trivial topology any set is $tgr$-closed set, so (b) follows (see theorem 3.1.5). Finally, in the trivial topology the only open sets are $\phi$, $X$ and $f^{-1}(X) = X$, $f^{-1}(\phi) = \phi$ which are $tgr$-closed sets, hence (c) follows. \qed
Remark 4.1.9. Composite of two contra $tgr$-continuous functions need not be contra $tgr$-continuous, for a counter example see the following example.

Example 4.1.10. Let $f : (\mathbb{R}, \tau) \to (\mathbb{R}, \sigma)$ and $g : (\mathbb{R}, \sigma) \to (\mathbb{R}, \nu)$ be identity functions, where $\tau$ is the cofinite topology, $\sigma$ is the trivial topology and $\nu$ is the discrete topology. Then (from proposition 4.1.8) both $f$, $g$ are contra $tgr$-continuous functions, but $g \circ f : (\mathbb{R}, \tau) \to (\mathbb{R}, \nu)$ which is also the identity function is not contra $tgr$-continuous since $\{0\}$ is open set in the discrete topology but $(g \circ f)^{-1}\{0\} = \{0\}$ is not $tgr$-closed in the cofinite topology. Hence, both $f$ and $g$ are contra $tgr$-continuous but $g \circ f$ is not.

Theorem 4.1.11. If $(X, \tau)$, $(Y, \sigma)$, $(Z, \nu)$ are topological spaces and $f : (X, \tau) \to (Y, \sigma)$ is contra $tgr$-continuous function, $g : (Y, \sigma) \to (Z, \nu)$ is continuous function, then $g \circ f : (X, \tau) \to (Z, \nu)$ is contra $tgr$-continuous function.

Proof. Let $V \subseteq Z$ be any open set, then $g^{-1}(V)$ is open subset of $Y$ (by the continuity of $g$), but $f : (X, \tau) \to (Y, \sigma)$ is contra $tgr$-continuous, so $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is $tgr$-closed subset of $X$. Hence, $g \circ f : (X, \tau) \to (Z, \nu)$ is contra $tgr$-continuous.

Remark 4.1.12. If $f : (X, \tau) \to (Y, \sigma)$ is continuous function and $g : (Y, \sigma) \to (Z, \nu)$ is contra $tgr$-continuous function, then $g \circ f : (X, \tau) \to (Z, \nu)$ need not be contra $tgr$-continuous function. For a counter example, take $f$, $g$ as in example (4.1.10), then $f$ is continuous and $g$ is contra continuous but $g \circ f$ is not contra $tgr$-continuous function.

Definition 4.1.13. Let $f : (X, \tau) \to (Y, \sigma)$, then $f$ is called $tgr$-irresolute if $f^{-1}(V)$ is $tgr$-closed in $X$ whenever $V$ is $tgr$-closed in $Y$.

Theorem 4.1.14. If $(X, \tau)$, $(Y, \sigma)$, $(Z, \nu)$ are topological spaces and $f : (X, \tau) \to (Y, \sigma)$ is $tgr$-irresolute function, $g : (Y, \sigma) \to (Z, \nu)$ is contra $tgr$-continuous function, then $g \circ f : (X, \tau) \to (Z, \nu)$ is contra $tgr$-continuous function.
Proof. Let $V \subseteq Z$ be any open set, then $g^{-1}(V)$ is $tgr$-closed subset of $Y$ (since $g$ is contra $tgr$-continuous), but $f : (X, \tau) \rightarrow (Y, \sigma)$ is $tgr$-irresolute, so $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is $tgr$-closed subset of $X$. Hence, $g \circ f : (X, \tau) \rightarrow (Z, \nu)$ is contra $tgr$-continuous. \qed

Remark 4.1.15. If $f : (X, \tau) \rightarrow (Y, \sigma)$ is contra $tgr$-continuous function and $g : (Y, \sigma) \rightarrow (Z, \nu)$ is $tgr$-irresolute function, then $g \circ f : (X, \tau) \rightarrow (Z, \nu)$ need not be contra $tgr$-continuous function, for a counter example take $f, g$ as in example (4.1.10).

Theorem 4.1.16. Let $f : X \rightarrow Y$ be a function and $g : X \rightarrow X \times Y$ be the graph function of $f$, then $f$ is contra $tgr$-continuous function if $g$ is contra $tgr$-continuous.

Proof. Let $V \subseteq Y$ be any open set, then $X \times V$ is open set in $X \times Y$, but $g$ is contra $tgr$-continuous, hence $g^{-1}(X \times V)$ is $tgr$-closed set in $X$. So,

$$g^{-1}(X \times V) = \{ x \in X : g(x) = (x, f(x)) \in X \times V \}$$

$$= \{ x \in X : x \in X \cap f^{-1}(V) \}$$

$$= X \cap f^{-1}(V) = f^{-1}(V)$$ is $tgr$-closed set in $X$.

Hence, $f$ is contra $tgr$-continuous. \qed

In the following theorems and examples we discuss the relations between contra $tgr$-continuous functions and other classes of functions, for example continuous, $LC$-continuous, $RC$-continuous, perfectly continuous, contra continuous, contra $rg$-continuous functions.

Theorem 4.1.17. Every $RC$-continuous function is contra $tgr$-continuous.

Proof. Straight forward from the fact that every regular closed set is $tgr$-closed. (See theorem 3.1.19). \qed

Remark 4.1.18. The converse of the above theorem need not be true, see the following example.
Example 4.1.19. Consider the identity function \( I : (\mathbb{R}, \tau) \to (\mathbb{R}, \sigma) \) where \( \tau \) is the cofinite topology and \( \sigma \) is the left ray topology, then \( I \) is contra \( tgr \)-continuous since any nonempty open set in the left ray topology is infinite and so it is \( tgr \)-closed set in the cofinite topology. However, \( f \) is not \( RC \)-continuous since \((-\infty, 0)\) is open in the left ray topology but \( I^{-1}(-\infty, 0) = (-\infty, 0) \) which is not regular closed in the cofinite topology.

Proof. A function \( f : (X, \tau) \to (Y, \sigma) \) is called contra \( rg \)-continuous function iff \( f^{-1}(V) \) is \( rg \)-closed set in \((X, \tau)\) whenever \( V \) is open set in \((Y, \sigma)\).

\( \square \)

Theorem 4.1.20. Every contra \( tgr \)-continuous function is contra \( rg \)-continuous function.

Proof. From theorem (3.1.16), we have every \( tgr \)-closed set is \( rg \)-closed set so the result follows.

\( \square \)

Remark 4.1.21. Contra \( rg \)-continuous functions need not be contra \( tgr \)-continuous, see the following example.

Example 4.1.22. Let \( X = \{a, b\} \), \( \tau = \{X, \phi, \{a\}\} \), \( \sigma = \{X, \phi, \{b\}\} \), and let \( I : (X, \tau) \to (Y, \sigma) \) be the identity function, then from example (4.1.3) \( I \) is not contra \( tgr \)-continuous function. However \( I \) is contra \( rg \)-continuous function, to prove this, notice that the only nonempty regular open set in \((X, \tau)\) is \( X \) hence any set in \((X, \tau)\) is \( rg \)-closed set, this means that \( I \) is contra \( rg \)-continuous.

Theorem 4.1.23. If \( f : (X, \tau) \to (Y, \sigma) \) is a continuous function and \( X \) is a hyperconnected space, then \( f \) is contra \( tgr \)-continuous.

Proof. Let \( V \subseteq Y \) be any open set. If \( f^{-1}(V) = \phi \), then it is \( tgr \)-closed. If \( f^{-1}(V) \neq \phi \), then it is a nonempty open set in \((X, \tau)\) (since \( f \) is continuous), so \( f^{-1}(V) \) is dense (since \( X \) is hyperconnected), hence by theorem (3.1.7) \( f^{-1}(V) \) is \( tgr \)-closed set, so \( f \) is contra \( tgr \)-continuous.

\( \square \)
Theorem 4.1.24. If \( f : (X, \tau) \to (Y, \sigma) \) is contra tgr-continuous function and \((X, \tau)\) is tgr-locally indiscrete, then \( f \) is continuous.

Proof. Suppose that \( f \) is contra tgr-continuous function and \((X, \tau)\) is tgr-locally indiscrete. Let \( V \subseteq Y \) be any open set, then \( f^{-1}(V) \) is tgr-closed set in \((X, \tau)\), so \( f^{-1}(V) \) is open in \((X, \tau)\) (since \((X, \tau)\) is locally indiscrete ). Hence, \( f \) is continuous. \( \square \)

Theorem 4.1.25. If \( f : (X, \tau) \to (Y, \sigma) \) is contra continuous function and \((X, \tau)\) is locally indiscrete, then \( f \) is contra tgr-continuous.

Proof. Let \( V \subseteq Y \) be any open set, then \( f^{-1}(V) \) is closed in \((X, \tau)\) (since \( f \) is contra continuous) and hence it is open (since \((X, \tau)\) is locally indiscrete ), so \( f^{-1}(V) \) is clopen and then it is regular closed and tgr-closed. Hence, \( f \) is contra tgr-continuous. \( \square \)

Theorem 4.1.26. Every perfectly continuous function is contra tgr-continuous function.

Proof. Notice that any clopen set is regular closed set then it is tgr-closed set, hence the result follows. \( \square \)

Remark 4.1.27. Contra tgr-continuous functions need not be perfectly continuous, see the following example.

Example 4.1.28. Let \( X = \{a, b\} \), let \( \tau = \{X, \phi, \{a\}\} \) and \( \sigma = \{X, \phi, \{b\}\} \). Define \( f : (X, \tau) \to (X, \sigma) \) by \( f(a) = b \) and \( f(b) = a \), then \( f \) is contra tgr-continuous but it is not perfectly continuous since \( \{b\} \) is open in \((X, \sigma)\) but \( f^{-1}(V) = \{a\} \) which is not clopen set in \((X, \tau)\).

Remark 4.1.29. Contra continuous functions and contra tgr-continuous functions are independent concepts, see the following example.
Example 4.1.30. Let $X = \{a, b\}$, $\tau = \{X, \phi, \{a\}\}$ and $\sigma = \{X, \phi, \{b\}\}$, define $f : (X, \tau) \to (X, \sigma)$ such that $f(a) = b$ and $f(b) = a$ then $f$ is contra tgr-continuous function but it is not contra continuous since $\{b\} \in \sigma$ but $f^{-1}\{b\} = \{a\}$ which is not closed in $(X, \tau)$. On the other hand the identity function $I : (X, \tau) \to (X, \sigma)$ is contra continuous function but it is not contra tgr-continuous.

Remark 4.1.31. Continuous functions and contra tgr-continuous functions are independent concepts, see the following example.

Example 4.1.32. Let $f : \mathbb{R}^1 \to \mathbb{R}^1$ be the identity function, then $f$ is continuous but it is not contra tgr-continuous since $(0, 1)$ is open in $\mathbb{R}^1$ but from example (3.1.4) $f^{-1}(0, 1) = (0, 1)$ is not tgr-closed in $\mathbb{R}^1$.

On the other hand, let $f : (\mathbb{R}, \tau) \to (\mathbb{R}, \sigma)$ be the identity function, where $\tau$ is the cofinite topology and $\sigma$ is the left ray topology. Then $f$ is contra tgr-continuous but not continuous, to prove this let $V \subseteq \mathbb{R}$ be any nonempty proper open set in the lefray topology, then $V = (-\infty, a)$ for some $a \in \mathbb{R}$, so $f^{-1}(-\infty, a) = (-\infty, a)$ which is tgr-closed in the cofinite topology, but $f$ is not continuous since $(-\infty, a)$ is open the left ray topology but $f^{-1}(-\infty, a) = (-\infty, a)$ which is not open set in the cofinite topology.

Example 4.1.33. Contra tgr-continuous functions and LC-continuous fuunctions are independent. Consider the identity function $I : (\mathbb{R}, \tau) \to (\mathbb{R}, \sigma)$ where $\tau$ is the cofinite topology and $\sigma$ is the left ray topology, then $I$ is contra tgr-continuous function but not LC-continuous, since $(-\infty, 0)$ is open in the left ray topology but it is not locally closed set in the cofinite topology, to prove that $(-\infty, 0)$ is not locally closed in the cofinite topology, assume on contrary that $(-\infty, 0)$ is locally closed, so $(-\infty, 0) = A \cup B$, for some open set $A$ and some closed set $B$, so $(-\infty, 0) \subseteq B$, but the closed sets are only $\mathbb{R}$, $\phi$, and the finite sets. Hence, $B = \mathbb{R}$, and $A = (-\infty, 0)$, so $(-\infty, 0)$ is open in the cofinite
topology, which is a contradiction. On the other hand, the identity function on the real line with the usual topology is $LC$-continuous (since it is continuous), but it is not contra $tgr$-continuous since $(0,1)$ is open but it is not $tgr$-closed.

**Definition 4.1.34.** Let $f : (X, \tau) \to (Y, \sigma)$, then $f$ is almost contra $tgr$-continuous if $f^{-1}(V)$ is $tgr$-closed in $X$ whenever $V$ is regular open in $Y$.

**Theorem 4.1.35.** Every contra $tgr$-continuous function is almost contra $tgr$-continuous function.

**Proof.** Follows from the fact that every regular open set is open. □

**Remark 4.1.36.** Almost contra $tgr$-continuous functions need not be contra $tgr$-continuous, see the following example.

**Example 4.1.37.** Let $f : \mathbb{R} \to (\mathbb{R}, \tau)$ be the identity function where $\tau$ is the left ray topology, then $f$ is almost contra $tgr$-continuous since the only regular open sets in the Left ray topology are $\mathbb{R}$, $\varnothing$. However, $f$ is not contra $tgr$-continuous since $(-\infty,0)$ is open in the left ray topology but $f^{-1}(-\infty,0) = (-\infty,0)$ is not $tgr$-closed set in $\mathbb{R}$.

**Theorem 4.1.38.** Every regular set connected function is almost contra $tgr$-continuous.

**Proof.** Recall that $f : X \to Y$ is called regular set connected iff $f^{-1}(V)$ is clopen in $X$ whenever $V$ is regular open in $Y$, then the result follows from the fact that any clopen set is regular closed and then it is $tgr$-closed. □

**Remark 4.1.39.** Almost contra $tgr$-continuous functions need not be regular set connected as shown in the following example.

**Example 4.1.40.** Let $X = \{a, b, c\}$, let $f : (X, \tau) \to (X, \sigma)$ be the identity function where $\tau = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$ and $\sigma = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}\}$. $f$ is not regular set
connected, since \{a\} is regular open in \(\sigma\), but \(f^{-1}(\{a\}) = \{a\}\) which is not clopen in \(\tau\). However, \(f\) is almost contra tgr-continuous. To show this, note that the only regular open sets in \(\sigma\) are \(\phi\), \{a\}, \{b\} and \(X\). \(f^{-1}(X) = X\) which is tgr-closed, \(f^{-1}(\phi) = \phi\) which is tgr-closed. Also, \(f^{-1}(\{b\}) = \{b\}\) which is regular closed, so it is tgr-closed in \(\tau\), and \(f^{-1}(\{a\}) = \{a\}\) is tgr-closed in \(\tau\).

### 4.2 Contra \(t^* gr\)-continuous functions

In this section we introduce the notation of contra \(t^* gr\)-continuous functions in topological spaces, then we discuss some examples and some characterizations. Also, we study the relations between contra \(t^* gr\)-continuous functions and other classes of functions. We should note that there is many results about contra \(t^* gr\)-continuous functions that is similar to contra tgr-continuous functions.

**Definition 4.2.1.** A function \(f : (X,\tau) \to (Y,\sigma)\) is called contra \(t^* gr\)-continuous iff \(f^{-1}(V)\) is \(t^* gr\)-closed in \(X\) whenever \(V\) is open in \(Y\).

**Example 4.2.2.** Let \(X = \{a, b\}\) and \(\tau = \{X, \phi, \{a\}\}\), \(\sigma = \{X, \phi, \{b\}\}\) and let \(f : (X,\tau) \to (Y,\sigma)\) be the identity function, then \(f\) is contra \(t^* gr\)-continuous function. However, if we define \(g : (X,\tau) \to (Y,\sigma)\) by \(f(a) = b\), \(f(b) = a\), then \(g\) is not contra \(t^* gr\)-continuous function since \(\{b\}\) is open set in \((Y,\sigma)\), but \(g^{-1}\{b\} = \{a\}\) is not \(t^* gr\)-closed set in \((X,\tau)\).

**Remark 4.2.3.** Contra tgr-continuous functions and contra \(t^* gr\)-continuous functions are independent as shown in the following example.

**Example 4.2.4.** Consider \(X = \{a, b\}\), \(\tau = \{X, \phi, \{a\}\}\), \(\sigma = \{X, \phi, \{b\}\}\) and \(f : (X,\tau) \to (X,\sigma)\) be the identity function, then \(f\) is contra \(t^* gr\)-continuous function but it is not
contra \( tgr \)-continuous, since \( \{b\} \) is open in \((X, \sigma)\) but \( f^{-1}\{b\} = \{b\} \) is not \( tgr \)-closed set in \((X, \tau)\). On the other hand, let \( g : (X, \tau) \to (X, \tau) \) be the identity function, then \( g \) is contra \( tgr \)-continuous function but it is not contra \( t^*gr \)-continuous since \( \{a\} \) is open set in \((X, \tau)\) but it is not \( t^*gr \)-closed set in \((X, \tau)\).

**Theorem 4.2.5.** Let \( f : X \to Y \) be any function, then the following are equivalent:

(i) \( f \) is contra \( t^*gr \)-continuous.

(ii) The inverse image of each closed set in \( Y \) is \( t^*gr \)-open in \( X \).

**Proof.** Follows directly from definition (4.2.1). \( \square \)

**Theorem 4.2.6.** If \( f : X \to Y \) is contra \( t^*gr \)-continuous then for every \( x \in X \), each \( F \in C(Y, f(x)) \) there exists \( U \in t^*grO(X, x) \) such that \( f(U) \subseteq F \) (ie), For each \( x \in X \), each closed subset \( F \) of \( Y \) with \( f(x) \in F \), there exists a \( t^*gr \)-open set \( U \) of \( Y \) such that \( x \in U \) and \( f(U) \subseteq F \).

**Proof.** Let \( f : X \to Y \) be a contra \( t^*gr \)-continuous function, let \( F \) be any closed set of \( Y \) and \( f(x) \in F \) where \( x \in X \). Then, \( f^{-1}(F) \) is \( t^*gr \)-open in \( X \), also \( x \in f^{-1}(F) \). Take \( U = f^{-1}(F) \), then \( U \) is a \( t^*gr \)-open set containing \( x \) and \( f(U) \subseteq F \). \( \square \)

**Theorem 4.2.7.** Let \((X, \tau)\) be a hyperconnected space, and let \( \beta \) be a base for a topological space \((Y, \sigma)\). Then, \( f : (X, \tau) \to (Y, \sigma) \) is contra \( t^*gr \)-continuous iff \( f^{-1}(B) \) is \( t^*gr \)-closed set in \((X, \tau)\) for any \( B \in \beta \).

**Proof.** \((\Rightarrow)\) It is clear since any basic open set is open, ie \( \beta \subseteq \sigma \).

\((\Leftarrow)\) Suppose that \( f^{-1}(B) \) is \( t^*gr \)-closed in \( X \) for any \( B \in \beta \). Let \( V \) be any open set in \( Y \), then \( V = \bigcup_{\alpha \in \Delta} B_\alpha \), where \( B_\alpha \in \beta \) for any \( \alpha \in \Delta \), so \( f^{-1}(B_\alpha) \) is \( t^*gr \)-closed in \( X \) for each \( \alpha \in \Delta \), but \( X \) is hyperconnected space, so by corollary (3.2.9), \( \bigcup_{\alpha \in \Delta} f^{-1}(B_\alpha) = f^{-1}(\bigcup_{\alpha \in \Delta} B_\alpha) = f^{-1}(V) \) is \( t^*gr \)-closed in \( X \). Hence, \( f \) is contra \( t^*gr \)-continuous. \( \square \)
Theorem 4.2.8. Suppose that $t^*grO(X, \tau)$ is closed under arbitrary unions. Then the following are equivalent for a function $f : (X, \tau) \to (Y, \sigma)$:

(1) $f$ is contra $t^*gr$-continuous.

(2) for each $x \in X$ and each $F \in C(Y, f(x))$, there exists a set $U \in t^*grO(X, x)$ such that $f(U) \subseteq F$.

Proof. (1 $\to$ 2) follows from theorem (4.2.6).

(2 $\to$ 1) Let $F$ be any closed set in $Y$, we will show that $f^{-1}(F)$ is a $t^*gr$-open set in $(X, \tau)$. Let $x \in f^{-1}(F)$, then $f(x) \in F$, so there exist a set $U_x \in t^*grO(X, \tau)$ such that $f(U_x) \subseteq F$. Hence, $f^{-1}(F) = \bigcup_{x \in f^{-1}(F)} U_x$ and it follows that $f^{-1}(F)$ is a $t^*gr$-open set in $(X, \tau)$. \hfill $\square$

Proposition 4.2.9. Let $f : (X, \tau) \to (Y, \sigma)$ be a function, then :

(a) If $\tau$ is the discrete topology, then $f$ is contra $t^*gr$-continuous.

(b) If $\tau$ is the trivial topology, then $f$ is contra $t^*gr$-continuous.

(c) If $\sigma$ is the trivial topology, then $f$ is contra $t^*gr$-continuous.

Proof. (a) follows from the fact that in the discrete topology any set is $t^*gr$-closed set. Similarly in the trivial topology any set is $t^*gr$-closed set, so (b) follows (see theorem 3.2.5). Finally, in the trivial topology the only open sets are $\phi, X$ and $f^{-1}(X) = X$ , $f^{-1}(\phi) = \phi$ which are $t^*gr$-closed sets, hence (c) follows. \hfill $\square$

Remark 4.2.10. Composite of two contra $t^*gr$-continuous functions need not be contra $t^*gr$-continuous, for a counter example see the following example.

Example 4.2.11. Let $f : (\mathbb{R}, \tau) \to (\mathbb{R}, \sigma)$ and $g : (\mathbb{R}, \sigma) \to (\mathbb{R}, \nu)$ be identity functions, where $\tau$ is the cofinite topology, $\sigma$ is the trivial topology and $\nu$ is the discrete topology. Then $f$ is contra $t^*gr$-continuous. Also, define then $g$ is contra $t^*gr$-continuous, but
\( g \circ f : (\mathbb{R}, \tau_{\text{cofinite}}) \to (\mathbb{R}, \tau_{\text{discrete}}) \) which is also the identity function is not contra \( t^*\text{gr}\)-continuous since \( \{0\} \) is open set in the discrete topology but \( (g \circ f)^{-1}\{0\} = \{0\} \) is not \( t^*\text{gr}\)-closed in the cofinite topology. Hence, both \( f \) and \( g \) are contra \( t^*\text{gr}\)-continuous but \( g \circ f \) is not.

**Theorem 4.2.12.** If \((X, \tau), (Y, \sigma), (Z, \nu)\) are topological spaces and \( f : (X, \tau) \to (Y, \sigma) \) is contra \( t^*\text{gr}\)-continuous function, \( g : (Y, \sigma) \to (Z, \nu) \) is continuous function, then \( g \circ f : (X, \tau) \to (Z, \nu) \) is contra \( t^*\text{gr}\)-continuous function.

**Proof.** Let \( V \subseteq Z \) be any open set, then \( g^{-1}(V) \) is open subset of \( Y \) (by the continuity of \( g \)), but \( f : (X, \tau) \to (Y, \sigma) \) is contra \( t^*\text{gr}\)-continuous, so \( f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V) \) is \( t^*\text{gr}\)-closed subset of \( X \). Hence, \( g \circ f : (X, \tau) \to (Z, \nu) \) is contra \( t^*\text{gr}\)-continuous. \( \square \)

**Remark 4.2.13.** If \( f : (X, \tau) \to (Y, \sigma) \) is continuous function and \( g : (Y, \sigma) \to (Z, \nu) \) is contra \( t^*\text{gr}\)-continuous function, then \( g \circ f : (X, \tau) \to (Z, \nu) \) need not be contra \( t^*\text{gr}\)-continuous function. For a counter example, take \( f \), \( g \) as in example (4.2.11), then \( f \) is continuous and \( g \) is contra \( t^*\text{gr}\)-continuous but \( g \circ f \) is not contra \( t^*\text{gr}\)-continuous function.

**Definition 4.2.14.** Let \( f : (X, \tau) \to (Y, \sigma) \), then \( f \) is called \( t^*\text{gr}\)-irresolute if \( f^{-1}(V) \) is \( t^*\text{gr}\)-closed in \( X \) whenever \( V \) is \( t^*\text{gr}\)-closed in \( Y \).

**Theorem 4.2.15.** If \((X, \tau), (Y, \sigma), (Z, \nu)\) are topological spaces and \( f : (X, \tau) \to (Y, \sigma) \) is \( t^*\text{gr}\)-irresolute function, \( g : (Y, \sigma) \to (Z, \nu) \) is contra \( t^*\text{gr}\)-continuous function, then \( g \circ f : (X, \tau) \to (Z, \nu) \) is contra \( t^*\text{gr}\)-continuous function.

**Proof.** Let \( V \subseteq Z \) be any open set, then \( g^{-1}(V) \) is \( t^*\text{gr}\)-closed subset of \( Y \) (since \( g \) is contra \( t^*\text{gr}\)-continuous), but \( f : (X, \tau) \to (Y, \sigma) \) is \( t^*\text{gr}\)-irresolute, so \( f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V) \) is \( t^*\text{gr}\)-closed subset of \( X \). Hence, \( g \circ f : (X, \tau) \to (Z, \nu) \) is contra \( t^*\text{gr}\)-continuous. \( \square \)
Remark 4.2.16. If \( f : (X, \tau) \to (Y, \sigma) \) is contra \( t^*gr \)-continuous function and \( g : (Y, \sigma) \to (Z, \nu) \) is \( t^*gr \)-irresolute function, then \( g \circ f : (X, \tau) \to (Z, \nu) \) need not be contra \( t^*gr \)-continuous function, for a counter example take \( f, g \) as in example (4.2.11).

**Theorem 4.2.17.** Let \( f : X \to Y \) be a function and \( g : X \to X \times Y \) be the graph function of \( f \), then \( f \) is contra \( t^*gr \)-continuous function if \( g \) is contra \( t^*gr \)-continuous.

**Proof.** Let \( V \subseteq Y \) be any open set, then \( X \times V \) is open set in \( X \times Y \), but \( g \) is contra \( t^*gr \)-continuous, hence \( g^{-1}(X \times V) \) is \( t^*gr \)-closed set in \( X \).

\[
g^{-1}(X \times V) = \{x \in X : g(x) = (x, f(x)) \in X \times V\}
= \{x \in X : x \in X \cap f^{-1}(V)\}
= X \cap f^{-1}(V) = f^{-1}(V) \text{ is } t^*gr \text{-closed set in } X.
\]

Hence, \( f \) is contra \( t^*gr \)-continuous. \qed

In the following theorems and examples we discuss the relations between contra \( t^*gr \)-continuous functions and other classes of functions, for example \( RC \)-continuous, contra \( gr \)-continuous, contra \( \pi gr \)-continuous, contra \( rg \)-continuous, contra \( rw \)-continuous, contra \( rwg \)-continuous, contra \(SWG \)-continuous.

**Remark 4.2.18.** Continuous functions and contra \( t^*gr \)-continuous functions are independent concepts, see the following example.

**Example 4.2.19.** Let \( f : \mathbb{R}^1 \to \mathbb{R}^1 \) be the identity function, then \( f \) is continuous but it is not contra \( t^*gr \)-continuous since \((0, 1)\) is open in \( \mathbb{R}^1 \) but from example (3.2.4) \( f^{-1}(0, 1) = (0, 1) \) is not \( t^*gr \)-closed in \( \mathbb{R}^1 \).

On the other hand, let \( f : (X, \tau) \to (X, \sigma) \) be the identity function, where \( X = \{a, b\} \) and \( \tau = \{X, \phi, \{a\}\} \), \( \sigma = \{X, \phi, \{b\}\} \), then \( f \) is contra \( t^*gr \)-continuous function but it is not continuous since \( \{b\} \) is open in \( (X, \sigma) \) but \( f^{-1}\{b\} = \{b\} \) is not open in \( (X, \tau) \).
Theorem 4.2.20. Every RC-continuous function is contra $t^*gr$-continuous.

Proof. Straight forward from the fact that every regular closed set is $t^*gr$-closed. See theorem (3.2.27). \hfill \Box

Remark 4.2.21. The converse of the above theorem need not be true, see the following example.

Example 4.2.22. Consider the identity function $I : (\mathbb{R}, \tau) \rightarrow (\mathbb{R}, \sigma)$ where $\tau$ is the trivial topology and $\sigma$ is the left ray topology, then $I$ is contra $t^*gr$-continuous (by proposition 4.2.9) but it is not RC-continuous since $(-\infty, 0)$ is open in the left ray topology but $I^{-1}(-\infty, 0) = (-\infty, 0)$ which is not regular closed in the trivial topology.

Theorem 4.2.23. Every contra $t^*gr$-continuous function is contra $gr$-continuous.

Proof. From theorem (3.2.14), we have every $t^*gr$-closed set is $gr$-closed set so the result follows. \hfill \Box

Corollary 4.2.24. (1) Every contra $t^*gr$-continuous function is contra $\pi gr$-continuous function.

(2) Every contra $t^*gr$-continuous function is contra $rg$-continuous function.

(3) Every contra $t^*gr$-continuous function is contra $wg$-continuous function.

(4) Every contra $t^*gr$-continuous function is contra $rw$-continuous function.

(5) Every contra $t^*gr$-continuous function is contra $rwg$-continuous function.

Proof. Follows from corollaries (3.2.17), (3.2.21), (3.2.22), (3.2.23) and (3.2.24). \hfill \Box

Theorem 4.2.25. Every contra $t^*gr$-continuous function is contra $swg$-continuous.

Proof. From theorem (3.2.25), we have every $t^*gr$-closed set is $swg$-closed set, so the result follows. \hfill \Box
Theorem 4.2.26. If $f : (X, \tau) \to (Y, \sigma)$ is contra continuous function and $(X, \tau)$ is locally indiscrete, then $f$ is contra $t^*gr$-continuous.

Proof. Let $V \subseteq Y$ be any open set, then $f^{-1}(V)$ is closed in $(X, \tau)$ and hence it is open (since $(X, \tau)$ is locally indiscrete ), so $f^{-1}(V)$ is clopen and then it is regular closed and $t^*gr$-closed. Hence, $f$ is contra $t^*gr$-continuous. \hfill \Box

Theorem 4.2.27. Every perfectly continuous function is contra $t^*gr$-continuous function.

Proof. Notice that any clopen set is regular closed set then it is $t^*gr$-closed set, hence the result follows. \hfill \Box

Remark 4.2.28. Contra $t^*gr$-continuous functions need not be perfectly continuous, see the following example.

Example 4.2.29. Let $X = \{a, b\}$, let $\tau = \{X, \phi, \{a\}\}$. Define $f : (X, \tau) \to (X, \tau)$ by $f(a) = b$ and $f(b) = a$, then $f$ is contra $t^*gr$-continuous but it is not perfectly continuous since $\{a\}$ is open in $(X, \tau)$ but $f^{-1}\{a\} = \{b\}$ which is not clopen set in $(X, \tau)$.

Definition 4.2.30. Let $f : (X, \tau) \to (Y, \sigma)$, then $f$ is almost contra $t^*gr$-continuous if $f^{-1}(V)$ is $t^*gr$-closed in $X$ whenever $V$ is regular open in $Y$.

Theorem 4.2.31. Every contra $t^*gr$-continuous function is almost contra $t^*gr$-continuous function.

Proof. Follows from the fact that every regular open set is open. \hfill \Box

Remark 4.2.32. Almost contra $t^*gr$-continuous functions need not be contra $t^*gr$-continuous, see the following example.
**Example 4.2.33.** Let $f : \mathbb{R}^1 \to (\mathbb{R}, \tau)$ be the identity function where $\tau$ is the left ray topology, then $f$ is almost contra $t^*gr$-continuous since the only regular open sets in the Left ray topology are $\mathbb{R}, \phi$. However, $f$ is not contra $t^*gr$-continuous since $(-\infty, 0)$ is open in the left ray topology but $f^{-1}(-\infty, 0) = (-\infty, 0)$ is not $t^*gr$-closed set in $\mathbb{R}^1$.

**Theorem 4.2.34.** Suppose that $(X, \tau)$ is a locally indiscrete topological space, and $(Y, \sigma)$ is any topological space, then a function $f : (X, \tau) \to (Y, \sigma)$ is contra $tgr$-continuous iff it is contra $t^*gr$-continuous.

**Proof.** From theorem (3.2.29), we have in a locally indiscrete topological space a set is $tgr$-closed iff it is $t^*gr$-closed, so the result follows. \hfill $\square$

The following diagram summarizes the relations between contra $tgr$, contra $t^*gr$-continuous functions and other classes of functions.

![Diagram](image)

Figure 4.1: Contra $tgr$ and contra $t^*gr$-continuous functions
Bibliography


