On linear maps preserving spectrum, spectral radius and essential spectral radius

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DECLARATION

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نتيجة الحكم على أطروحة ماجستير

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حوالي الدوال الخطية الحافظة للطيف، لنصف قطر الطيف ونقص قطر الطيف الأساسي

On linear maps preserving spectrum, spectral radius and essential spectral radius

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وبعد المناولة أوصت اللجنة بمراجعة بالبحث درجة الماجستير في كلية العلوم/ قسم الرياضيات.

والجنة إذ تمنحها هذه الدرجة فإنها توصيها بثقتها وذوقاً طاعته وان تذكرها علمها في خدمة دينها ووطنها.

١٢٠٨ إبراهيم ديزاب المسمار

نائب الرئيس لشئون البحث العلمي والدراسات العليا

أ.د. عبد الرؤوف على المناعمة
Dedicated

To...

My parents,

My husband,

My lovely princes Hala,

My brothers,

My friends,

And all knowledge seekers...
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Abstract

In this thesis, we focus our study on a part of the linear maps on algebras of operators that preserve certain properties of operators. We study linear maps preserving spectrum. Let X and Y be Banach spaces. We show that a spectrum preserving surjective linear map $\phi$ from $B(X)$ to $B(Y)$ is either of the form $\phi(T) = ATA^{-1}$ for an isomorphism $A$ of $X$ onto $Y$ or the form $\phi(T) = BTB^{-1}$ for an isomorphism $B$ of $X'$ onto $Y$.

After this, we study linear maps preserving the spectral radius. Let $X$ be a complex Banach space. We show that if $\phi: B(X) \rightarrow B(X)$ is a surjective linear map such that $T$ and $\phi(T)$ have the same spectral radius for every $T \in B(X)$, then $\phi = c\theta$ where $\theta$ is either an algebra-automorphism or an antiautomorphism of $B(X)$ and $c$ is a complex constant such that $|c|=1$.

Finally, we characterize linear maps from $B(H)$ onto itself that preserve the essential spectral radius. Where $B(H)$ is the algebra of all bounded linear operators on an infinite dimensional complex Hilbert space $H$. 
Introduction

Functional analysis is a branch of mathematics, and specially of analysis, concerned with the study of spaces of functions and operators acting on them.
It has its historical roots in the study of transformations, such as the Fourier transform, and in the study of differential and integral equations. This usage of the word functional goes back to the calculus of variations, implying a function whose argument is a function. Its use in general has been attributed to mathematician Stefan Banach.

In the modern view, functional analysis is seen as the study of complete normed vector spaces over the real or complex numbers. Such spaces are called Banach spaces. An important example is a Hilbert space, where the norm arise from an inner product. These spaces are of fundamental importance in many areas, including the mathematical formulation of quantum mechanics. [13]

Spectral theory is one of the main branches of modern functional analysis and its applications. Roughly speaking, it is concerned with certain inverse operators, their general properties and their relations to the original operators. [12]

By a linear preserver we mean a linear map of an algebra A into itself which, roughly speaking, preserves certain properties of some elements in A. Linear preserver problems concern the characterization of such maps. Automorphisms and anti-automorphisms certainly preserve various properties of the elements. Therefore, it is not surprising that these two types of maps often appear in the conclusions of the results. Over the last few decades there has been a considerable interest in the so called linear preserver problems.

Over the past few years, there has been a considerable interest in linear mappings on algebras of operators that preserve certain properties of
operators. In particular, a problem how to characterize linear maps that preserve the spectrum of each operator has attracted the attention of many mathematicians.

In 1985, Jafarian and Sourour proved that a surjective linear spectrum-preserving map of \( B(X) \) onto \( B(Y) \), where \( X \) and \( Y \) are Banach spaces and \( B(X) \) the algebra of all bounded linear operators on \( X \), is either an algebra-isomorphism or an antiisomorphism.

A similar result for finite-dimensional spaces was obtained much earlier by Marcus and Moyls in 1959. Recently in 1994, Aupetit and Mouton extended the result of Jafarian and Sourour to primitive Banach algebras with minimal ideals.

In 1995, Šemrl and Bresar characterize the surjective linear maps preserve the spectral radius. Recently in 2007, Mbekhta characterized linear maps from \( B(H) \) onto itself that preserve the set of Fredholm operators in both directions. Where \( B(H) \) is the algebra of all bounded linear operators on an infinite dimensional complex Hilbert space \( H \). Further, Mbekhta and Šemrl characterized linear maps from \( B(H) \) onto itself preserving semi-Fredholm operators in both directions, and improved the recently obtained characterization of linear preservers of generalized invertibility that they obtained jointly with Rodman in 2006.

In 2007, Bendaoud, Bourhim and Sarh extend the above mentioned results to more general settings where a surjective linear map preserves the essential spectral radius. [20]

In this work, we focus our study on a surjective linear map preserving the spectrum and the spectral radius of each operator in \( B(X) \). It is our goal to describe the general form of these linear maps, where it is introduced in [1] and [29]. Next, we characterize linear maps from \( B(H) \) onto itself that preserve the essential spectral radius, where it is introduced in [21].

This thesis is organized as follows.

Chapter 1 consists of three sections. where we will introduce some concepts that are necessary for understanding this thesis and we will study some important definitions in functional analysis, this includes the Banach spaces, bounded linear operators, Hilbert spaces, Banach algebra and some related definitions and results.

Chapter 2 consists of two sections. In the first one we will study the Spectrum of bounded linear operators and some of its properties. In the second section we will talk about a surjective linear map preserving spectrum.
Chapter 3 consists of two sections. In the first one we will study the Spectral radius of bounded linear operators and some of its properties. In the second section we will characterize a surjective linear map preserving spectral radius.

Chapter 4 is divided into two sections, section 1 is talking about essential spectral radius and some related definitions, and section 2 is talking about a linear map from B(H) onto itself that preserve the essential spectral radius.

Throughout our study, all vector spaces and algebras are assumed to be over \( \mathbb{C} \), the complex field. \( \mathbb{R} \) and \( \mathbb{N} \) denotes the set of real numbers and natural numbers, respectively. B(X) will denote the algebra of all bounded linear operators on a complex Banach space X. We write \( X' \) for the dual of X and \( T^* \) for the adjoint of \( T \in B(X) \). B(H) is the algebra of all bounded linear operators on an infinite dimensional complex Hilbert space H. \( R_\lambda(T) \), \( \rho(T) \), \( \sigma(T) \), \( r_\sigma(T) \) and \( r_e(T) \) denotes the resolvent operator, resolvent, spectrum, spectral radius and essential spectral radius of \( T \in B(X) \), respectively. \( x \otimes f \) will denote the rank one operator, where \( x \in X \) and \( f \in X' \).
Chapter 1
Preliminaries

This chapter consists of three sections, where we give some basic definitions and theorems that are related to Banach spaces and Banach algebra. These concepts and results are necessary and important to understand this thesis.

In section 1.1 we give the definitions of Banach space and linear bounded operator. Also we give some related results.

In section 1.2 we define Hilbert space and some related concepts that are used in the next chapters.

In section 1.3 we give some facts about Banach algebra.

1.1 Banach Space

**Definition 1.1.1.** [12] A metric space is a pair \((X, d)\), where \(X\) is a non-empty set and \(d\) is a metric on \(X\); that is a function on \(X \times X\) such that for all \(x, y, z \in X\), we have:

1. \(d(x, y) \geq 0\).
2. \(d(x, y) = 0\) if and only if \(x = y\).
3. \(d(x, y) = d(y, x)\).
4. \(d(x, y) \leq d(x, z) + d(z, y)\).

**Definition 1.1.2.** [12] A sequence \((x_n)\) in a metric space \((X, d)\) is said to be convergent if there is \(x \in X\) such that \(\lim_{n \to \infty} d(x_n, x) = 0\). In this case \(x\) is called the limit of \(x_n\) or \((x_n)\) converges to \(x\) and we write \(\lim_{n \to \infty} x_n = x\) or simply, \(x_n \to x\). If \((x_n)\) is not convergent, it is said to be divergent.
Definition 1.1.3. [12] A sequence \((x_n)\) in a metric space \((X, d)\) is said to be **Cauchy** if for every \(\varepsilon > 0\) there is \(k = k(\varepsilon) \in \mathbb{N}\) such that \(d(x_n, x_m) < \varepsilon\) for all \(n, m > k\).

Definition 1.1.4. [12] The space \((X, d)\) is said to be **complete**, if every Cauchy sequence in \(X\) converges in \(X\).

Definition 1.1.5. [12] A **normed space** \(X\) is a vector space with a norm defined on it. A norm on a real or complex vector space \(X\) is a real-valued function on \(X\) whose value at an \(x \in X\) is defined by \(\| x \|\) which has the properties

1. \(\| x \| \geq 0\).
2. \(\| x \| = 0\) if and only if \(x = 0\).
3. \(\| \alpha x \| = |\alpha| \| x \|\).
4. \(\| x + y \| \leq \| x \| + \| y \|\). For all \(x, y \in X\) and scalar \(\alpha\).

Definition 1.1.6. [12] A **complete normed space** is said to be **Banach space**.

**Note.** A norm on \(X\) defines a metric \(d\) on \(X\) which is given by \(d(x, y) = \|x - y\|\) where \(x, y \in X\) and called the metric induced by the norm. Hence normed spaces and Banach spaces are metric spaces.

Definition 1.1.7. [12] Let \(X\) and \(Y\) be normed spaces both real or both complex. An operator \(T : X \rightarrow Y\) is called linear operator if for all \(x, y \in \text{D}(T)\) and all scalars \(\alpha\)

\[T(\alpha x + y) = \alpha Tx + Ty\]

The operator \(T\) is called a bounded linear operator if \(T\) is linear and there is a number \(c\) such that,

\[\| Tx \| \leq c \| x \| \quad \text{for all } x \in X.\]

The smallest \(c\) such that the above inequality holds for all nonzero \(x \in \text{D}(T)\) is defined as \(\|T\|\), thus

\[\| T \| = \sup\{ \frac{\|Tx\|}{\|x\|} : x \in \text{D}(T), x \neq 0 \}\]

If \(\text{D}(T) = \{0\}\) then we define \(\| T \| = 0\).

Definition 1.1.8. [12] Let \((X, d)\) and \((Y, \tilde{d})\) be metric spaces. A mapping \(T : X \rightarrow Y\) is said to be continuous at \(x_0 \in X\) if for every \(\varepsilon > 0\) there is \(\delta > 0\) such that \(\tilde{d}(Tx, Tx_0) < \varepsilon\) for all \(x\) satisfying \(d(x, x_0) < \delta\). \(T\) is said to be continuous if it is continuous at every point of \(X\).
**Definition 1.1.9.** [9] Let $X$ and $Y$ be normed spaces and let $T$ be a linear operator from $X$ to $Y$, then the **kernel** of $T$ (also null space of $T$) is denoted by $\text{Ker}T$ and defined by

$$\text{Ker}T = \{ x \in X : Tx = 0 \}.$$ 

The **cokernel** of $T$ is the quotient space $Y/\text{range}(T)$ of the codomain of $T$ by the image of $T$. The dimension of the cokernel is called the **corank** of $T$.

**Theorem 1.1.10.** If $X$ and $Y$ are normed spaces, then $B(X,Y)$ the space of all bounded linear operators from $X$ into $Y$, is a normed space with norm defined by

$$\| T \| = \sup\{ \frac{\| T(x) \|}{\| x \|} : x \in X, x \neq 0 \} = \sup\{ \| Tx \| : x \in X, \| x \| = 1 \}$$

for each $T \in B(X,Y)$. Furthermore, if $Y$ is complete, then $B(X,Y)$ is complete.

**Proof.** See [12]

**Definition 1.1.11.** [12] A bounded linear functional $f$ is a bounded linear operator with domain in a normed space $X$ and range in the scalar field $K$ of $X$.

**Definition 1.1.12.** [12] Let $X'$ denote the Banach space of all bounded linear functionals on $X$ which is called the **dual space** of $X$.

**Note.** [3] If $x' \in X'$ then the value of $x'$ at the point $x$ is denoted by $x'x$. It will sometimes be convenient to have another notation. we shall write $\langle x, x' \rangle$ for the value of $x'$ at $x$.

**Definition 1.1.13.** [3] Let $X$ and $Y$ be normed spaces and $T : X \to Y$ be a bounded linear operator. The operator $T^* : Y' \to X'$ given by $T^*y' = x'$ or its defined by the formula

$$\langle x, T^*y' \rangle = \langle Tx, y' \rangle$$

for each $x \in X$, $x' \in X'$ and $y' \in Y'$. The operator $T^*$ is called the transpose of $T$ (or the **adjoint** of $T$).

**Theorem 1.1.14.** (Hahn-Banach Theorem) If $X$ is a normed space, $\{x_1, x_2, \ldots, x_n\}$ is a linearly independent subset of $X$, and $a_1, a_2, \ldots, a_n$ are arbitrary scalars, then there is an $f$ in $X'$ such that $f(x_k) = a_k$ for all $1 \leq k \leq n$.

**Proof.** See [16]

**Theorem 1.1.15.** (Bounded linear functionals). Let $X$ be a normed space and let $x_o \neq 0$ be any element of $X$. Then there exists a bounded linear functional $f$ on $X$ such that

$$\| f \| = 1 \quad \text{and} \quad f(x_o) = \| x_o \| .$$
**Proof.** See [12]

**Definition 1.1.16.** [12] Let X and Y be normed spaces and $T : D(T) \to Y$ be a linear operator with $D(T) \subseteq X$. Then T is called a closed linear operator if its graph 

$$G(T) = \{ (x, y) : x \in D(T), y = Tx \}$$

is closed in the normed space $X \times Y$, where the two algebraic operations of the vector space $X \times Y$ are defined as

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2); \quad \alpha (x, y) = (\alpha x, \alpha y) \quad (\alpha \text{ is scalar})$$

and the norm on $X \times Y$ is defined by

$$\| (x, y) \| = \| x \| + \| y \|.$$

**Theorem 1.1.17. Closed Graph Theorem.** Let X and Y be Banach spaces and let $T : D(T) \to Y$ be a closed linear operator with $D(T) \subseteq X$. If $D(T)$ is closed in $X$, then $T$ is bounded.

**Proof.** See [12]

**Theorem 1.1.18.** Let $T : D(T) \to Y$ be a linear operator with $D(T) \subseteq X$ and X and Y be normed spaces. Then, T is closed if and only if it has the following property: if $x_n \to x$, where $x_n \in D(T)$ for all $n$ and $Tx_n \to y$, then $x \in D(T)$ and $Tx = y$.

**Proof.** See [12]

**Lemma 1.1.19.** Let $T : D(T) \to Y$ be a bounded linear operator with $D(T) \subseteq X$, where X and Y are normed spaces. Then,

(a) If $D(T)$ is a closed subset of X, then T is closed.

(b) If T is closed and Y is complete, then $D(T)$ is a closed subset of X.

**Proof.** See [12]

**Lemma 1.1.20.** If the inverse $T^{-1}$ of a closed linear operator exists, then $T^{-1}$ is a closed linear operator.

**Proof.** See [12]

**Theorem 1.1.21. Open Mapping Theorem, Bounded Inverse Theorem.** A bounded linear operator from a Banach space X onto a Banach space Y is an open mapping. Hence if T is bijective, then $T^{-1}$ is continuous and thus is bounded.

**Proof.** See [12]
Definition 1.1.22. [12] Two vector spaces are said to be isomorphic if there is a bijective linear map between the two vector spaces. Two normed spaces are said to be isomorphic if there is a bijective linear map that preserves the norm between the two normed spaces. If the vector space $X$ is isomorphic with a subspace of a vector space $Y$, then we say that $X$ is embeddable in $Y$.

Lemma 1.1.23. Let $X$ be a normed space. Then the canonical mapping $C : X \rightarrow X''$ defined by $C(x) = g_x$, where $g_x(f) = f(x)$ (for all $f \in X'$) is an isomorphism from $X$ onto $R(C)$, the range of $C$.

Proof. See [12]

Definition 1.1.24. [12] A normed space $X$ is said to be reflexive if $R(C) = X''$, where $C$ as in Lemma 1.1.23.

Theorem 1.1.25. If a normed space $X$ is reflexive, then it is complete.

Proof. See [12]

Definition 1.1.26. [12] A vector space $X$ is said to be the direct sum of two subspaces $Y$ and $Z$ of $X$, written $X = Y \oplus Z$, if each $x \in X$ has a unique representation $x = y + z$, $y \in Y$ and $z \in Z$. Then $Z$ is called the algebraic complement of $Y$ in $X$ and vice versa, and $Y, Z$ is called a complementary pair of subspaces in $X$.

Proposition 1.1.27. If $Y$ is a subspace of a vector space $X$, then there exists a subspace $Z$ of $X$ such that $X = Y \oplus Z$.

Proof. See [12]

Definition 1.1.28. [12] (Invariant subspace) A subspace $Y$ of a normed space $X$ is said to be invariant under a linear operator $T : X \rightarrow X$ if $T(Y) \subset Y$.

Definition 1.1.29. [12] A metric space $X$ is said to be compact if every sequence in $X$ has a convergent subsequence. A subset $M$ of $X$ is said to be compact if $M$ is compact considered as a subspace of $X$; that is every sequence in $M$ has a convergent subsequence whose limit belongs to $M$.

Definition 1.1.30. [12] Let $X$ and $Y$ be normed spaces. An operator $T : X \rightarrow Y$ is called a compact linear operator (or completely continuous linear operator) if $T$ is linear and if for every bounded subset $M$ of $X$, the image $T(M)$ is relatively compact, that is, the closure $\overline{T(M)}$ is compact. Equivalently, an operator $T : X \rightarrow Y$ is said to be compact if for any bounded sequence $\{x_n\}$ in $X$, $\{T(x_n)\}$ has a convergent subsequence.
Lemma 1.1.31. Let X and Y be normed spaces. Then every compact linear operator \( T : X \rightarrow Y \) is bounded, hence continuous.

Proof. See [12]

Notes. (1) The compact linear operators from a normed space X to a normed space Y form a vector space.

Proof. See [12]

(2) The set of compact operators from X to Y, denoted \( K(X,Y) \), is a closed subspace of \( B(X,Y) \).

Proof. See [6]

Theorem 1.1.32. Let \( T : X \rightarrow X \) be a compact linear operator and \( S : X \rightarrow X \) a bounded linear operator on a normed space X. Then TS and ST are compact.

Proof. See [12]

Theorem 1.1.33. Let \( T : X \rightarrow Y \) be a linear operator. If \( T \) is compact, so is its adjoint operator \( T^\times : Y' \rightarrow X' \); here X and Y are normed spaces.

Proof. See [12]

Definition 1.1.34. [32] A polynomial given by \( p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \), is called \textbf{monic} if \( a_n = 1 \).

Definition 1.1.35. [32] Let \( T \in B(X) \) and X is a normed space. Then the \textbf{minimal} polynomial of T is the unique monic polynomial \( p \) of smallest degree such that \( p(T) = 0 \).

1.2 Hilbert Space

Definition 1.2.1. [12] An \textbf{inner product space} is a vector space X with an inner product defined on X. An inner product on X is a mapping of \( X \times X \) into the scalar field \( k \) of X such that for all \( x, y \), and \( z \) in X and any scalar \( \alpha \) we have,

\[
\begin{align*}
\text{(Ip1)} \quad & < x + y, z > = < x, z > + < y, z > \\
\text{(Ip2)} \quad & < \alpha x, y > = \alpha < x, y >
\end{align*}
\]
\[(Ip3) \quad \langle x, y \rangle = \overline{\langle y, x \rangle}, \text{ the complex conjugate of } \langle y, x \rangle\]

\[(Ip4) \quad \langle x, x \rangle \geq 0, \text{ and } \langle x, x \rangle = 0 \text{ if and only if } x = 0.\]

A complete inner product space is said to be a **Hilbert space**.

**Notes.** (1) An inner product on \(X\) defines a norm on \(X\) given by \(\| x \| = \sqrt{\langle x, x \rangle}\) and a metric on \(X\) given by \(d( x, y ) = \| x - y \| = \sqrt{\langle x - y, x - y \rangle}\).

(2) Inner product spaces are normed spaces and Hilbert spaces are Banach spaces.

**Example 1.2.2.** [12] **Hilbert sequence space** \(\ell^2\) which is the set of all sequences of numbers \((\xi_i)\) such that \(\sum_{i=1}^{\infty} |\xi_i|^2\) is finite.

This space is a Hilbert spaces with inner product defined by

\[\langle x, y \rangle = \sum_{i=1}^{\infty} \xi_i \overline{\gamma_i}.\]

This inner product induces the norm

\[\| x \| = \left( \sum_{i=1}^{\infty} |\xi_i|^2 \right)^{1/2}, \text{ where } x = (\xi_i) \text{ and } y = (\gamma_i).\]

**Theorem 1.2.3.** (Riesz-Frechet) Suppose that \(H\) is a Hilbert space over a field (real or complex) \(F\). Then for every continues linear functional \(f: H \to F\), there exists a unique \(x_0 \in H\) such that

\[\| f \| = \| x_0 \| \text{ and } f(x) = \langle x, x_0 \rangle \text{ for every } x \in H.\]

**Proof:** see [12]

**Note.** By Riesz-Frechet Theorem, for every Hilbert space \(H\), \(H\) is isomorphic with \(H^*\).

**Definition 1.2.4.** Let \(H_1\) and \(H_2\) be two Hilbert spaces and \(T: H_1 \to H_2\) a bounded linear operator. The **Hilbert adjoint** operator \(T^*\) of \(T\) is the operator \(T^*: H_2 \to H_1\) such that for all \(x \in H_1\) and all \(y \in H_2\),

\[\langle Tx, y \rangle = \langle x, T^* y \rangle.\]

An operator \(T\) is called **self adjoint** if \(T^* = T\).

**Theorem 1.2.5.** The Hilbert adjoint operator \(T^*\) of \(T\) exists, is unique, is linear and is bounded with norm \(\| T^* \| = \| T \|\).

**Proof.** See [12]
Theorem 1.2.6. Let $H_1$ and $H_2$ be Hilbert spaces, $S, T: H_1 \to H_2$ bounded linear operators and $\alpha$ any scalar. Then we have,

a) $\langle T^*y, x \rangle = \langle y, Tx \rangle$ for all $x \in H_1$ and $y \in H_2$.

b) $(S + T)^* = S^* + T^*$ and $(\alpha T)^* = \bar{\alpha} T^*$.

c) $(T^*)^* = T$, and in the case $H_1 = H_2$, $(ST)^* = T^*S^*$.

d) $\| T^*T \| = \| TT^* \| = \| T \|^2$.

e) $T^*T = 0$ if and only if $T = 0$.

Proof. See [12]

Definition 1.2.7. [12] An element $x$ in an inner product space $X$ is said to be orthogonal to $y \in X$ if $\langle x, y \rangle = 0$ and we write $x \perp y$. If $A, B$ are subsets of $X$, then $x \perp A$ if $x \perp a$ for all $a \in A$, and $A \perp B$ if $a \perp b$ for all $a \in A$ and all $b \in B$.

Remark 1.2.8. Let $T \in B(H_1,H_2)$ where $H_1$ and $H_2$ are Hilbert spaces, then the orthogonal complement of range($T$),

$\text{R}(T)^\perp = \text{Ker}(T^*)$, where $T^*$ is the Hilbert adjoint of $T$.

Proof. $x \in \text{ker} T^*$ if and only if $T^*x = 0$

if and only if $\langle T^*x,y \rangle = 0$ for all $y \in H$

if and only if $\langle x,T y \rangle = 0$

if and only if $x \in (\text{range}T)^\perp$.

Definition 1.2.9. [12] An orthogonal set $M$ in an inner product space $X$ is a subset $M$ of $X$ whose elements are pairwise orthogonal. An orthonormal subset $M$ of $X$ is an orthogonal set in $X$ whose elements have norm 1.

That is, for all $x, y \in M$,

$$\langle x, y \rangle = \begin{cases} 0 & \text{if } x \neq y \\ 1 & \text{if } x = y \end{cases}$$

A set of basis vectors $B = \{ u_1, u_2, u_3, \ldots, u_n \}$ for $X$ is called an orthonormal basis if they are

(i) Orthogonal, that is $\langle u_i, u_j \rangle = 0$, for $i \neq j$.

(ii) Normalized, that is $\| u_j \| = 1$, for $j = 1, 2, 3, \ldots, n$.

Theorem 1.2.10. (Projection Theorem). Let $Y$ be any closed subspace of a Hilbert space $H$. Then $H = Y \oplus Z$, where $Z = Y^\perp = \{ z \in H : z \perp Y \}$.

Proof. See [12]
1.3 Banach Algebra

Definition 1.3.1. [12] An algebra $A$ over a field $k$ is a vector space $A$ over $k$ such that for all $x, y \in A$, a unique product $xy \in A$ is defined with the properties:

1. $(xy)z = x(yz)$
2. $(x+y)z = xz + yz$
3. $x(y+z) = xy + xz$
4. $\alpha(xy) = (\alpha x)y = x(\alpha y)$

for all $x, y, z \in A$ and scalar $\alpha \in k$.

Notes. (1) $A$ is called an algebra with unity if there exists $e \in A$ such that $ex = xe = x$ for all $x \in A$. $A$ is called unital if $\|e\| = 1$.

(2) $A$ is called a commutative algebra if $ab = ba$ for all elements $a$ and $b$ in $A$.

Definition 1.3.2. [12] A normed algebra $A$ is a normed space which is an algebra such that

$$\|xy\| \leq \|x\| \|y\|$$

for all $x, y \in A$.

If $A$ has a unit $e$, then $\|e\| = 1$.

The Banach algebra $A$ is a complete normed algebra which is complete considered as a normed space.

Example 1.3.3. Space $B(X)$. The Banach space $B(X)$ of all bounded linear operators on a complex Banach space $X \neq \{0\}$ is a Banach algebra with identity $I$, the multiplication being composition of operators. $B(X)$ is not commutative, unless dim $X = 1$.

Proof. See [12]

Definition 1.3.4. [30] Let $A$ be an algebra and $B \subseteq A$. Then $B$ is said to be a subalgebra if $B$ itself is an algebra with respect to the operations of $A$.

Definition 1.3.5. [30] If $A$ is an algebra, a left ideal of $A$ is a subalgebra $I$ of $A$ such that $ax \in I$ whenever $a \in A$, $x \in I$. A right ideal of $A$ is a subalgebra $I$ of $A$ such that $xa \in I$ whenever $a \in A$, $x \in I$. A (two sided) ideal is a subalgebra of $A$ that is both a left ideal and a right ideal.

That is $I \subseteq A$ is called an ideal (two sided) if

(i) $I$ is a subspace i.e., if $a, b \in I$ and $\alpha \in \mathbb{C}$, then $\alpha a + b \in I$.

(ii) $I$ is an ideal in the algebra i.e., $a \in I$ and $x \in A$ implies that $ax, xa \in I$.

An ideal $I$ is said to be maximal if $I \neq \{0\}$, $I \neq A$ and if $J$ is any ideal of $A$ such that $I \subseteq J$, then either $J = I$ or $J = A$.
Proposition 1.3.6. If I is a closed ideal in a Banach algebra, then A/I is a Banach algebra with the quotient norm
\[ \|\tilde{a}\| = \|a + I\| = \inf_{b \in I} \|a + b\| \]
for any \( a \in A \).

Proof. See [30]

Proposition 1.3.7. The set of all compact operators on a Hilbert space \( H \), denoted \( K(H) \), is a closed two-sided ideal in \( B(H) \).

Proof. See [30]

Theorem 1.3.8. (N. Jacobson) Let \( A \) be an algebra with unit \( 1 \) and let \( x, y \in A \), \( \alpha \in \mathbb{C} \), with \( \alpha \neq 0 \). Then \( \alpha - xy \) is invertible in \( A \) if and only if \( \alpha - yx \) is invertible in \( A \).

Proof. See [30]

Theorem 1.3.9. Let \( A \) be a Banach algebra with unit \( 1 \). Then the following sets are identical:
1) The intersection of all maximal left ideals of \( A \).
2) The intersection of all maximal right ideals of \( A \).
3) \( \{ x \in A : 1 - zx \text{ is invertible for all } z \in A \} \).
4) \( \{ x \in A : 1 - xz \text{ is invertible for all } z \in A \} \).

Proof. See [30]

Definition 1.3.10. [30] If \( A \) is a unital Banach algebra, then the set having properties (1) - (4) is called the radical (or more exactly, the Jacobson radical) \( \text{rad}(A) \) of \( A \). If \( \text{rad}(A) = \{ 0 \} \), we say that \( A \) is semi-simple.

Definition 1.3.11. [20] If \( A \) and \( B \) are algebras over a field \( K \), then we will call a map \( \varphi : A \rightarrow B \) a homomorphism if
\[
\varphi(\alpha x) = \alpha \varphi(x) \\
\varphi(x + y) = \varphi(x) + \varphi(y) \\
\varphi(xy) = \varphi(x) \varphi(y)
\]
for all \( \alpha \) in \( K \) and \( x, y \) in \( A \).

If \( \varphi \) is bijective then \( \varphi \) is said to be an isomorphism between \( A \) and \( B \).

Definition 1.3.12. [20] A bijective linear map \( \varphi \) between two algebras, \( A \) and \( B \) is called an anti-isomorphism if
\[
\varphi(xy) = \varphi(y) \varphi(x) \quad \text{for all } x \text{ and } y \text{ in } A.
\]
Definition 1.3.13. [20] An automorphism is an isomorphism from an algebra to itself. An inner-automorphism of an algebra $X$ is a function $\phi : X \to X$ defined for all $x$ in $X$ by
$$\phi(x) = a^{-1}xa,$$
where $a$ is a given fixed element of $X$. 

Definition 1.3.14. [20] Let $A$ be a Banach algebra. An element $x \in A$ is called nilpotent if $x^m = 0$ for some natural number $m \geq 1$.

Definition 1.3.15. [6] Let $A$ be a Banach algebra. An involution on $A$ is a map $*: A \to A$ satisfying the following properties:
(i) $(a*)^* = a$ for all $a \in A$,
(ii) $(\alpha a + \beta b)^* = \bar{\alpha} a^* + \bar{\beta} b^*$ for all $a, b \in A$ and all $\alpha, \beta \in \mathbb{C}$,
(iii) $(ab)^* = b^*a^*$ for all $a, b \in A$.

The pair $(A, *)$ is called an involutive Banach algebra.

Notes. [6] (1) If $A$ is an involutive Banach algebra and $L \subseteq A$ has the property that $x^* \in L$ whenever $x \in L$, then we say that $L$ is self-adjoint.

(2) If $A$ and $B$ are involutive Banach algebras and $\phi : A \to B$ is a homomorphism satisfying $\phi(x^*) = \phi(x)^*$, for all $x \in A$, we say that $\phi$ is a $*$-homomorphism.

(3) If $A$ is an involutive Banach algebra and for all $x \in A$ we have $\|x^*\| = \|x\|$, we say that the involution is isometric.

Definition 1.3.16. [6] By a C*-algebra we shall mean an involutive Banach algebra $A$ which satisfies the C*-equation
$$\|x^*x\| = \|x\|\|x^*\| = \|x\|^2$$
for every $x \in A$.

Remarks. [6] (1) In a C*-algebra the involution is isometric.

Proof. Using the C* equation, we have that
for any $x \in A$, $\|x^*x\| = \|x\|\|x^*\| = \|x\|^2$, so $\|x^*\| = \|x\|$. 

(2) If a C*-algebra $A$ has a unit $e$, then $e = e^*$.

Proof. In any unital algebra, the unit is unique. Now, for all $a \in A$, $ae^* = (ea^*)^* = (a^*)^* = a$ and similarly $a = e*a$ so $e^*$ is the identity, showing that $e = e^*$.

Examples 1.3.17. [6]
(1) From Theorem 1.2.6 an example of a C*-algebra is the algebra \( B(H) \) of bounded linear operators defined on a complex Hilbert space \( H \); here \( x^* \) denotes the adjoint operator of the operator \( x : H \to H \). Furthermore, any subalgebra \( A \) of \( B(H) \) that is closed under adjoints (that is, \( T^* \in A \) whenever \( T \in A \)) and is closed in the norm sense (hence complete) is an example of a C*-algebra.

(2) Let \( H \) be a Hilbert space and \( T \in B(H) \) be a compact operator. Denote by \( K(H) \) the set of all compact operators in \( B(H) \). \( K(H) \) is a closed ideal of \( B(H) \) by proposition 3.1.4. Moreover if \( T \in K(H) \), then \( T^* \in K(H) \) by Theorem 1.1.33. Therefore \( K(H) \) is a C*-subalgebra of \( B(H) \) by (1) above.

(3) The algebraic quotient of a C*-algebra by a closed proper two-sided ideal, with the natural norm, is a C*-algebra.

**Definition 1.3.18.** [6] A C*-algebra \( A \) is simple if it has no nontrivial closed ideals.

**Definition 1.3.19.** [6] A C*-algebra \( A \) is prime if, whenever \( J \) and \( K \) are ideals of \( A \) with \( J \cap K = \{0\} \), either \( J \) or \( K \) is \( \{0\} \). (i.e \( \{0\} \) is a prime ideal in \( A \)).
Chapter 2

Linear Maps Preserving Spectrum

This chapter consists of two sections, in the first one we study spectrum and some of its properties.

In the second section we study linear maps that preserve spectrum.

2.1 Spectrum and some of its properties

Definition 2.1.1. [12] Resolvent. Let $X \neq \{0\}$ be a complex normed space and $T: \mathcal{D}(T) \to X$ a linear operator with domain $\mathcal{D}(T) \subset X$. With $T$ we associate the operator $T_{\lambda} = T - \lambda I$ where $\lambda$ is a complex number and $I$ is the identity operator on $\mathcal{D}(T)$. If $T_{\lambda}$ has an inverse, we denote it by $R_{\lambda}(T)$, that is

$$R_{\lambda}(T) = (T - \lambda I)^{-1},$$

and call it the resolvent operator of $T$ or, simply, the resolvent of $T$.

Definition 2.1.2. [12] (Regular value, resolvent set, spectrum). Let $X \neq \{0\}$ be a complex normed space and $T: \mathcal{D}(T) \to X$ a linear operator with domain $\mathcal{D}(T) \subset X$. A regular value $\lambda$ of $T$ is a complex number such that

(R1) $R_{\lambda}(T)$ exists,

(R2) $R_{\lambda}(T)$ is bounded,

(R3) $R_{\lambda}(T)$ is defined on a set which is dense in $X$.

The resolvent set $\rho(T)$ of $T$ is the set of all regular values $\lambda$ of $T$. Its complement $\sigma(T) = \mathbb{C} - \rho(T)$ in the complex plane $\mathbb{C}$ is called the spectrum of $T$, and a $\lambda \in \sigma(T)$ is called a spectral value of $T$.

Remark 2.1.3. [12] The spectrum $\sigma(T)$ is partitioned into three disjoint sets:
(i) The point spectrum or discrete spectrum $\sigma_p(T)$ is the set such that $R_\lambda(T)$ does not exist. A $\lambda \in \sigma_p(T)$ is called an eigenvalue of $T$. A scalar $\lambda \in \mathbb{C}$ is an eigenvalue of $T$ if there exists a nonzero vector $x$ in $X$ such that $Tx = \lambda x$.

(ii) The continuous spectrum $\sigma_c(T)$ is the set such that $R_\lambda(T)$ exists and satisfies (R3) but not (R2), that is, $R_\lambda(T)$ is unbounded.

(iii) The residual spectrum $\sigma_r(T)$ is the set such that $R_\lambda(T)$ exists (and may be bounded or not) but does not satisfy (R3), that is, the domain of $R_\lambda(T)$ is not dense in $X$.

Notes. [12] (1) Each pair of $\rho(T)$, $\sigma_p(T)$, $\sigma_c(T)$ and $\sigma_r(T)$ are disjoint and their union is the whole complex plane:

$$\mathbb{C} = \rho(T) \cup \sigma(T) = \rho(T) \cup \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T).$$

(2) In the case of a finite dimensional normed space $X$ we have

$$\sigma_c(T) = \sigma_r(T) = \emptyset$$

and so $\sigma_p(T) = \sigma(T)$.

(3) If $X$ is infinite dimensional, then $T$ can have spectral values which are not eigen values. This can be seen from the following example.

Example. [12] The right-shift operator. Let $T : \ell^2 \rightarrow \ell^2$ be defined by

$$T((\xi_1, \xi_2, \xi_3, ....)) = (0, \xi_1, \xi_2, \xi_3, ....)$$

where $x = (\xi_1, \xi_2, \xi_3, ....)$ is any element in $\ell^2$. Then $0 \in \sigma_r(T)$.

Proof. First note that $T$ is bounded, because for any $x \in \ell^2$,

$$\|Tx\|^2 = \sum_{j=1}^\infty |\xi_j|^2 \leq \|x\|^2$$

and so $\|Tx\| \leq \|x\|$. Hence $\|T\| = 1$.

Let $T_0x = 0$. Then $(0, \xi_1, \xi_2, \xi_3, ....) = (0, 0, 0, 0, ....)$ and so $\xi_1 = \xi_2 = \xi_3 .... = 0$; that is $x = 0$. Therefore $\ker(T_0) = 0$, so that $R_0(T) = T_0^{-1} = T^{-1}$ exists and $R_0(T) : T(X) \rightarrow X$ is given by $T^{-1}(0, \xi_1, \xi_2, \xi_3, ....) = (\xi_1, \xi_2, \xi_3, ....)$. Hence $0 \notin \sigma_p(T)$. Since $(1, 0, 0, 0, ....) \in \ell^2$ but $(1, 0, 0, 0, ....) \notin \overline{T(X)}$, then $R(T_0) = T(X) = \{ (\eta_1) \in \ell^2 : \eta_1 = 0 \}$ is not dense in $\ell^2$. 

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Hence by definition of $\sigma_r(T)$, we have $0 \in \sigma_r(T)$; that is $0 \in \sigma(T)$ but $0 \notin \sigma_p(T)$.

**Lemma 2.1.4** [12] Let $X$ be a complex Banach space, $T: X \rightarrow X$ a linear operator, and $\lambda \in \rho(T)$. Assume that (a) $T$ is closed or (b) $T$ is bounded. Then $R_{\lambda}(T)$ is defined on the whole space $X$ and is bounded.

**Proof:** (a) Suppose that $T$ is closed. Then $D(T) = D(T)$ with $x_n \rightarrow x$ and $T_{\lambda}x_n \rightarrow y$.

Then $Tx_n = (Tx_n - \lambda x_n) + \lambda x_n = T_{\lambda}x_n + \lambda x_n$.

Therefore, $\lim Tx_n = \lim T_{\lambda}x_n + \lim \lambda x_n = y + \lambda x$.

That is $Tx_n \rightarrow y + \lambda x$ and $x_n \rightarrow x$, hence by Theorem 1.1.18, $Tx = y + \lambda x$, so that $T_{\lambda}x = (T - \lambda I)x = y$ and so $x \in D(T_{\lambda}) = D(T)$. Therefore $T_{\lambda}$ is closed.

By Lemma 1.1.20 $R_{\lambda}$ is closed. But $\lambda \in \rho(T)$, then $R_{\lambda}$ is bounded, so by Lemma 1.1.19(b) $D(R_{\lambda})$ is closed. But $D(R_{\lambda}) = X$ for any $\lambda \in \rho(T)$, hence $D(R_{\lambda}) = D(R_{\lambda}) = X$.

(b) Since $D(T) = X$ is closed and $T$ is bounded, then by Lemma 1.1.19(a) $T$ is closed and so the statement follows from part (a) of this proof.

**Theorem 2.1.5** [12] Let $T \in B(X)$, where $X$ is a Banach space. If $\|T\| < 1$, then $(I - T)^{-1}$ exists as a bounded linear operator on the whole space $X$ and

$$(I - T)^{-1} = \sum_{j=0}^{\infty} T^j = I + T + T^2 + \ldots.$$ 

[where the series on the right is convergent in the norm on $B(X)$].

**Proof:** Since $\|T\| < 1$, then $\sum_{j=0}^{\infty} \|T^j\|$ is convergent.

But $\|T\| \leq \|T\| \|T^{j+1}\| = \|T\| \|T\| \|T^j\| \leq \ldots \leq \|T\|^j$.

Then $\sum_{j=0}^{\infty} \|T^j\|$ is convergent; that is $\sum_{j=0}^{\infty} T^j$ is absolutely convergent. Since $X$ is complete, then $B(X)$ is complete by Theorem 1.1.10, therefore $\sum_{j=0}^{\infty} T^j$ is convergent say $\sum_{j=0}^{\infty} T^j = S$.

Note that $(I - T)(I + T + \ldots + T^n) = (I + T + \ldots + T^n)(I - T) = I - T^{n+1}$.

We now let $n \rightarrow \infty$, then $T^{n+1} \rightarrow 0$ because $\|T\| < 1$. Hence we obtain
\[(I - T)S = S(I - T) = I. \] Therefore, \([I - T]^{-1} = S = \sum_{j=0}^{\infty} T^j.\]

**Theorem 2.1.6** [12] (Spectrum closed). The resolvent set \(\rho(T)\) of a bounded linear operator \(T\) on a complex Banach space \(X\) is open; hence the spectrum \(\sigma(T)\) is closed.

**Proof.** If \(\rho(T) = \emptyset\), then it is open.

Let \(\rho(T) \neq \emptyset\). For a fixed \(\lambda_o \in \rho(T)\) and any \(\lambda \in \mathbb{C}\) we have

\[
T - \lambda I = T - \lambda_o I - (\lambda - \lambda_o)I \\
= (T - \lambda_o I) [I - (\lambda - \lambda_o)(T - \lambda_o I)^{-1}].
\]

Let \(V = I - (\lambda - \lambda_o)(T - \lambda_o I)^{-1}\), then \(T_{\lambda} = T_{\lambda_o} V \quad \ldots \quad (I)\)

Since \(\lambda_o \in \rho(T)\) and \(T\) is bounded, then by Lemma 2.1.4 (b) \(R_{\lambda_o} = T_{\lambda_o}^{-1} \in B(X)\).

Now we show that

\[
U(\lambda_o) = \{ \lambda \in \mathbb{C} : |\lambda - \lambda_o| < \frac{1}{\|R_{\lambda_o}\|} \}
\]

is contained in \(\rho(T)\), and so \(\rho(T)\) will be open, because \(\lambda_o\) was arbitrary and fixed in \(\rho(T)\).

Let \(\lambda \in U(\lambda_o)\). Then \(\|\lambda - \lambda_o\| R_{\lambda_o} \| < 1\) and so by Theorem 2.1.5, \(V = I - (\lambda - \lambda_o) R_{\lambda_o}\) has an inverse,

\[
V^{-1} = \sum_{j=0}^{\infty} [\lambda - \lambda_o j] R_{\lambda_o}^{-1} = \sum_{j=0}^{\infty} (\lambda - \lambda_o)^j R_{\lambda_o}^{-j} \quad \text{and} \quad V^{-1} \in B(X).
\]

But \(T_{\lambda_o}^{-1} = R_{\lambda_o} \in B(X)\), then by (I)

\[
R_{\lambda} = T_{\lambda}^{-1} = (T_{\lambda_o} V)^{-1} = V^{-1} T_{\lambda_o}^{-1} = V^{-1} R_{\lambda_o} \text{ exists.}
\]

That means \(\lambda \in \rho(T)\). Therefore \(U(\lambda_o) \subseteq \rho(T)\).

Hence \(\rho(T)\) is open, so that its complement \(\sigma(T) = \mathbb{C} - \rho(T)\) is closed.

**Theorem 2.1.7** [12] The spectrum \(\sigma(T)\) of a bounded linear operator \(T: X \rightarrow X\) on a complex Banach space \(X\) is compact and lies in the disk given by

\[
|\lambda| \leq \|T\|
\]

Hence the resolvent set \(\rho(T)\) of \(T\) is not empty.

[ Recall that a subset of the complex plane is compact if and only if it is closed and bounded ]
**Proof.** Let \( \lambda \in \{ \lambda \in \mathbb{C} : |\lambda| > \|T\| \} \). Then \( \left\| \frac{1}{\lambda} T \right\| = \frac{\|T\|}{|\lambda|} < 1 \), and so by Theorem 2.1.5

\[ R_\lambda = (T - \lambda I)^{-1} \]

\[ = -\frac{1}{\lambda} (I - \frac{1}{\lambda} T)^{-1} \]

\[ = \frac{1}{|\lambda|} \sum_{j=0}^{\infty} \left( \frac{T}{\lambda} \right)^j \quad \text{and} \quad R_\lambda \in B(X). \]

Hence \( \lambda \in \rho(T) \). Therefore \( \{ \lambda \in \mathbb{C} : |\lambda| > \|T\| \} \subseteq \rho(T) \) which implies that \( \{ \lambda \in \mathbb{C} : |\lambda| \leq \|T\| \} \supseteq \sigma(T) \), so that \( \sigma(T) \) is bounded and by Theorem 2.1.6 \( \sigma(T) \) is closed. Hence \( \sigma(T) \) is compact.

**Definition 2.1.8** [12] A metric space is said to be **connected** if it is not the union of two disjoint nonempty open subsets. A subset of a metric space is said to be **connected** if it is connected regarded as a subspace.

**Definition 2.1.9** [12] A **domain** \( G \) in the complex plane \( \mathbb{C} \) is an open connected subset \( G \) of \( \mathbb{C} \).

**Definition 2.1.10** [12] A complex valued function \( h \) of a complex variable \( \lambda \) is said to be **holomorphic** (or analytic) on a domain \( G \) of the complex \( \lambda \)-plane if \( h \) is defined and differentiable on \( G \), that is, the derivative \( h' \) of \( h \), defined by

\[ h'(\lambda) = \lim_{\Delta \lambda \to 0} \frac{h(\lambda + \Delta \lambda) - h(\lambda)}{\Delta \lambda} \]

exists for every \( \lambda \in G \).

The function \( h \) is said to be holomorphic at a point \( \lambda_0 \in \mathbb{C} \) if \( h \) is holomorphic on some \( \epsilon \)-neighborhood of \( \lambda_0 \).

**Notes.** [12] (1) \( h \) is holomorphic on \( G \) if and only if at every \( \lambda_0 \in G \) it has a power series representation

\[ h(\lambda) = \sum_{j=0}^{\infty} c_j (\lambda - \lambda_0)^j \]

with a nonzero radius of convergence.

(2) Here, holomorphicity is defined over an open set, however, differentiability could only at one point. If \( h \) is holomorphic over the entire complex plane, we say that \( h \) is **entire**.
Definition 2.1.11 [12] A vector valued function or operator function is a mapping $S : \Lambda \rightarrow B(X)$ where $\Lambda$ is any subset of the complex $\lambda$-plane and $X$ is an normed space.

Definition 2.1.12 [12] Let $\Lambda$ be an open subset of $\mathbb{C}$ and $X$ a complex Banach space. Then $S : \Lambda \rightarrow B(X)$ given by $S(\lambda) = S_{\lambda}$ is said to be locally holomorphic on $\Lambda$ if $\forall \, x \in X$ and $f \in X'$ the function $h : \Lambda \rightarrow \mathbb{C}$ defined by $h(\lambda) = f(S_{\lambda}x)$ is holomorphic at every $\lambda_{o} \in \Lambda$. $S$ is said to be holomorphic on $\Lambda$ if $S$ is locally holomorphic on $\Lambda$ and $\Lambda$ is a domain. $S$ is said to be holomorphic at a point $\lambda_{o}$ $\in \mathbb{C}$ if $S$ is holomorphic on some $\epsilon$-neighborhood of $\lambda_{o}$.

Theorem 2.1.13. For any Banach space $X$, $T \in B(X)$ and every $\lambda_{o} \in \rho(T)$, the resolvent $R_{\lambda}(T)$ is represented by

$$R_{\lambda}(T) = \sum_{j=0}^{\infty} (\lambda - \lambda_{o})^{j} R_{\lambda_{o}} \lambda^{j+1}$$

The series being absolutely convergent for every $\lambda$ in the open disk given by $|\lambda - \lambda_{o}| < \frac{1}{\|R_{\lambda_{o}}\|}$ in the complex plane. This disk is a subset of $\rho(T)$.

Proof. See [12]

Theorem 2.1.14. [12] The resolvent $R_{\lambda}(T)$ of a bounded linear operator $T : X \rightarrow X$ on a complex Banach space $X$ is holomorphic at every $\lambda_{o} \in \rho(T)$. Hence it is locally holomorphic on $\rho(T)$.

Proof. By Theorem 2.1.13, if $\lambda_{o} \in \rho(T)$, then

$$R_{\lambda}(T) = \sum_{j=0}^{\infty} R_{\lambda_{o}} (T)^{j+1} (\lambda - \lambda_{o})^{j} \ldots (1)$$

Converges absolutely for all $\lambda$ satisfying $|\lambda - \lambda_{o}| < \frac{1}{\|R_{\lambda_{o}}\|}$.

Now, $\forall \, x \in X$ and $\forall \, f \in X'$, define $h : \rho(T) \rightarrow \mathbb{C}$ by $h(\lambda) = f(R_{\lambda}(T)(x))$.

Then by (1)

$$h(\lambda) = f(R_{\lambda}(T)(x)) = \sum_{j=0}^{\infty} f (R_{\lambda_{o}} (T)^{j+1} (\lambda - \lambda_{o})^{j} x) \quad \text{(since $f$ is continuous)}$$

$$= \sum_{j=0}^{\infty} (\lambda - \lambda_{o})^{j} f (R_{\lambda_{o}} (T)^{j+1} x)$$
= \sum_{j=0}^{\infty} c_j \left( \lambda - \lambda_0 \right)^j.

Where \( c_j = f \left( R_{\lambda_0}(T)^j x \right) \) with radius of convergence \( \frac{1}{R_{\lambda_0}} \neq 0 \). Hence \( R_{\lambda} \) is holomorphic at \( \lambda_0 \in \rho(T) \) and so it is holomorphic at every \( \lambda \in \rho(T) \). Hence \( R_{\lambda} \) is locally holomorphic on \( \rho(T) \).

**Theorem 2.1.15.** [12] If \( X \neq \{0\} \) is a complex Banach space and \( T \in B(X) \), then \( \sigma(T) \neq \emptyset \).

**Proof:** If \( T = 0 \), then \( T_{\lambda} x = 0 \) for \( x \neq 0 \) only when \( \lambda = 0 \), that is \( \sigma(T) = \{0\} \neq \emptyset \).

Let \( T \neq 0 \), then \( \|T\| \neq 0 \).

Now if \( |\lambda| \geq 2\|T\| \), then \( \frac{1}{|\lambda|} \leq \frac{1}{2\|T\|} < \frac{1}{\|T\|} \) and hence by Theorem 2.1.7

\[
R_{\lambda} = \frac{-1}{\lambda} \sum_{j=0}^{\infty} \left( \frac{1}{\lambda} T \right)^j \text{ is convergent. Moreover, } \|R_{\lambda}\| \leq \frac{1}{|\lambda|} \sum_{j=0}^{\infty} \left| \frac{1}{\lambda} T \right|^j \ldots \ldots (1)
\]

But \( (|\lambda| - \square T \square)^j \left( \|T\| + \left\| \frac{T}{\lambda} \right\|^2 + \left\| \frac{T}{\lambda} \right\|^3 + \ldots \right) \)

\[
= |\lambda| + \square T \square \left( \|T\| + \left\| \frac{T}{\lambda} \right\|^2 + \left\| \frac{T}{\lambda} \right\|^3 + \ldots \right) - \|T\| \square \left( \left\| \frac{T}{\lambda} \right\|^2 + \left\| \frac{T}{\lambda} \right\|^3 + \ldots \right)
\]

\[
= |\lambda| - \left| \|T\| \right| \frac{\left| \|T\| \right|^{n+1}}{|\lambda|^n}
\]

\[
= |\lambda| \left( 1 - \left| \|T\| \right| \right)
\]

Then as \( n \to \infty \), we have \( (|\lambda| - \|T\|)^{\sum_{j=0}^{\infty} \frac{1}{|\lambda|} T} = |\lambda| \) because \( \|T\| \leq \frac{1}{2} \).

Hence \( \frac{1}{|\lambda|} \sum_{j=0}^{\infty} \|T\| = \frac{1}{|\lambda| - \|T\|} \ldots \ldots (2) \)

By (1) and (2), \( \|R_{\lambda}\| \leq \frac{1}{|\lambda| - \|T\|} \leq \frac{1}{\|T\|} \ldots \ldots (3) \) because \( |\lambda| \geq 2\|T\| \).
Hence the series is absolutely convergent.

Now, suppose that $\sigma(T) = \emptyset$. Then $\rho(T) = \mathbb{C}$.

Hence $R_\lambda$ is holomorphic for all $\lambda \in \mathbb{C}$ by Theorem 2.1.14. Consequently, for a fixed $x \in X$ and a fixed $f \in X'$, the function $h : \mathbb{C} \to \mathbb{C}$ defined by $h(\lambda) = f(R_\lambda x)$ is holomorphic on $\mathbb{C}$, that is, $h$ is an entire function. Since holomorphy implies continuity, $h$ is continuous and thus bounded on the compact disk $\{ \lambda \in \mathbb{C} : |\lambda| \leq 2\|T\| \}$. But by (3), for $|\lambda| \geq 2\|T\|$, we have,

$$|h(\lambda)| = \left| f \left( R_\lambda x \right) \right|$$
\[ \leq \|f\| \left\| R_\lambda x \right\| \]
\[ \leq \|f\| \left\| R_\lambda \right\| \left\| x \right\| \]
\[ \leq \|f\| \frac{\|x\|}{\|T\|} \]

Hence $h$ is bounded on $\{ \lambda \in \mathbb{C} : |\lambda| \geq 2\|T\| \}$. Therefore $h$ is bounded on $\mathbb{C}$ which is entire, then by Liouville's theorem, which states that an entire function which is bounded on the whole complex plane is a constant. Since $x \in X$ and $f \in X'$, in $h$ were arbitrary, $h = constant$ implies that $R_\lambda$ is independent of $\lambda$, and so is $R_\lambda^{-1} = T - \lambda I$ is independent of $\lambda$ which is impossible.

Therefore $\sigma(T) \neq \emptyset$.

**Theorem 2.1.16.** [12] **Spectral Mapping Theorem for Polynomials.** Let $X$ be a complex Banach space, $T \in B(X)$ and $p(\lambda) = a_n \lambda^n + a_{n-1} \lambda^{n-1} + \ldots + a_0$ ($a_n \neq 0$), then

$$\sigma(p(T)) = p(\sigma(T))$$

That is, the spectrum $\sigma(p(T))$ of the operator $p(T) = a_n T^n + a_{n-1} T^{n-1} + \ldots + a_0 I$ consists precisely of all those values which the polynomial $p$ assumes on the spectrum $\sigma(T)$ of $T$.

**Proof:** By Theorem 2.1.15 $\sigma(T) \neq \emptyset$.

If $n = 0$, then $p(\lambda) = \alpha_0$, so that

$$p(\sigma(T)) = \{ p(\lambda) \in \mathbb{C} : \lambda \in \sigma(T) \} = \{\alpha_0\} \text{ and}$$

$$\sigma(p(T)) = \sigma(\alpha_0 I) = \{ \lambda \in \mathbb{C} : (\alpha_0 I - \lambda I)x = 0, x \neq 0 \} = \{ \lambda \in \mathbb{C} : (\alpha_0 - \lambda)Ix = 0, x \neq 0 \}$$
= \{ \lambda \in \mathbb{C} : \lambda = \alpha_0 \} = \{ \alpha_0 \}

Therefore, \( p(\sigma(T)) = \sigma(p(T)) = \{ \alpha_0 \} \) in the case \( n = 0 \).

If \( n > 0 \), we show that

(a) \( \sigma(p(T)) \subset p(\sigma(T)) \) and (b) \( p(\sigma(T)) \subset \sigma(p(T)) \).

(a) Let \( \mu \in \mathbb{C} \) be fixed and arbitrary, \( S = p(T) \) and let \( S_\mu = P(T) - \mu I \).

Suppose that \( S_\mu^{-1} \) exists. Then \( S_\mu^{-1} \) is the resolvent operator of \( p(T) \). Since \( X \) is complex, then the polynomial given by \( S_\mu(\lambda) = p(\lambda) - \mu \) must be factored completely into linear terms say,

\[
S_\mu(\lambda) = p(\lambda) - \mu = \alpha_n (\lambda - \gamma_1) (\lambda - \gamma_2) \ldots \ldots (\lambda - \gamma_n) \ldots \ldots(1)
\]

where \( \gamma_1, \gamma_2, \ldots, \gamma_n \) are the zeros of \( S_\mu \).

Corresponding to (1) we have
\[
S_\mu = p(T) - \mu I = \alpha_n (T - \gamma_1 I)(T - \gamma_2 I) \ldots \ldots (T - \gamma_n I).
\]

If each \( \gamma_i \in \rho(T) \), then each \( T - \gamma_i I \) has a bounded inverse which, by Lemma 2.1.4 is defined on all of \( X \), because \( T \in B(X) \) and \( X \) is complex Banach space. So that \( S_\mu^{-1} \) is bounded and is defined on all of \( X \) and

\[
S_\mu^{-1} = \frac{1}{\alpha_n} (T - \gamma_n I)^{-1} \ldots \ldots (T - \gamma_1 I)^{-1}
\]

Hence, \( \mu \in \rho(p(T)) \).

Therefore, we conclude that if \( \mu \in \sigma(p(T)) \), then \( \mu \notin \rho(p(T)) \) and so \( \gamma_i \notin \rho(T) \) for some \( i \); that is \( \gamma_i \in \sigma(T) \) for some \( i \).

Now by (1) \( s_\mu(\gamma_i) = p(\gamma_i) - \mu = 0 \), hence \( \mu = p(\gamma_i) \in p(\sigma(T)) \).

Therefore, \( \sigma(p(T)) \subset p(\sigma(T)) \).

(b) We show that \( p(\sigma(T)) \subset \sigma(p(T)) \).

Let \( \kappa \in p(\sigma(T)) \). Then \( \kappa = p(\beta) \) for some \( \beta \in \sigma(T) \). There are two cases:

(A) \( T - \beta I \) has no inverse or (B) \( T - \beta I \) has an inverse.

(A) If \( T - \beta I \) has no inverse, since \( \kappa = p(\beta) \), then \( p(\beta) - \kappa = 0 \) and so \( \beta \) is a zero of the polynomial given by

\[
s_\kappa(\lambda) = p(\lambda) - \kappa.
\]

It follows that we can write
$s_κ(λ) = p(λ) - κ = (λ - β)g(λ),$

where $g(λ)$ denotes the product of the other $n-1$ linear factors and $α_n$. Corresponding to this representation we have

$$S_κ = p(T) - κI = (T - βI)g(T) \quad \ldots \quad (2)$$

Since the factors of $g(T)$ all commute with $(T - βI)$, we also have

$$S_κ = g(T)(T - βI).$$

Now we show that $S_κ^{-1}$ does not exist (by contradiction).

Suppose that $S_κ^{-1}$ exists, then

$$I = (T - βI)g(T)S_κ^{-1} = S_κ^{-1}g(T)(T - βI).$$

Hence $(T - βI)$ has an inverse, which is a contradiction. Therefore $S_κ^{-1}$ does not exist, so $κ \in σ(p(T)).$

In this case we have $p(σ(T)) \subseteq σ(p(T)).$

(B) If $(T - βI)^{-1}$ exists, then for the range of $T - βI$ we must have

$$R(T - βI) \neq X \quad \ldots \quad (3)$$

because if $R(T - βI) = X$, then by Theorem 1.1.21 applied to $T - βI$, $(T - βI)^{-1}$ is bounded and so $β \in ρ(T)$ which is a contradiction.

By (2) and (3) we obtain $R(S_κ) \neq X.$

Then $κ \in σ(p(T))$, because if $κ \notin σ(p(T))$, then $κ \in ρ(p(T))$ and so Lemma 2.1.4(b) applied to $p(T)$ implies that $R(S_κ) = X.$

Therefore, in this case also $p(σ(T)) \subseteq σ(p(T)).$

From all of the above we have, $p(σ(T)) = σ(p(T)).$ \[\square\]

**Remark:** (1) In particular,

$$σ(T^n) = σ(T)^n \quad \text{for every} \quad n ≥ 0,$$

that is, $ν \in σ(T^n) = \{ λ^n \in ℂ : λ \in σ(T) \}$ if and only if $ν \in σ(T^n)$, and

$$σ(αT) = ασ(T) \quad \text{for every} \quad α \in ℂ,$$
that is, \( \nu \in \sigma(T) = \{ \alpha \lambda \in \mathbb{C} : \lambda \in \sigma(T) \} \) if and only if \( \nu \in \sigma(\alpha T) \).

(2) If \( T \) is a nilpotent operator (i.e., if \( T^n = 0 \) for some \( n \geq 1 \)), then \( \sigma(T) = \sigma_p(T) = \{0\} \), and so \( r_{\sigma(T)} = 0 \).

**Proof.** Suppose \( T \) is nilpotent, with \( T^n = 0 \) for some \( n \geq 1 \). If \( \lambda \in \sigma(T) \), then \( \lambda^n \in \sigma(T^n) = \sigma(0) = \{0\} \) by (1) above, so that \( \lambda = 0 \).

**Definition 2.1.17.** [31] Let \( X \) and \( Y \) be two normed spaces then \( T \in B(X, Y) \) is said to be an operator of **finite rank** if \( \dim R(T) \) is finite.

**Definition 2.1.18.** [18] We say that an operator \( T : X \to X \) is a **rank one operator** if there exist \( x \in X \) and \( f \in X' \) so that \( Ty = \langle y, f \rangle x \). We use the notation \( T = x \otimes f \).

**Note 2.1.19.** [1] The **duality** between a Banach space and its dual is denoted by \( \langle x, f \rangle = f(x) \), for any \( x \in X \) and any \( f \in X' \).

**Lemma 2.1.20.** If \( K : H \to B \) is a finite rank operator where \( H \) and \( B \) are Hilbert spaces, then \( K \) is compact.

**Proof.** See [31]

**Theorem 2.1.21.** Let \( X \) be a normed space, and let \( T \in B(X) \) be compact and let \( \lambda \) be a nonzero complex number. Then, exactly one of the followings hold:

1. \( T - \lambda I \) is invertible
2. \( \lambda \) is an eigenvalue of \( T \).

This result, known has Fredholm Alternative,

**Proof.** See [12].

**Lemma 2.1.22.** [1] Let \( X \) be Banach space and \( A \in B(X) \). Then \( \sigma(T+ A) \subseteq \sigma(T) \) for every \( T \in B(X) \) if and only if \( A = 0 \).

**Proof.** Let \( A = 0 \), then \( \sigma(T+ A) = \sigma(T+0) = \sigma(T) \).

Conversely assume that \( A \neq 0 \), and let \( x \) be a vector in \( X \) such that \( Ax = y \neq 0 \). Since \( y \neq 0 \), then there exists an \( f \in X' \) such that \( \langle x, f \rangle = 1 \) and \( \langle y, f \rangle \neq 0 \).

Let \( T = (x-y) \otimes f \), then \( (T+A)x = Tx + Ax \)

\[ = (x-y) \otimes f x + Ax \]

\[ = \langle x, f \rangle (x-y) + y \]
\[(x - y) + y = x\] 
(since \(\langle x, f \rangle = 1\))

so \(1 \in \sigma(T + A)\).

But \(\sigma(T) = \{0, \langle x - y, f \rangle \}\) and

\[
\langle x - y, f \rangle = \langle x, f \rangle - \langle y, f \rangle = 1 - \langle y, f \rangle \neq 1, \text{ so } 1 \notin \sigma(T).
\]

Therefore, \(\sigma(T + A) \notin \sigma(T)\).

2.2 Linear Maps Preserving Spectrum

**Definition 2.2.1.** [1] Let \(X\) and \(Y\) be Banach spaces, we say that \(\phi : B(X) \rightarrow B(Y)\) is a **spectrum-preserving** linear map from \(B(X)\) to \(B(Y)\) if

\[
\sigma(\phi(T)) = \sigma(T) \quad \text{for every } T \in B(X).
\]

**Lemma 2.2.2.** [1] Let \(X\) and \(Y\) be Banach spaces. If \(\phi\) is a spectrum-preserving linear map from \(B(X)\) to \(B(Y)\), then \(\phi\) is injective.

**Proof.** Let \(\phi\) be a linear map from \(B(X)\) to \(B(Y)\) which preserves the spectrum.

Let \(A \in B(X)\) such that \(\phi(A) = 0\), then for every \(T \in B(X)\),

\[
\sigma(T + A) = \sigma(\phi(T + A)) = \sigma(\phi(T) + \phi(A)) = \sigma(\phi(T) + 0) = \sigma(\phi(T)) = \sigma(T)
\]

By Lemma 2.1.22, \(A = 0\). Therefore \(\phi\) is injective.
In what follows, we will assume that $\phi$ is a surjective linear map preserving the spectrum from $B(X)$ to $B(Y)$.

**Lemma 2.2.3.** [1] Let $X$, $Y$ and $\phi$ be as in Lemma 2.2.2. If $\phi$ is surjective, then $\phi(I) = I$.

**Proof.** Since $I \in B(Y)$ and $\phi$ is surjective, there exists an $S \in B(X)$ such that $\phi(S) = I$.

For any $T \in B(X)$, we have

$$
\sigma(T+(S-I)) = \sigma(\phi(T+S-I)) \quad \text{(by Definition 2.2.1)}
$$

$$
= \sigma(\phi(T-I) + \phi(S))
$$

$$
= \sigma(\phi(T-I) + 1)
$$

$$
= \sigma(\phi(T-I) + 1) \quad \text{(by Theorem 2.1.16 where $a_0 = 1$ and $\phi(T-I)$ is the operator)}
$$

$$
= \sigma(T-I) + 1 \quad \text{(by Definition 2.2.1)}
$$

$$
= 1 + \sigma(T) - 1 \quad \text{(by Theorem 2.1.16)}
$$

$$
= \sigma(T) \quad \text{for every $T \in B(X)$}.
$$

Then $S - I = 0$ by Lemma 2.1.22. Therefore $S = I$ and so $\phi(I) = I$.

**Lemma 2.2.4.** [1] Let $X$ be a Banach space. Then for $T \in B(X)$, $x \in X$, $f \in X'$ and $\lambda \notin \sigma(T)$, we have

$$
\lambda \in \sigma(T+x \otimes f) \text{ if and only if } \langle (\lambda - T)^{-1} x, f \rangle = 1.
$$

**Proof.** First of all since $\lambda \notin \sigma(T)$, then $(\lambda - T)^{-1} \in B(X)$.

Let $\langle (\lambda - T)^{-1} x, f \rangle = 1 \quad \ldots \quad (1)$, then

$$
(T + x \otimes f) (\lambda - T)^{-1} x = T(\lambda - T)^{-1} x + x \otimes f ((\lambda - T)^{-1} x)
$$

$$
= T(\lambda - T)^{-1} x + \langle (\lambda - T)^{-1} x, f \rangle x
$$

$$
= T(\lambda - T)^{-1} x + x \quad \text{(by (1))}
$$

$$
= (\lambda - T)^{-1} (Tx + (\lambda - T)x)
$$

$$
= \lambda (\lambda - T)^{-1} x,
$$

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Therefore, \( \lambda \) is an eigenvalue of \( T + x \otimes f \), thus \( \lambda \in \sigma(T + x \otimes f) \).

Conversely, let \( \lambda \in \sigma(T + x \otimes f) \), then by the invariant of the Fredholm alternative \( \lambda \) is an eigenvalue of \( T + x \otimes f \) and so there exists a nonzero vector \( u \in X \) such that \((T + x \otimes f)u = \lambda u\).

Then, \( \lambda u = Tu + (x \otimes f)u \)
\[ = Tu + \langle u, f \rangle x \text{. Then } \]
\[ \langle u, f \rangle x = \lambda u - Tu = (\lambda - T)u \]
Therefore \( u = \langle u, f \rangle (\lambda - T)^{-1} x \)
\[ = \langle (\lambda - T)^{-1} x, f \rangle u \]
It follows that \( \langle (\lambda - T)^{-1} x, f \rangle = 1 \).

**Definition 2.2.5.** [10] An operator is Fredholm if its range is closed and its kernel and cokernel are finite-dimensional. The essential spectrum of \( T \), usually denoted \( \sigma_e(T) \), is the set of all complex numbers \( \lambda \) such that \( T - \lambda I \) is not a Fredholm operator.

**Remark.** The essential spectrum is a subset of the spectrum \( \sigma \), and its complement is the discrete spectrum, so \( \sigma_e(T) = \sigma(T) - \sigma_p(T) \).

The following theorem, which may be of independent interest, gives a spectral characterization of rank one operators.

**Theorem 2.2.6.** [1] Let \( X \) be Banach space and \( A \in B(X), A \neq 0 \). The following conditions are equivalent.

(i) \( A \) has rank 1.

(ii) \( \sigma(T + A) \cap \sigma(T + cA) \subseteq \sigma(T) \) for every \( T \in B(X) \) and for every scalar \( c \neq 1 \).

**Proof.**

(i) \( \Rightarrow \) (ii) Suppose \( A \) has rank 1, then by Definition 2.1.18, \( A = x \otimes f \) for any \( x \in X \) and any \( f \in X' \). Let \( T \in B(X) \) and \( c \neq 1 \), suppose that \( \lambda \notin \sigma(T) \), we show that \( \lambda \notin \sigma(T + A) \cap \sigma(T + cA) \) for all \( c \neq 1 \).

Since \( \lambda \notin \sigma(T) \), then \((\lambda - T)^{-1} \in B(X) \). Now, if \( \lambda \in \sigma(T + cA) \), then by Lemma 2.2.4, \( c <(\lambda - T)^{-1} x, f> = 1 \).

But \( c \neq 1 \), then \( <(\lambda - T)^{-1} x, f> \neq 1 \) and so by Lemma 2.2.4, \( \lambda \notin \sigma(T + A) \).
Therefore, $\lambda \not\in \sigma(T+A) \cap \sigma(T+cA)$.

Hence, $\sigma(T+A) \cap \sigma(T+cA) \subseteq \sigma(T)$.

(ii) $\Rightarrow$ (i) Suppose for contraposition that $\text{rank } A \geq 2$. We will show that condition (ii) is not satisfied.

Case I: $A$ is a scalar $\alpha I$, $\alpha \neq 0$.

Let $T$ be an operator with $\sigma(T) = \{0, \alpha\}$. But $\sigma(A) = \{\alpha\}$ so $\sigma(T+A) = \{\alpha, 2\alpha\}$ and $\sigma(T+2A) = \{2\alpha, 3\alpha\}$.

Thus, $\sigma(T+A) \cap \sigma(T+2A) = \{2\alpha\} \not\subseteq \sigma(T)$.

Case II: $A$ is not a scalar.

We will construct a nilpotent operator $N$ with $N^3 = 0$ and a scalar $c \neq 1$ such that $\sigma(N + A) \cap \sigma(N + cA)$ contains a nonzero scalar and so we conclude

$$\sigma(N + A) \cap \sigma(N + cA) \not\subseteq \sigma(N) = \{0\}$$

Subcase II.1: Suppose that there exists a vector $u$ in $X$ such that $u, Au, A^2u$ are linearly independent. Let $U$ be the linear span of $\{u, Au, A^2u\}$ and let $V$ be a (closed) complement of $U$ in $X$ (by Proposition 1.1.2).

Define a linear operator $N$ on $X$ by

$$Nu = u - Au,$$

$$NAu = Au - 2A^2u,$$

$$NA^2u = -u/2 + 3Au/2 - 2A^2u,$$

$$Nv = 0 \text{ for } v \in V.$$

We have, $\| Nu \| = \| u - Au \| \leq \| u\| - \| Au \|

\leq \| u\| - c \| u\| \quad (\text{since } A \in B(X), \text{ so } \exists \text{ a scalar } c \text{ such that } \|Au\| \leq c \|u\| )$

\[ = k \| u\| \text{ where } k = 1 - c \]

$$\| NAu \| = \| Au - 2A^2u \| \leq \| Au \| - 2 \| Au \|^2$$

\[ = \| Au \| (1 - 2\| Au \|) \]

\[ \leq \| Au \| (1 - 2c \| u\| ) \]
\[
= k \| \text{Au} \| \quad \text{where } k = 1 - 2c \|u\|
\]

Similarly, \( \| NA^2u \| \leq k \| A^2u \| \) and \( \| Nv \| = \| 0 \| = 0 \). Therefore \( N \in B(X) \)

We have \( N^3v = 0 \), \( \forall v \in V \) and

\[
N^3u = N^2(u - \text{Au}) = N(\text{Nu} - \text{NAu})
\]

\[
= N(u - \text{Au} - \text{Au} + 2A^2u)
\]

\[
= Nu - 2\text{NAu} + 2 \text{NA}^2u
\]

\[
= u - \text{Au} - 2(\text{Au} - 2A^2u) + 2(-u/2 + 3Au/2 - 2A^2u)
\]

\[
= u - 3\text{Au} + 4A^2u - u + 3\text{Au} - A^2u = 0
\]

Similarly, \( N^3\text{Au} = 0 \) and \( N^3A^2u = 0 \). Therefore,

\[
N^3x = N^3(\alpha_1u + \alpha_2\text{Au} + \alpha_3A^2u + \beta v) \quad \text{for some scalars } \alpha_1, \alpha_2, \alpha_3, \beta
\]

\[
= \alpha_1 N^3u + \alpha_2 N^3\text{Au} + \alpha_3 N^3A^2u + \beta N^3v = 0 \quad \forall x \in X.
\]

Now, \((N + A)u = Nu + Au = u - \text{Au} + Au = u\)

and \((N + 2A)\text{Au} = \text{NAu} + 2A\text{Au} = \text{Au} - 2A^2u + 2A^2u = \text{Au}\).

But \(u \neq 0\) and \( \text{Au} \neq 0 \), therefore, \( 1 \in \sigma(N + A) \) and \( 1 \in \sigma(N + 2A) \),

that is; \( 1 \in \sigma(N + A) \cap \sigma(N + 2A) \not\subset \sigma(N) = \{0\} \)

Subcase II.2: For every \( x \in X \), the vectors \( x, Ax, A^2x \) are linearly dependent.

First we show that \( A \) satisfies a quadratic polynomial equation \( p(A) = 0 \).

Since \( A \) is not a scalar, there exists a vector \( u_1 \) in \( X \) such that \( u_1 \) and \( \text{Au}_1 \), are linearly independent. Therefore the minimal polynomial \( p \) of \( u_1 \) is quadratic.

Now let \( x \in X \) and consider the restriction of \( A \) to the invariant subspace \( U_1 = \text{span} \{u_1, \text{Au}_1, x, Ax\} \). Let \( q \) be the minimal polynomial of \( A|U_1 \). By a standard result in linear algebra, there exists a vector \( u \in U_1 \) such that \( q \) is also the minimal polynomial of \( u \) and so by our assumption, \( \deg q \leq 2 \). On the other hand, \( p \) divides \( q \), so \( q = p \) and \( p(A|U_1) = 0 \); in particular, \( p(A) x = 0 \). Since \( x \) is arbitrary, we have \( p(A) = 0 \).

We now consider four subcases according as

\[
p(t) = (t - \alpha)(t - \beta) \text{ or } (t - \alpha)^2 \text{ or } t(t - \alpha) \text{ or } t^2, \quad \text{where } \alpha \neq 0 \neq \beta \neq \alpha.
\]
By the standard decomposition of algebraic operators and since rank $A \geq 2$ and $A$ is not a scalar, we see that $A$ has a finite-dimensional invariant subspace $W$ such that $A|_W$ has a matrix representation
\[
\begin{pmatrix}
\alpha & 0 \\
0 & \beta
\end{pmatrix}, \begin{pmatrix}
\alpha & 1 \\
0 & \alpha
\end{pmatrix}, \begin{pmatrix}
\alpha & 0 & 0 \\
0 & \alpha & 0 \\
0 & 0 & 0
\end{pmatrix}, \text{or } \begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix} \oplus \begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}
\]
respectively.

We consider a complement $Z$ of $W$ in $X$ and an operator $N$ such that $N|Z = 0$ and $N|W$ has matrix representation
\[
\begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}, \begin{pmatrix}
0 & 0 \\
\alpha^2 & 0
\end{pmatrix}, \begin{pmatrix}
0 & 0 & 0 \\
0 & 2\alpha & 2\alpha \\
0 & -2\alpha & -2\alpha
\end{pmatrix}, \text{or } \begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix} \oplus \begin{pmatrix}
0 & 0 \\
2 & 0
\end{pmatrix}
\]
respectively.

Let $c$ be $\alpha \beta^{-1}$ or 4 or 2 or 2, respectively.

Then $N^2 = 0$ and $\sigma(N+ A) \cap \sigma(N+ cA)$ includes a nonzero scalar, namely $\alpha$ or $2\alpha$ or $2\alpha$ or $\sqrt{2}$, respectively.

Corollary 2.2.7. [1] Let $X$ be a Banach space and let $A \in B(X)$. The following conditions are equivalent.

(i) $A$ has rank 1 or is a scalar.

(ii) $\sigma(N+ A) \cap \sigma(N+ cA) \subseteq \{0\}$ for every nilpotent operator $N$ satisfying $N^3 = 0$ and every scalar $c \neq 1$.

Proof. By Theorem 2.2.6 and its proof.

In infinite-dimensional spaces, this result can be refined as follows:

Corollary 2.2.8. [1]. If $X$ is infinite-dimensional, then rank $A \leq 1$ if and only if $\sigma(N+ A) \cap \sigma(N+ cA) = \{0\}$ for every nilpotent operator $N$ with $N^3 = 0$ and every scalar $c \neq 1$. Furthermore, $A$ is a nonzero scalar if and only if the above intersection is empty.

Proof. This follows from the fact that for a nilpotent $N$ and a compact $K$, the essential spectrum of $N+ K$ is $\{0\}$ and so the spectrum includes 0.
We now return to our study of the mapping $\phi$.

**Lemma 2.2.9.** [1] Let $X$ be a Banach space and $\phi$ be a spectrum preserving linear map from $B(X)$ to $B(Y)$. If $R \in B(X)$ has rank one, then $\phi(R)$ has rank 1.

**Proof.** Since $\phi$ is injective by Lemma 2.2.2, then $\phi^{-1}$ exists and so if $T \in B(X)$, then $\exists S \in B(X)$ such that $T = \phi^{-1}(S)$. But $\phi$ preserves the spectrum, so $\sigma(T) = \sigma(\phi(T))$ which implies $\sigma(\phi^{-1}(S)) = \sigma(\phi(\phi^{-1}(S))) = \sigma(S)$; that is $\phi^{-1}$ preserves the spectrum.

Now, if $R \in B(X)$ has rank one, then by Theorem 2.2.6 $\sigma(T+R) \cap \sigma(T+cR) \subseteq \sigma(T)$ for every $T \in B(X)$ and every $c \neq 1$.

Hence,

\[
\sigma(T + \phi(R)) \cap \sigma(T + c\phi(R)) = \sigma(\phi^{-1}(T + \phi(R))) \cap \sigma(\phi^{-1}(T + c\phi(R))) \\
= \sigma(\phi^{-1}(T) + \phi^{-1}(\phi(R))) \cap \sigma(\phi^{-1}(T) + c\phi^{-1}(\phi(R))) \\
= \sigma(S + R) \cap \sigma(S + cR) \\
\subseteq \sigma(S) \quad \text{(by Theorem 2.2.6)} \\
= \sigma(\phi(T)) \\
= \sigma(T) \quad \text{(since $\phi$ preserves the spectrum)}
\]

Therefore by Theorem 2.2.6 $\phi(R)$ has rank 1. \hfill \square

**Theorem 2.2.10.** [1] Let $X$ and $Y$ be Banach spaces. If $\phi : B(X) \to B(Y)$ is a spectrum-preserving surjective linear mapping, then either

(i) there is a bounded invertible operator $A : X \to Y$ such that $\phi(T) = ATA^{-1}$ for every $T \in B(X)$ or

(ii) there is a bounded invertible operator $B : X' \to Y$ such that $\phi(T) = BTB^{-1}$ for every $T \in B(X)$.

**Proof.** For every nonzero $x \in X$ and every nonzero $f \in X'$ consider the sets $L_x = \{x \otimes h : h \in X'\}$ and $R_f = \{u \otimes f : u \in X\}$.

Each of $L_x$ and $R_f$ is a linear subspace of $B(X)$ consisting of rank one operators and is maximal among such spaces.
It follows that for every \( x \in X \), \( \phi(L_x) = \{ \phi(x \otimes h) : h \in X' \} \) is either an \( L_y \) for some \( y \in Y \) or an \( R_g \) for some \( g \in Y' \).

Furthermore, we cannot have \( \phi(L_u) = L_y \) and \( \phi(L_v) = R_g \) simultaneously for some \( u \) and \( v \) in \( X \) since \( L_y \cap R_g = y \otimes g \) is a one-dimensional space while \( L_u \cap L_v \) has dimension 0 if \( u \neq v \) or \( \dim X' \) if \( u = v \).

So we have two cases:

**Case I.** For every \( x \in X \), there exists a \( y \in Y \) such that \( \phi(L_x) = L_y \), so \( \phi(x \otimes f) = y \otimes g \). The mapping \( f \rightarrow g \) is linear, so \( g = C_x f \) for a linear transformation \( C_x : X' \rightarrow Y' \).

**Claim:** the space \( \{ C_x : x \in X \} \) has dimension 1.

If this is not the case, then there exist \( x_1, x_2 \in X \), \( y_1, y_2 \in Y \) and two linearly independent transformations \( C_1, C_2 \) such that

\[
\phi(x_1 \otimes f) = y_1 \otimes C_1 f \quad \text{and} \quad \phi(x_2 \otimes f) = y_2 \otimes C_2 f \quad \text{for every} \quad f.
\]

It follows that

\[
y_1 \otimes C_1 f + y_2 \otimes C_2 f = \phi(x_1 \otimes f) + \phi(x_2 \otimes f)
\]

\[
= \phi(x_1 \otimes f + x_2 \otimes f) \quad \text{(since} \ \phi \ \text{is linear)}
\]

\[
= \phi((x_1 + x_2) \otimes f)
\]

and so has rank 1 for every \( f \).

Since \( C_1 \) and \( C_2 \) are linearly independent we must have that \( y_1 \) and \( y_2 \) are linearly dependent.

and so there exists a scalar \( k \) such that \( y_2 = ky_1 \).

Now,

\[
L_{y_1} = \{ y_1 \otimes h : h \in Y' \} = \{ ky_1 \otimes h : h \in Y' \} = \{ y_2 \otimes h : h \in Y' \} = L_{y_2},
\]

implying that \( \phi^{-1}(L_{y_1}) = L_{x_1} = \phi^{-1}(L_{y_2}) = L_{x_2} \) and so \( x_1 \) and \( x_2 \) are linearly dependent.

However, in this case, we get that \( C_1 \) and \( C_2 \) are linearly dependent which is a contradiction. This establishes the fact that \( \dim \{ C_x : x \in X \} = 1 \) and so by absorbing a constant in the first term of the tensor product, we have one linear transformation \( C : X' \rightarrow Y' \) such that \( \phi(x \otimes f) = y \otimes C f \).
Now the mapping $x \mapsto y$ is linear, and we have $\phi(x \otimes f) = Ax \otimes Cf$ where $A$ is a linear transformation from $X$ to $Y$.

**Claim**: Both $A$ and $C$ are bijections.

Let $Ax_1 = Ax_2$ and $Cf_1 = Cf_2$ for $x_1, x_2 \in X$ and $f_1, f_2 \in X'$, then

$$Ax_1 \otimes Cf_1 = Ax_2 \otimes Cf_2 \text{ so } \phi(x_1 \otimes f_1) = \phi(x_2 \otimes f_2).$$

But $\phi$ is injective so $x_1 \otimes f_1 = x_2 \otimes f_2$, thus $x_1 = x_2$ and $f_1 = f_2$.

Hence, $A$ and $C$ are injective.

Now let $y \in Y$ and $g \in Y'$, then $y \otimes g \in B(Y)$, but $\phi$ is onto so $\exists x \otimes f \in B(X)$, such that $\phi(x \otimes f) = y \otimes g = Ax \otimes Cf$, so $y = Ax$ for some $x \in X$ and $g = Cf$ for some $f \in X'$.

Hence $A$ and $C$ are surjective.

Now, let $T$ be an arbitrary operator on $X$, then

$$\phi(T + x \otimes f) = \phi(T) + \phi(x \otimes f)$$

$$= \phi(T) + Ax \otimes Cf$$

And so $\sigma(T + x \otimes f) = \sigma(\phi(T + x \otimes f))$

$$= \sigma(\phi(T) + Ax \otimes Cf)$$

Let $\lambda$ be a complex number with $\lambda \notin \sigma(T)$, by Lemma 2.2.4, we have that

$$< (\lambda - T)^{-1}x , f > = 1 \text{ if and only if } < (\lambda - \phi(T))^{-1}Ax , Cf > = 1,$$

and so, by linearity, we have

$$< (\lambda - T)^{-1}x , f > = < (\lambda - \phi(T))^{-1}Ax , Cf > \quad (*)$$

for every $x \in X$, $f \in X'$ and $\lambda \notin \sigma(T)$.

By Theorem 1.1.17, we can easily establish the fact that $A$ and $C$ are bounded.

Now, for every nonzero complex number $z$ in some neighborhood $\{ z : |z| < \delta \}$ of 0, replacing $\lambda$ with $l/z$ in $(*)$, we get

$$< (l/z - T)^{-1}x , f > = < (l/z - \phi(T))^{-1}Ax , Cf >$$

Then,

$$< (l - zT)^{-1}x , f > = < (l - z\phi(T))^{-1}Ax , Cf >$$

And so,

$$< (l-zT)^{-1}A^{-1}y , f > = < (l - z \phi(T))^{-1}y , Cf >$$
Each side of the above equation is analytic in \( \{ z: 0 < |z| < \delta \} \) with a removable singularity at 0.

Taking the limit as \( z \to 0 \), we get \( \langle A^{-1}y, f \rangle = \langle y, Cf \rangle \).

Taking the derivative at \( z = 0 \), we get \( \langle TA^{-1}y, f \rangle = \langle \phi(T)y, Cf \rangle \).

Thus,
\[
\langle TA^{-1}y, f \rangle = \langle A^{-1}\phi(T)y, f \rangle
\]
and we get \( TA^{-1} = A^{-1}\phi(T) \).

Therefore \( \phi(T) = ATA^{-1} \).

**Case II.** For every \( x \in X \), there exists a \( g \in Y' \) such that \( \phi(L_x) = R_g \).

By a proof similar to the above, we get an isomorphism \( B \) from \( X' \) onto \( Y \) such that \( \phi(T) = BTB^{-1} \) for every \( T \in B(X) \).

**Theorem 2.2.11.** Two algebras \( B(X) \) and \( B(Y) \) are algebraically isomorphic if and only the corresponding spaces \( X, Y \) are isomorphic; more over \( V = \phi(U) \) being the isomorphism between \( B(X) \) and \( B(Y) \) there exists an isomorphism \( A(x) = y \) such that \( \phi(U) = AUA^{-1} \) for every \( U \in B(X) \).

**Proof.** See [22].

**Corollary 2.2.12.** [1] If \( \phi: B(X) \to B(Y) \) is a surjective linear map, then \( \phi \) preserves the spectrum if and only if it is an algebra-isomorphism or an anti-isomorphism.

**Proof.** Suppose that \( \phi: B(X) \to B(Y) \) is a surjective linear map which preserves the spectrum, then by theorem 2.2.10 and theorem 2.2.11 \( \phi \) is an algebraisomorphism or an antiisomorphism.

**Corollary 2.2.13.** [1]. Every automorphism of the algebra \( B(X) \) is inner.

**Proof.** Let \( X \) be banach space and \( \phi \) is an automorphism of the algebra \( B(X) \), then by corollary 2.2.11 \( \phi \) preserves the spectrum so by theorem 2.2.10 there is a bounded invertible operator \( A: X \to X \) such that \( \phi(T) = ATA^{-1} \) for every \( T \in B(X) \), and so by Definition 1.3.13 \( \phi \) is inner.

**Remark 2.2.14.** [1] If \( \phi \) takes the form (ii) of Theorem 2.2.10, the surjectivity of \( \phi \) implies that every operator on \( X' \) is a dual of an operator on \( X \). This, in turn, implies that \( X \) is reflexive. Since \( Y \) must be isomorphic to \( X' \), it follows that \( Y \) too is reflexive.
Remark 2.2.15. [1] When \( X = Y = H \), a Hilbert space, case (ii) of Theorem 2.2.10 takes the form \( \phi(T) = CT^*C^{-1} \), where \( C \in B(H) \) and \( T^* \) denotes the Hilbert adjoint of \( T \) relative to a fixed but arbitrary orthonormal basis.
Chapter 3
Linear Maps Preserving Spectral Radius

This chapter consists of two sections, in the first one we study spectral radius and some of its properties. In the second section we study linear maps that preserve spectral radius.

3.1 Spectral radius and some of its properties

**Definition 3.1.1.** [12] The **spectral radius** $r_{\sigma}(T)$ of an operator $T \in B(X)$ on a complex Banach space $X$ is the radius

$$r_{\sigma}(T) = \sup_{\lambda \in \sigma(T)} |\lambda|$$

of the smallest closed disk centered at the origin of the complex $\lambda$-plane and containing $\sigma(T)$.

**Lemma 3.1.2.** [8] For any operator $T \in B(X)$ on a complex Banach space $X$, $r_{\sigma}(T^n) = r_{\sigma}(T)^n$ for every $n \geq 0$ and $r_{\sigma}(\alpha T) = |\alpha| r_{\sigma}(T)$.

**Proof.** Take an arbitrary nonnegative integer $n$. Theorem 2.1.16 ensures that $\sigma(T^n) = \sigma(T)^n$. Hence $\nu \in \sigma(T^n)$ if and only if $\nu = \lambda^n$ for some $\lambda \in \sigma(T)$, and so

$$r_{\sigma}(T^n) = \sup_{\nu \in \sigma(T^n)} |\nu|$$

$$= \sup_{\lambda \in \sigma(T)} |\lambda^n|$$

$$= \sup_{\lambda \in \sigma(T)} |\lambda|^n$$

$$= (\sup_{\lambda \in \sigma(T)} |\lambda|)^n$$

$$= r_{\sigma}(T)^n.$$
Let $\alpha$ be arbitrary complex number, Then by Theorem 2.1.16 $\sigma(\alpha T) = \alpha \sigma(T)$. Hence $\nu \in \sigma(\alpha T)$ if and only if $\nu = \alpha \lambda$ for some $\lambda \in \sigma(T)$, and so
\[
 r_\sigma(\alpha T) = \sup_{\nu \in \sigma(\alpha T)} |\nu| \\
= \sup_{\lambda \in \sigma(T)} |\alpha \lambda| \\
= |\alpha| \sup_{\lambda \in \sigma(T)} |\lambda| \\
= |\alpha| r_\sigma(T).
\]

**Remarks 3.1.3.** [8] (1) If $\lambda \in \sigma(T)$, then $|\lambda| \leq \|T\|$ . This follows by Theorem 2.1.7. Thus $r_\sigma(T) \leq \|T\|$.

Therefore, for every operator $T \in B(X)$, and for each nonnegative integer $n$,
\[
0 \leq r_\sigma(T^n) = r_\sigma(T)^n \leq \|T^n\| \leq \|T\|^n.
\]

(2) If $T$ is a nilpotent operator (i.e., if $T^n = 0$ for some $n \geq 1$), then $\sigma(T) = \sigma_p(T) = \{0\}$, and so $r_\sigma(T) = 0$.

**Proof.** Suppose $T$ is nilpotent, with $T^n = 0$ for some $n \geq 1$. If $\lambda \in \sigma(T)$, then $\lambda^n \in \sigma(T^n) = \sigma(0) = \{0\}$ by the spectral mapping theorem, so that $\lambda = 0$.

Hence, $r_\sigma(T) = \sup_{\lambda \in \sigma(T)} |\lambda| = 0$.

**Definition 3.1.4.** [12] If an operator $T$ satisfies $r_\sigma(T) = 0$ (and hence $\sigma(T) = \{0\}$), we declare $T$ **quasinilpotent**.

**Remark.** By Remarks 3.1.3 and Definition 3.1.4 we conclude that every nilpotent operator is quasinilpotent.

**Theorem 3.1.5.** Let $A$ be a Banach algebra. Suppose $r \in A$ is such that $r_\sigma(a + r) = 0$ for all $a$ quasi-nilpotent $a$. Then $r \in \text{rad } A$.

**Proof.** See [19]

**Theorem 3.1.6** [8] **(Gelfand–Beurling Formula)**
\[
r_\sigma(T) = \lim \|T^n\|^\frac{1}{n}.
\]

**Proof.** By remark 3.1.3(1) above we have $r_\sigma(T)^n \leq \|T^n\|$. Then
\[
r_\sigma(T) \leq \lim_{n \to \infty} \frac{1}{n} \|T^n\| \leq \lim_{n \to \infty} \sqrt[2n]{\|T^n\|} \ldots \ldots (**)
\]
A power series \( \sum_{n=0}^{\infty} c_n \kappa^n \) converges absolutely for \(|k| < r\) with radius of convergence \( r \) given by the well-known Hadmard formula \( \frac{1}{r} = \lim_{n \to \infty} \sqrt[n]{|c_n|} \).

Setting \( \kappa = \frac{1}{\lambda} \), then if \(|\kappa| < r\), we have

\[
R_\lambda = -\frac{1}{\lambda} \sum_{n=0}^{\infty} \left( \frac{1}{\lambda} \right)^n = -\frac{1}{\lambda^2} \sum_{n=0}^{\infty} T^n \kappa^n.
\]

Then writing \(|c_n| = \|T^n\|\) we obtain

\[
\left\| \sum_{n=0}^{\infty} T^n \kappa^n \right\| \leq \sum_{n=0}^{\infty} \|T^n\| \|\kappa^n\|,
\]

Then by the Hadmard formula we have absolute convergence for \(|\kappa| < r\) and so

\[
|\lambda| = \frac{1}{|\kappa|} > \frac{1}{r} = \lim_{n \to \infty} \sqrt[n]{\|T^n\|}.
\]

But \( R_\lambda \) is locally holomorphic on \( \rho(T) \) in the complex \( \lambda \)-plane, as was mentioned before in theorem 2.1.14.

To \( \rho(T) \) there corresponds a set in the complex \( \kappa \)-plane, call it \( M \). Then it is known from complex analysis that the radius of convergence \( r \) is the radius of the largest open circular disk about \( \kappa = 0 \) which lies entirely in \( M \).

Hence \( \frac{1}{r} \) is the radius of the smallest circle about \( \lambda = 0 \) in the \( \lambda \)-plane whose exterior lies entirely in \( \rho(T) \).

By definition this means that \( \frac{1}{r} = r_\sigma(T) \).

Hence by (1) and (*),

\[
r_\sigma(T) = \frac{1}{r} = \lim_{n \to \infty} \sqrt[n]{\|T^n\|} \geq \lim_{n \to \infty} \sqrt[n]{\|T^n\|} \geq r_\sigma(T)
\]

Therefore,

\[
r_\sigma(T) = \lim_{n \to \infty} \sqrt[n]{\|T^n\|}.
\]

**Note.** Recall that a function \( f \) on a metric space \( X \) is called **continuous** at \( x \) if for any given positive \( \varepsilon \) one can find an open ball \( B(x, \delta) \) of radius \( \delta \) about \( x \),

\( B(x, \delta) = \{ y : d(x, y) < \delta \} \), such that \( f(x) - \varepsilon \leq f(y) \leq f(x) + \varepsilon \) (\( y \in \Delta(x, \delta) \)).

The inequality above is two-sided. If only one-sided inequality holds then the function is said to be semi-continuous.

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**Definition 3.1.7** [34] Let $D$ be domain (connected open set) in $\mathbb{C}$ and $\mathbb{C}^2(D)$ denotes the space of functions on $D$ with continuous second order derivatives, then a function $h : D \rightarrow \mathbb{R}$ is called **harmonic** on $D$ if $h \in \mathbb{C}^2(D)$ and $h_{xx} + h_{yy} = 0$ on $D$, where $h_{xx}$ and $h_{yy}$ are the second order partial derivatives of $h$.

**Example 3.1.8.** [34] Let $z \in \mathbb{C}$, then

1. $h(z) = |z|^2 = x^2 + y^2$ is not harmonic anywhere on $\mathbb{C}$ as $h_{xx} + h_{yy} = 4$.
2. $h(z) = \text{Re}(z^2) = x^2 - y^2$ is harmonic on $\mathbb{C}$.

**Definition 3.1.9** [34] Let $(X, d)$ be a metric space. A function $f : X \rightarrow (-\infty, \infty)$ is called **upper semi-continuous** at $x \in X$ if for any given positive $\varepsilon$ one can find a positive $\delta$ such that

$$f(y) \leq f(x) + \varepsilon \quad (y \in B(x, \delta))$$

Note that upper semi-continuous functions are allowed to take value $-\infty$. This is consistent with the definition above.

**Remarks.** [34] (1) A function $f$ is said to be upper semi-continuous on $X$ if it has this property at every $x \in X$. An equivalent definition for $f$ to be upper semi-continuous on $X$ is to require that

$$\limsup_{y \to x} f(y) \leq f(x) \quad \forall x \in X.$$  

(2) A function $g$ is lower semi-continuous iff $-g$ is upper semi-continuous.

(3) Obviously, every continuous function is also upper semi-continuous. But the converse is not true. e.g $(f(z) = \ln |z|, z \in \mathbb{C})$, is upper semi-continuous but not continuous at $z = 0$).

(4) Obviously, if $f$ and $g$ are upper semi-continuous functions, so is their sum $f + g$ and $\max(f(z), g(z))$.

**Definition 3.1.10** [34] Let $U \subseteq \mathbb{C}$ be open. We call the function $u : U \rightarrow (-\infty, \infty)$ is **superharmonic** if $u$ is upper semi-continuous in $U$ and satisfies the local submean inequality. Namely, for any $w \in U$ there exists $\rho > 0$ such that $\forall r \in (0, \rho)$

$$u(w) \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} u(w + re^{i\theta})d\theta$$

**Note.** $v$ is superharmonic iff $-v$ is subharmonic.

**Theorem 3.1.11.** Let $U$ be an open subset of $\mathbb{C}$, and $u \in \mathbb{C}^2(D)$ Then $u$ is subharmonic on $U$ if and only if $u_{xx} + u_{yy} \geq 0$.

**Proof.** See [34]
Proposition 3.1.12 [34]
(i) If \( u \) and \( v \) are subharmonic and \( a, b \geq 0 \), then \( au + bv \) is subharmonic.
(ii) If \( u, v \) are subharmonic, then \( \max(u, v) \) is subharmonic.

Proof. The proof of (i) follows from the above theorem and (ii) is obvious.

Theorem 3.1.13 (Liouville’s theorem for subharmonic functions) Let \( u \) be subharmonic in \( \mathbb{C} \) and suppose that
\[
\limsup_{z \to \infty} \frac{u(z)}{\ln |z|} \leq 0.
\]
Then, \( u \) is constant in \( \mathbb{C} \). E.g., every subharmonic function on \( \mathbb{C} \) which is bounded above must be constant.

Proof. See [34].

Lemma 3.1.14 Let \( f \) be an analytic function from a domain \( D \) of \( \mathbb{C} \) into a Banach space \( X \). Then \( \lambda \to \log \|f(\lambda)\| \) is subharmonic on \( D \).

Proof. See [34].

Theorem 3.1.15 (E. Vesentini) Let \( f \) be an analytic function from a domain \( D \) of \( \mathbb{C} \) into a Banach algebra \( A \). Then both \( \lambda \to r_{\sigma}(f(\lambda)) \) and \( \lambda \to \log r_{\sigma}(f(\lambda)) \) are subharmonic functions on \( D \).

Proof. See [34].

Theorem 3.1.16 (I. Kaplansky) Let \( X \) be a complex vector space and let \( T \) be a linear operator from \( X \) into \( X \). Suppose that there exists an integer \( n \geq 1 \) such that \( \xi, T\xi, \ldots, T^{n-1}\xi \) are linearly dependent for all \( \xi \in X \). Then \( T \) is algebraic of degree less than or equal to \( n \).

Proof. See [14].

### 3.2 Linear Maps Preserving Spectral Radius
**Definition 3.2.1.** [29] Let $X$ and $Y$ be Banach spaces, we say that a linear map $\phi : B(X) \to B(Y)$ preserves spectral radius from $B(X)$ to $B(Y)$ if $r_\sigma(\phi(T)) = r_\sigma(T)$ for every $T \in B(X)$.

**Remark.** [29] In particular, a surjective linear map $\phi : B(X) \to B(X)$ satisfies $r_\sigma(\phi(A)) = r_\sigma(A)$ for every $A \in B(X)$ is continuous. This is a result of the next theorem.

**Theorem 3.2.2.** Let $A$ be a complex Banach algebra and $B$ be a semi-simple complex Banach algebra. Suppose that $T$ is a linear mapping from $A$ onto $B$ such that $r_\sigma(Tx) \leq r_\sigma(x)$ for every $x$ in $A$. Then $T$ is continuous.

**Proof.** See [5]

**Lemma 3.2.3.** [29] Suppose that a surjective linear map $\phi : B(X) \to B(X)$ satisfies $r_\sigma(\phi(A)) = r_\sigma(A)$ for every $A \in B(X)$, then $\phi$ is injective.

**Proof.** Suppose for contrary that $\phi(A) = 0$ for some $A \neq 0$. Then there exists $x \in X$ such that $y = Ax \neq 0$.
But $r_\sigma(A) = r_\sigma(\phi(A)) = 0$ so $x$ and $y$ are linearly independent (since it is not the case, then $x$ and $y$ are linearly dependent and so $y = \alpha x$ for some $\alpha \neq 0$ (since $y \neq 0$). Then $\exists x \neq 0$ such that $Ax = \alpha x$. Hence $\alpha \in \sigma(A)$ contradiction (since $r_\sigma(A) = 0$)).

Let $V$ be a topological complement of the linear span of $\{x, y\}$ in $X$.
Define $N \in B(X)$ by
$$N = N = x - y \quad \text{and} \quad Nv = 0, \ v \in V.$$  
Now, $N^2x = N^2y = N(x-y) = Nx - Ny = 0$ and $N^2v = 0$, then $N^2 \neq 0 \ \forall x \in X$, and so $r_\sigma(N) = r_\sigma(\phi(N)) = 0$.

On the other hand, $(A+N)x = Ax + Nx = y + x - y = x$ yields $1 \in \sigma(A+N)$ and $r_\sigma(A+N) \geq 1$,
but $r_\sigma(\phi(N)) = r_\sigma(A+N) \geq 1$ . a contradiction. $\square$

**Lemma 3.2.4.** [29] Suppose that a surjective linear map $\phi : B(X) \to B(X)$ satisfies $r_\sigma(\phi(A)) = r_\sigma(A)$ for every $A \in B(X)$, then $\phi(I) = cI$, where $c \in \mathbb{C}$ and $|c| = 1$.

**Proof.** We want to show that $\phi(I)x = cx \ \forall x \in X$. It is equal to show that $x$ and $\phi(I)x$ are linearly dependent for every $x \in X$.
Suppose this is not true for some $x \in X$.

Let $W$ be a topological complement of the linear span of $\{x, \phi(I)x\}$ in $X$.
Define a linear operator $M$ by
$$M(x) = 2x - \phi(I)x,$$
$$M(\phi(I)x) = 4x - 2\phi(I)x,$$
\[ M(w) = 0, \ w \in W. \]

Now, \[ \| M(x) \| = \| 2x - \phi(I)x \| \]
\[ \leq \| 2x \| - \| \phi(I)x \| \]
\[ \leq 2\| x \| - \| \phi(I)\| \| x \| \]
\[ = (2 - \| \phi(I)\| )\| x \| \]
\[ = c\| x \| \]

\[ \| M(\phi(I)x) \| = \| 4x - 2\phi(I)x \| \]
\[ \leq 4\| x \| - 2\| \phi(I)x \| \]
\[ = c\| \phi(I)x \| \]

\[ \| M(w) \| = 0, \] that is \( M \) is bounded. Hence \( M \in B(X). \)

Now, \( M^2x = M(2x - \phi(I)x) \)
\[ = 2Mx - M\phi(I)x \]
\[ = 2(2x - \phi(I)x) - (4x - 2\phi(I)x) = 0, \]

\[ M^2\phi(I)x = M(4x - 2\phi(I)x) \]
\[ = 4Mx - 2M\phi(I)x \]
\[ = 4(2x - \phi(I)x) - 2(4x - 2\phi(I)x) = 0 \]

and \( M^2w = 0, \) so \( M^2 = 0 \) \( \forall x \in X. \) that is \( M \) is nilpotent.

Since \( \phi \) is surjective we have \( M = \phi(R) \) for some \( R \in B(X) \) such that \( r_\sigma(M) = r_\sigma(\phi(M)) = r_\sigma(R) = 0 \) (By remark 3.1.3(2))

Now, \( r_\sigma(I+R) = 1 + r_\sigma(R) = 1 + 0 = 1 \) (by Theorem 2.1.16 where \( a_0 = 1 \) and \( R \) is the operator).

Therefore \( r_\sigma(\phi(I+R)) = r_\sigma(\phi(I) + \phi(R)) \)
\[ = r_\sigma(\phi(I) + M) = 1. \]

But we have \( (\phi(I)+M)x = \phi(I)x + 2x - \phi(I)x = 2x, \) so \( 2 \in \sigma(\phi(I)+M) \) which contradicts \( r_\sigma(\phi(I)+M) = 1. \)

**Remark.** There is no loss of generality in assuming \( c = 1, \) and hence, \( \phi(I) = I. \)

**Definition 3.2.5.** [29] Let \( A \in B(X) \) then \( \pi(A) \) is the set of all numbers \( \lambda \) in the spectrum of \( A \) such that \( r_\sigma(A) = |\lambda|. \)

**Lemma 3.2.6.** [29] Suppose that a surjective linear map \( \phi: B(X) \to B(X) \) satisfies \( r_\sigma(\phi(A)) = r_\sigma(A) \) for every \( A \in B(X), \) then \( \pi(\phi(A)) = \pi(A) \) for every \( A \in B(X). \)
**Proof.** Let $A \in B(X)$ and $\lambda \in \pi(A)$, then $\lambda \in \sigma(A)$ and $r_{\sigma}(A) = |\lambda|$. Then we have
\[
\begin{align*}
    r_{\sigma}(\phi(A) + \lambda I) &= r_{\sigma}(\phi(A)) + r_{\sigma}(\lambda I) \\
    &= r_{\sigma}(A + \lambda I) \\
    &= r_{\sigma}(A) + r_{\sigma}(\lambda I) \\
    &= |\lambda| + |\lambda| \\
    &= 2|\lambda| = 2r_{\sigma}(A).
\end{align*}
\]
Therefore, there is $\alpha \in \sigma(\phi(A))$ such that $|\alpha + \lambda| = 2|\lambda|$. But $r_{\sigma}(\phi(A)) = r_{\sigma}(A)$, so we have $|\alpha| \leq |\lambda|$, then
\[
2|\lambda| = |\alpha + \lambda| \\
\leq |\alpha| + |\lambda| \\
\leq |\lambda| + |\lambda|.
\]
But then $\alpha = \lambda$. Thus, $\lambda \in \sigma(\phi(A))$ and $r_{\sigma}(\phi(A)) = |\lambda|$, so that $\lambda \in \pi(\phi(A))$. This shows that $\pi(A) \subseteq \pi(\phi(A))$.

Since $\phi^{-1}$ also preserves the spectral radius, it follows that $\pi(\phi(A)) \subseteq \pi(A)$. Therefore $\pi(\phi(A)) = \pi(A)$.

**Lemma 3.2.7.** [29] Let $\phi: B(X) \to B(X)$ be a surjective linear map which satisfies $r_{\sigma}(\phi(A)) = r_{\sigma}(A)$ for every $A \in B(X)$ and let $B \in B(X)$ be such that $B^k = 0$ for some $k \geq 2$. If $A \in B(X)$ satisfies $AQ^iA = 0$, $i = 0, 1, 2, \ldots, k-1$, where $Q = \phi(B)$, then
\[
r_{\sigma}(\lambda Q^k + AQ^{k-1} + QAQ^{k-2} + Q^2AQ^{k-3} + \ldots + Q^{k-1}A) = 0
\]
for every $\lambda \in \mathbb{C}$.

**Proof.** Let $B_1 = AQ^{k-1} + QAQ^{k-2} + Q^2AQ^{k-3} + \ldots + Q^{k-1}A$ and $B_2 = Q^k$. Since $r_{\sigma}(A + \lambda Q)^k = r_{\sigma}((A + \lambda Q)^k)$ and $A^2 = AQA = AQ^2A = \ldots = AQ^{k-2}A = 0$ (by the hypotheses), then
\[
r_{\sigma}(A + \lambda Q)^k = r_{\sigma}((A + \lambda Q)^k) \\
= |\lambda|^{k-1} r_{\sigma}(AQ^{k-1} + QAQ^{k-2} + Q^2AQ^{k-3} + \ldots + Q^{k-1}A + \lambda Q^k) \\
= |\lambda|^{k-1} r_{\sigma}(B_1 + \lambda B_2).
\]
On the other hand,
\[
r_{\sigma}(A + \lambda Q)^k = r_{\sigma}(\phi^{-1}(A + \lambda Q))^k \\
= r_{\sigma}(\phi^{-1}(A) + \lambda \phi^{-1}(Q))^k \\
= r_{\sigma}(\phi^{-1}(A) + \lambda B)^k \\
= r_{\sigma}(\phi^{-1}(A)^k + \lambda [\phi^{-1}(A)^{k-1}B + \ldots + B \phi^{-1}(A)^{k-1}] \\
+ \lambda^2 [\phi^{-1}(A)^{k-2}B^2 + \ldots + B^2 \phi^{-1}(A)^{k-2}] + \ldots \\
+ \lambda^{k-1} [\phi^{-1}(A)B^{k-1} + \ldots + B^{k-1} \phi^{-1}(A)] + B^k)
\]
As $B^k = 0$, this yields
\[
r_{\sigma}(A + \lambda Q)^k = r_{\sigma}(A_0 + \lambda A_1 + \ldots + \lambda^{k-1}A_{k-1}),
\]
where
\[ A_0 = \phi^{-1}(A)^k, \quad A_1 = \phi^{-1}(A)^{k-1}B + \ldots + B \phi^{-1}(A)^k, \]
\[ \ldots, \quad A_{k-1} = \phi^{-1}(A)B^{k-1} + \ldots + B^{k-1} \phi^{-1}(A). \]

Thus we have showed that
\[ |\lambda|^{-k+1} r_\sigma(B_1 + \lambda B_2) = r_\sigma(A_0 + \lambda A_1 + \ldots + \lambda^{k-1} A_{k-1}). \]

Therefore, for \(|\lambda| \geq 1\) we have
\[ r_\sigma(B_1 + \lambda B_2) = |\lambda|^{-k+1} r_\sigma(A_0 + \lambda A_1 + \ldots + \lambda^{k-1} A_{k-1}), \]
\[ \leq |\lambda|^{-k+1}( ||A_0|| + ||A_1|| + \ldots + ||A_{k-1}||) \]
\[ \leq ||A_0|| + ||A_1|| + \ldots + ||A_{k-1}||. \]

On the other hand, for \(|\lambda| \leq 1\) we have
\[ r_\sigma(B_1 + \lambda B_2) \leq ||B_1 + \lambda B_2|| \]
\[ \leq ||B_1|| + ||B_2|| \]
\[ \leq ||B_1|| + ||B_2||. \]

Thus we have proved that the function \(\lambda \mapsto r_\sigma(B_1 + \lambda B_2)\) is bounded.

As it is a subharmonic function by Theorem 3.1.15, it follows by theorem 3.1.13 that \(r_\sigma(B_1 + \lambda B_2) = r_\sigma(B_1)\) for every \(\lambda \in \mathbb{C}\).

\textbf{Claim:} \(B_1^{k+1} = 0\).

\textbf{Proof.} \(B_1 A = AQ^{k-1}A + QAQ^{k-2} A + Q^2AQ^{k-3} A + \ldots + Q^{k-1} AA = 0\) (by the hypotheses).

Now, \(B_1^2 = (AQ^{k-1} + QAQ^{k-2} + Q^2AQ^{k-3} + \ldots + Q^{k-1} A)^2 \]
\[ = AQ^{k-1}AQ^{k-1} + AQ^{k-2}AQ^{k-2} + \ldots + QAQ^{k-2}AQ^{k-1} \]
\[ + \ldots + QAQ^{k-2}Q^{k-1} A + \ldots + Q^{k-1} AAQ^{k-1} + \ldots + Q^{k-1} A Q^{k-1} A \]
\[ = X_{k-2} AQ^{k-2} + X_{k-3} AQ^{k-3} + \ldots + X_0 A \quad \text{for some } X_i \in B(X) \]

and so \(B_1^2 QA = X_{k-2} AQ^{k-2} QA + X_{k-3} AQ^{k-3} QA + \ldots + X_0 A QA = 0 = B_1^2 A\).

This further implies that \(B_1^3\) is of the form
\[ B_1^3 = Y_{k-3} AQ^{k-3} + Y_{k-4} AQ^{k-4} + \ldots + Y_0 A \quad \text{for some } Y_i \in B(X), \]

and consequently, \(B_1^3 QA = B_1^3 A = 0\).

Repeating this procedure we will have \(B_1^{k+1} = 0\).

Hence \(r_\sigma(B_1) = 0\), but then \(r_\sigma(B_1 + \lambda B_2) = 0, \lambda \in \mathbb{C}\), as desired. \(\square\)
Lemma 3.2.8. [29] Let $\phi: B(X) \to B(X)$ be a surjective linear map which satisfies $r_\sigma(\phi(A)) = r_\sigma(A)$ for every $A \in B(X)$ and let $B \in B(X)$ and $B^k = 0$ for some $k \geq 2$, then $\phi(B^{k-1}) = 0$.

Proof. Suppose that $\phi(B)^{2k-1} = Q^{2k-1} \neq 0$, where $Q = \phi(B)$. Let $p$ be a complex polynomial of degree not exceeding $2k-1$. Since $B^k = 0$, then $r_\sigma(B) = r_\sigma(\phi(B)) = r_\sigma(Q) = 0$, it follows that $p(Q) \neq 0$. By theorem 3.1.16, there is $u \in X$ such that the vectors $u, Qu, Q^2u, ..., Q^{2k-1}u$ are linearly independent.

Therefore, $Q^{k-1}u \notin Z$, the linear span of $[u, Qu, ..., Q^{k-2}u, Q^{k-1}u, ..., Q^{2k-2}u, Q^{2k-1}u - Q^{k-1}u]$.

Whence there exists $f \in X'$ such that $f(Z) = \{0\}$ and $f(Q^{k-1}u) = 1$.

Thus,

$$f(u) = 0, f(Qu) = 0, ..., f(Q^{k-2}u) = 0, f(Q^{k-1}u) = 1, f(Q^{k}u) = 0, f(Q^{k+1}u) = 0, ..., f(Q^{2k-2}u) = 0, f(Q^{2k-1}u) = 1.$$  

Set $A = (u - Q^k u) \otimes f$, then

$$Au = ((u - Q^ku) \otimes f)u = f(u)(u - Q^ku) = 0,$$

$$AQu = ((u - Q^ku) \otimes f)Qu = f(Qu)(u - Q^ku) = 0,$$

$$AQ^{k-2}u = ((u - Q^ku) \otimes f)Q^{k-2}u = f(Q^{k-2}u)(u - Q^ku) = 0,$$

$$AQ^{k-1}u = ((u - Q^ku) \otimes f)Q^{k-1}u = f(Q^{k-1}u)(u - Q^ku),$$

That is $AQ^i u = 0$ for $i = 0, 1, ..., k-2$.

while $AQ^{k-1}u = u - Q^ku$. Whence

$$(Q^k + AQ^{k-1} + QAQ^{k-2} + ... + Q^{k-1}A)u = Q^k u + (u - Q^ku) + 0 + ... + 0 = u,$$

so that $1 \in \sigma(Q^k + AQ^{k-1} + QAQ^{k-2} + ... + Q^{k-1}A)$ and so

$$r_\sigma(Q^k + AQ^{k-1} + QAQ^{k-2} + ... + Q^{k-1}A) \geq 1.$$  

However, for $i = 0, 1, ..., k-1$ we have

$$AQ^i A = ((u - Q^ku) \otimes f) Q^i ((u - Q^ku) \otimes f) = ((u - Q^ku) \otimes f) ((Q^i u - Q^{k+i}u) \otimes f) = ((u - Q^ku) \otimes f) (f(Q^iu - Q^{k+i}u)) = 0, \text{ for } f(Q^iu - Q^{k+i}u) = 0.$$  

Therefore, by lemma 3.2.7 we must have

$$r_\sigma(Q^k + AQ^{k-1} + QAQ^{k-2} + ... + Q^{k-1}A) = 0.$$  

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With this contradiction the proof is completed.

**Note.** [29] As $\phi^{-1}$ also preserves the spectral radius, we have in fact shown that $\phi$ preserves nilpotents in both directions (that is, $N \in B(X)$ is nilpotent if and only if $\phi(N)$ is nilpotent), and so we can apply the next theorem.

**Theorem 3.2.9.** [29] Let $X$ be a nontrivial complex Banach space and $B_0(X)$ be the linear span of all nilpotent operators in $B(X)$. Suppose that a surjective linear map $\phi: B_0(X) \rightarrow B_0(X)$ preserves nilpotent operators in both directions. Then there exist a nonzero complex number $d$ and either

(i) a bounded bijective linear operator $T: X \rightarrow X$ such that $\phi(A) = dTAT^{-1}$ for every $A \in B_0(X)$; or

(ii) a bounded bijective linear operator $T: X' \rightarrow X$ such that $\phi(A) = dTA'T^{-1}$ for every $A \in B_0(X)$. In this case $X$ must be reflexive.

**Remarks 3.2.10.** [29] (1) In general, $B_0(X)$ is a proper subspace of $B(X)$. Namely, there exist Banach spaces $X$ such that $B(X)$ has a nonzero multiplicative linear bounded operator, whose kernel then obviously contains $B_0(X)$.

(2) On the other hand, there are important examples of Banach spaces, for instance, infinite dimensional Hilbert spaces, satisfying $B_0(X) = B(X)$. For such spaces Theorem 3.2.11 follows immediately from Theorem 3.2.9 and our observation that spectral radius preserving maps also preserve nilpotents.

(3) Suppose first that the situation (i) in Theorem 3.2.9 occurs. In this case we define $\Phi: B(X) \rightarrow B(X)$ by

$$\Phi(A) = T^{-1} \phi(A) T.$$ 

If (ii) occurs, then $X$ is reflexive. Thus, the natural embedding $K: X \rightarrow X''$ is bijective. In this case we define $\Phi$ by

$$\Phi(A) = K^{-1} T' \phi(A)' (T^{-1})' K.$$ 

In either case $\Phi$ is a bijective linear mapping preserving the spectral radius, and satisfying

$$\Phi(I) = I, \quad \Phi(N) = dN$$

for every nilpotent operator $N \in B(X)$.

**Theorem 3.2.11.** [29] Suppose that a surjective linear map $\phi: B(X) \rightarrow B(X)$ satisfies $r_\sigma(\phi(A)) = r_\sigma(A)$ for every $A \in B(X)$. Then there exist a complex number $c$ such that $|c| = 1$ and either

(i) a bounded bijective linear operator $T: X \rightarrow X$ such that $\phi(A) = cTAT^{-1}$ for
every $A \in B(X)$; or
(ii) a bounded bijective linear operator $T : X' \to X$ such that $\phi(A) = cTA'T^{-1}$
for every $A \in B(X)$. In this case $X$ must be reflexive.

**Proof.** Clearly, by Remark 3.2.10.(3) the theorem will be proved by showing that $d = 1$ and $\Phi(A) = A$ for every $A \in B(X)$.

**Step 1.** $d = 1$ and $\Phi(P) = P$ for every rank one projection $P$.

**Proof.** A rank one projection can be written as $x \otimes f$ where $f(x) = 1$.
Let us first show that $y = \Phi(x \otimes f)x$ and $x$ are linearly depending. Suppose this is not true.
Then we have
\[
\Phi(x \otimes f)x = \alpha x + z
\]
where $\alpha = f(y) \in \mathbb{C}$ and $0 \neq z \in \ker f$.

Now pick $g \in X'$ such that $g(x) = 0$ and $g(z) = 1$, and set $w = \Phi(x \otimes f)z$.

We have
\[
w = \beta x + \gamma z + u,
\]
where $\beta = f(w)$, $\gamma = g(w)$ and $u = w - f(w)x - g(w)z \in \ker f \cap \ker g$.
Now define
\[
\delta = (2 - \alpha)(2 - \gamma) - \beta,
\]
\[
N = d^{-1}[(\delta x - u) \otimes g],
\]
\[
A = N + (x \otimes f).
\]
Clearly, $N^2 = 0$, so that $\Phi(N) = dN$ and $\Phi(A) = \Phi(x \otimes f) + dN$.

From $A^3 = A^2$ it follows that $r_0(A) \leq 1$ this, however, contradicts the equality
\[
\Phi(A)((2 - \gamma)x + z) = 2((2 - \gamma)x + z).
\]

Thus we have proved that for any $x \in X$ and $f \in X'$ such that $f(x) = 1$ there is $\lambda(x, f) \in \mathbb{C}$ such that
\[
\Phi(x \otimes f)x = \lambda(x, f)x.
\]

Now pick $x, y \in X$ and $f, g \in X'$ such that $f(x) = g(y) = 1$ and $f(y) = g(x) = 0$.
The operators $y \otimes f$, $x \otimes g$ and $(x - y) \otimes (f + g)$ are nilpotents, so that the identity
\[
x \otimes f - y \otimes g = y \otimes f - x \otimes g + (x - y) \otimes (f + g)
\]
yields
\[
\Phi(x \otimes f) - \Phi(y \otimes g) = d(x \otimes f - y \otimes g).
\]
(Note that we have actually proved the following: if P and Q are rank one projections such that PQ = QP = 0 then \( \Phi(P) - \Phi(Q) = d(P - Q) \).)

Applying the last identity we get

\[
\Phi(x \otimes f)y = \lambda(y, g)y - dy.
\]

Thus, we have shown that given a rank one projection \( x \otimes f \) and \( y \in \text{Ker} f \), the vector \( \Phi(x \otimes f)y \) lies in the linear span of \( \{ y \} \). Consequently, the restriction of \( \Phi(x \otimes f) \) to \( \text{Ker} f \) is a scalar multiple of the identity.

We resume the arguments above in the following statement:

For any rank one projection \( P \) there exist \( \lambda_P, \mu_P \in \mathbb{C} \) such that \( \lambda_P \neq \mu_P \) and \( \Phi(P) = \lambda_P P + \mu_P (I - P) \).

Therefore, \( \sigma(\Phi(P)) = \{ \lambda_P, \mu_P \} \).
Since \( \pi(\Phi(P)) = \pi(P) = \{ 1 \} \), it follows that either \( \lambda_P = 1 \) and \( |\mu_P| < 1 \) or \( \mu_P = 1 \) and \( |\lambda_P| < 1 \).

Next we have

\[
\{ -1, 1 \} = \pi(I - 2P) = \pi(I - 2 \Phi(P)) \subseteq \{ 1 - 2\lambda_p, 1 - 2\mu_p \},
\]
and hence \( \{ -1, 1 \} = \{ 1 - 2\lambda_p, 1 - 2\mu_p \} \).

But then one concludes that

either \( \lambda_P = 0 \) or \( \mu_P = 0 \), that is, either \( \Phi(P) = P \) or \( \Phi(P) = I - P \).

Now let P and Q be rank one projections satisfying

\( PQ = QP = 0 \).

As observed above, we then have

\( \Phi(P) - \Phi(Q) = d(P - Q) \).

Whence it follows that, either \( \Phi(P) = P, \Phi(Q) = Q \) and \( d = 1 \) or \( \Phi(P) = I - P, \Phi(Q) = I - Q \) and \( d = -1 \).

However, in the latter case we have

\( \Phi(P + Q) = 2I - (P + Q) \).

Assuming that the dimension of \( X \) is greater of 2 we see that \( r_\sigma(\Phi(P+Q)) = 2 \), while \( r_\sigma(P+Q) = 1 \) a contradiction.

The two dimensional case is easy to treat.
Anyway, the general finite dimensional case follows almost immediately from [28].

**Step 2.** $\Phi(A) = A$ for every $A \in B(X)$.

**Proof.** Suppose that $Ax = x$ for some $A \in B(X)$ and a nonzero $x \in X$. Given $\lambda \in \mathbb{C}$, $|\lambda| > \|A\|$, we then have $$(\lambda I - A)^{-1}x = 1/(\lambda - 1)x.$$ Pick $f \in X'$ such that $f(x) = 1$. Applying Lemma 2.2.4 we see that $\lambda \in \sigma(A + \mu(x \otimes f))$ if and only if $\mu f((\lambda I - A)^{-1}x) = 1$, that is, $\lambda = 1 + \mu$.

Thus, we have proved that $$\sigma(A + \mu(x \otimes f)) \subseteq \{\lambda \in \mathbb{C} \mid |\lambda| \leq \|A\|\} \cup \{1+\mu\}.$$ Therefore, if $|\lambda| > \|A\| + 1$, then $$1 + \mu \in \pi(A + \mu(x \otimes f)) = \pi(\Phi(A) + \mu(x \otimes f)),$$ so, in particular, $1 + \mu \in \sigma(\Phi(A) + \mu(x \otimes f))$.

Applying Lemma 2.2.4 again, it follows that $\mu f((I + \mu I - \Phi(A))^{-1}x) = 1$. That is, $\mu( f((\mu I - (\Phi(A) - I))^{-1}x)) = 1$, which can be written in the form $$\sum_{k=0}^{\infty} \mu^k f \left( \left( (\Phi(A) - I)^{-1}x \right) \right) = 1.$$ Whence $f((\Phi(A) - I)x) = 0$, that is, $f(\Phi(A)x) = 1$.

Now suppose that $g \in X'$ satisfies $g(x) = 0$. Then $(f + g)(x) = 1$ which implies $(f + g)(\Phi(A)x) = 1$ and so $g(\Phi(A)x) = 0$. This shows that $x$ and $\Phi(A)x$ are linearly dependent, in fact, $\Phi(A)x = x$ for $f(x) = 1$ and $f(\Phi(A)x) = 1$.

Thus, we have shown that $Ax = x$ implies $\Phi(A)x = x$.

Now let $y \in X$ and $A \in B(X)$ be arbitrary. Set $z = Ay$ and pick an operator $F$ of rank one such that $Fy = y - z$. Then, of course, $\Phi(F) = F$, and so $(A + F)y = y$ yields $$(\Phi(A) + F)y = y,$$ that is, $\Phi(A)y = z$.

Thus, $\Phi(A) = A$.  

\[\square\]
Chapter 4

Linear Maps Preserving The Essential Spectral Radius

This chapter consists of two sections. In the first one we study essential spectral radius. In the second section we study linear maps that preserve essential spectral radius.

4.1 Essential spectral radius

Throughout this chapter, H will denote an infinite dimensional complex Hilbert space, B(H) will denote the algebra of all bounded linear operators on H and K(H) will denote the algebra of all linear compact operators on H.

**Definition 4.1.1.** [21] Let H be a Hilbert space and T ∈ B(H), then T is said to be **left Fredholm** (resp. **right Fredholm**) if range(T) is closed and dim(Ker(T)) is finite (resp. dim(coker(T) = H/range(T)) is finite).

We say that T is **semi-Fredholm** if it is left Fredholm or right Fredholm and T is said to be **Fredholm** if it is both left and right Fredholm.

**Note.** [27] The dimension of the cokernel is called the codimension, and it is denoted by codimT. That is an operator T ∈ B(H) is called Fredholm if R(T) is closed and \( \max \{ \dim \ker(T), codim R(T) \} < \infty \).

**Definition 4.1.2.** [27] If T is Fredholm, we will define the **index** of T (which is denoted by Ind(T)) to be the number,

\[
\text{index}(T) = \dim \ker(T) - \dim \coker(T) \quad \ldots \ldots (1)
\]

\[
= \dim \ker(T) - \dim \ker(T^*) \quad \ldots \ldots (2)
\]
**Notes.** Equations (1) and (2) above are the same since, (using range(T ) is closed and applying Theorem 1.2.10 and Remark 1.2.8)

\[ H = R(T ) \oplus R(T ^\perp) = R(T ) \oplus \ker(T^*) \]

so that coker(T ) = H/R(T ) \cong \ker(T^*).

**Lemma 4.1.3.** The requirement that range(T ) is closed in Definition 4.1.1 is redundant.

**Proof.** See [27].


1. If X and Y are finite-dimensional Hilbert spaces, then every operator from X to Y is Fredholm.
   **Proof.** Let T \in B(X,Y), then T is bounded and dim (ker(T )) < \infty, dim (coker(T )) < \infty. Hence T is Fredholm.

2. Let H be a Hilbert space, then any invertible operator T \in B(H) is Fredholm, and its index is 0.
   **Proof.** Let T be any invertible operator in B(H), then ker(T ) = {0} and R(T ) = H. Hence dim (ker(T )) = 0 < \infty, dim (coker(T )) = 0 < \infty. Hence T is Fredholm, and its index is 0.

3. Right-shift in \ell^2: Consider the operator, T : (\xi_1, \xi_2, \xi_3, \ldots) \rightarrow (0, \xi_1, \xi_2, \xi_3, \ldots). Then, kerT = {0}. Also, Range T = \{(\xi_i) : \xi_1 = 0\}. It is closed and codim Range T = 1. Hence, Ind T = -1.

**Definition 4.1.5.** [21] For every T \in B(H) we set

- \(\sigma_e(T) = \{\lambda \in \C : (T - \lambda I) \text{ is not Fredholm}\}\),
- \(\sigma_{le}(T) = \{\lambda \in \C : (T - \lambda I) \text{ is not left Fredholm}\}\),
- \(\sigma_{re}(T) = \{\lambda \in \C : (T - \lambda I) \text{ is not right Fredholm}\}\),
- \(\sigma_{SF}(T) = \{\lambda \in \C : (T - \lambda I) \text{ is not semi-Fredholm}\}\).

These are called the **essential spectrum**, the **left essential spectrum**, the **right essential spectrum**, and the **semi-Fredholm spectrum**, respectively, of T.

**Remarks 4.1.6.** [27]

1. \(\sigma_{SF}(T) = \sigma_{le}(T) \cap \sigma_{re}(T)\) and \(\sigma_e(T) = \sigma_{le}(T) \cup \sigma_{re}(T)\).
   **Proof.**
   \[ \sigma_{SF}(T) = \{\lambda \in \C : (T - \lambda I) \text{ is not semi-Fredholm}\} \quad \text{(by definition 4.1.5)} \]
   \[ = \{\lambda \in \C : (T - \lambda I) \text{ is neither left Fredholm nor right Fredholm}\} \quad \text{(by definition 4.1.1)} \]
\[ \{ \lambda \in \mathbb{C} : (T - \lambda I) \text{ is not left Fredholm} \} \cap \{ \lambda \in \mathbb{C} : (T - \lambda I) \text{ is not right Fredholm} \} \]
\[ = \sigma_{le}(T) \cap \sigma_{re}(T) \quad (\text{by definition 4.1.5}) \]
\[ \sigma_e(T) = \{ \lambda \in \mathbb{C} : (T - \lambda I) \text{ is not Fredholm} \} \]
\[ = \{ \lambda \in \mathbb{C} : \dim(\text{Ker}(T - \lambda I)) = \infty \text{ or } \text{codim}(T - \lambda I) = \infty \} \quad (\text{by definition 4.1.1}) \]
\[ = \{ \lambda \in \mathbb{C} : (T - \lambda I) \text{ is not invertible} \} \subseteq \sigma(T). \]

**Definition 4.1.7.** [27] Let H be a Hilbert space, B(H) be the bounded linear operators on H and let K(H) be the closed ideal of B(H) consisting of all compact operators on H. The quotient algebra \( C(H) = B(H)/K(H) \) is called the **Calkin algebra.**

**Remark 4.1.8.** [27] The Calkin Algebra \( C(H) = B(H)/K(H) \) is a prime semi simple C*-algebra.

**Proof.** By Proposition 1.3.6, \( C(H) \) is Banach algebra. Moreover since \( B(H) \) and \( K(H) \) are C*-algebras (by example 1.3.17 (1) and (2) ), then by (3) in example 1.3.17) the Calkin algebra \( C(H) = B(H)/K(H) \) is C*-algebra. By definition 1.3.10, we have
\[ \text{Rad}(C(H)) = \{ T \in C(H) : 1 - TS \text{ is invertible for all } S \in C(H) \} = K(H), \]
Hence \( C(H) \) is semi-simple.

**Theorem 4.1.9.** (Atkinson’s theorem) \( T \) is a Fredholm operator in \( B(H) \) if and only if \( \pi(T) \) is invertible in \( C(H) \) (where \( \pi \) is the quotient map from \( B(H) \) onto \( C(H) \)), or for \( T \in B(H) \) the following are equivalent:
1. \( T \) is Fredholm.
2. \( \exists S \in B(H) \) such that \( I - TS \) and \( I - ST \) are of finite rank.
3. \( \exists S \in B(H) \) such that \( I - TS \) and \( I - ST \) are compact, equivalently \( T \) is invertible modulo compact operators.

**Proof.** See [11]

**Remark 4.1.10.** [27] If \( T \in B(H) \), then the essential spectrum \( \sigma_e(T) \) is the spectrum of \( \pi(T) \) in \( C(H) \), i.e.
\[ \sigma_e(T) = \{ \lambda \in \mathbb{C} : \pi(\lambda - T) \text{ is not invertible in } C(H) \} = \sigma(\pi(T)), \]

where \( \pi \) is the natural homomorphism from \( B(H) \) onto \( C(H) \).

**Proof.** \( \sigma_e(T) = \{ \lambda \in \mathbb{C} : (T - \lambda I) \text{ is not Fredholm} \} \)

\[ = \{ \lambda \in \mathbb{C} : \pi(\lambda - T) \text{ is not invertible in } C(H) \} \text{ (by Theorem 4.1.9)} \]

\[ = \sigma(\pi(T)). \]

**Corollary 4.1.11.** [7] If \( F \in B(H) \) is Fredholm then \( F^* \) is Fredholm and

\[ \text{index}(F) = - \text{index}(F^*). \]

**Proof.** Choose \( T \in B(H) \) such that both \( TF - I \) and \( FT - I \) are compact.

Then by Theorem 1.1.33, \( F^*T^* - I \) and \( T^*F^* - I \) are compact which implies that \( F^* \) is Fredholm.

On the other hand, we have

\[ \text{index}(F) = \dim \text{Ker}(F) - \dim \text{Ker}(F^*) \text{ (by (1) in definition 4.1.2)} \]

and

\[ \text{index}(F^*) = \dim \text{Ker}(F^*) - \dim \text{Ker}(F^{**}) \]

\[ = \dim \text{Ker}(F^*) - \dim \text{Ker}(F) \]

Hence,

\[ \text{index}(F) = - \text{index}(F^*). \]

**Lemma 4.1.12** If \( F \) is a finite rank operator on a Hilbert space \( H \), then

\[ \text{index}(I + F) = 0. \]

**Proof.** See [7]

**Theorem 4.1.13.** If we are given two Fredholm operators \( T_1 \in B(H) \) and \( T_2 \in B(H) \), then \( T_2T_1 \in B(H) \) is also a Fredholm operator, and it satisfies

\[ \text{index}T_2T_1 = \text{index}T_1 + \text{index}T_2. \]

**Proof.** See [7]

**Theorem 4.1.14.** [11] Let \( T \in B(H) \) be a Fredholm and let \( K \in K(H) \). Then \( T + K \) is a Fredholm operator, and \( \text{ind}(T + K) = \text{ind}(T) \).

**Proof.** Let \( T \in B(H) \) be a Fredholm operator, and let \( K \in K(H) \). Then by Theorem 4.1.9 there exists \( S \in B(H) \) and operators \( K_1 \) and \( K_2 \) which are compact on \( H \) such that

\[ ST = I + K_1 \text{ and } TS = I + K_2. \]

We see that
\[ S(T + K) = ST + SK = I + K_1 + SK = I + K' \]
\[ (T + K)S = TS + KS = I + K_2 + KS = I + K'' \]

where \( K' \) and \( K'' \) are compact operators.

By Theorem 4.1.9 we conclude that \( T + K \) is a Fredholm operator.

\( I + K_1 \) has index 0 by lemma 4.1.12, so by Theorem 4.1.13,

\[ 0 = \text{index}(I + K_1) = \text{index}(ST) = \text{index}S + \text{index}T \]

This tells us that \( \text{index}S = -\text{index}T \).

Since \( K' \) is also a finite rank operator, \( \text{index}(I + K') = 0 \)
so that

\[ 0 = \text{index}(S(T + K)) = \text{index}S + \text{index}(T + K) = -\text{index}T + \text{index}(T + K) \]

Hence \( \text{index}(T + K) = \text{index}T \).

**Definition 4.1.15.** [21] Let \( T \in B(H) \). The radius \( r_e(T) \) of the essential spectrum \( \sigma_e(T) \) is defined by

\[ r_e(T) = \sup \{ |\lambda| : \lambda \in \sigma_e(T) \} \]

**Remark 4.1.16.** \( r_e(T) = r_o(\pi(T)) \)

**Proof.** \( r_e(T) = \sup \{ |\lambda| : \lambda \in \sigma_e(T) \} \) (by definition 4.1.15)
\[ = \sup \{ |\lambda| : \lambda \in \sigma(\pi(T)) \} \) (since \( \sigma_e(T) = \sigma(\pi(T)) \) by remark 4.1.10)
\[ = r_o(\pi(T)). \]

**Definition 4.1.17.** [21] A linear map \( \phi \) from \( B(H) \) into itself preserves the essential spectral radius if \( r_e(\phi(T)) = r_e(T) \) for every \( T \in B(H) \).

**Definition 4.1.18.** [20] A mapping \( \phi \) from an algebra \( A \) into another algebra \( B \) is said to be a **Jordan homomorphism** from \( A \) into \( B \) if

1. \( \phi(a + b) = \phi(a) + \phi(b) \),
2. \( \phi(ab + ba) = \phi(a)\phi(b) + \phi(b)\phi(a) \), for all \( a \) and \( b \) in \( R \).

Observe that the definition of a Jordan homomorphism given above is equivalent to the statement that \( \phi(a^2) = (\phi(a))^2 \) for every \( a \in A \).

**Note.** [15] It is of course clear that every homomorphism and every anti-homomorphism is a Jordan homomorphism.
**Theorem 4.1.19.** A Jordan automorphism of a prime ring of characteristic different from 2 and 3 is either an automorphism or an anti-automorphism. We say that a ring $R$ is of characteristic greater than $n$ if $n!x = 0$ implies $x = 0$.

Recall that a nonzero ring $R$ is called a prime ring if for any two ideals $A$ and $B$ of $R$, $AB = \{0\}$ implies $A = \{0\}$ or $B = \{0\}$.

**Proof.** See [15]

**Remark 4.1.20.** Let $H$ be a Hilbert space. Then

$$\sigma_c(T + K) = \sigma_c(T)$$

and so $r_c(T + K) = r_c(T)$, for all $T \in B(H)$ where $K \in K(H)$.

**Proof.** $\sigma_c(T + K) = \sigma(\pi(T + K))$ (by remark 4.1.10)

$$= \sigma(\pi(T) + \pi(K))$$

$$= \sigma(\pi(T)) \text{ (since } \pi(K) = 0)$$

$$= \sigma_c(T).$$

4.2 **Linear Maps Preserving The Essential Spectral Radius**

Throughout this chapter, $H$ will denote an infinite dimensional complex Hilbert space, $B(H)$ will denote the algebra of all bounded linear operators on $H$, $K(H)$ will denote the algebra of all linear compact operators on $H$ and $C(H) = B(H)/K(H)$ will denote the Calkin algebra.

In the proof of the main result we will use a result concerning linear maps preserving the usual spectral radius. This is a theorem by Lin and Mathieu [35, Corollary 2.6] that concerns surjective spectral radius preserving maps between two unital $C^*$-algebras such that one of them is purely infinite with real rank zero.

**Theorem 4.2.1.** Let $C_1$ and $C_2$ be two unital $C^*$-algebras such that one of them is purely infinite with real rank zero and let $\phi : C_1 \to C_2$ be a surjective linear
map preserving the spectral radius. Then there are a central unitary \( \lambda \in \mathbb{C} \) and a Jordan isomorphism \( J : C_1 \rightarrow C_2 \) such that \( \phi(a) = \lambda J(a) \) for all \( a \in C_1 \).

**Proof.** See [35]

**Remark.** For our purposes, the only relevant example of an algebra having the latter properties in the previous Theorem is the Calkin algebra. Therefore in the next lemma we state theorem 4.2.1 only for the Calkin algebra.

**Lemma 4.2.2.** [21] Let \( C(H) \) be the Calkin algebra and let \( \phi : C(H) \rightarrow C(H) \) be a surjective linear map preserving the spectral radius. Then there are \( \lambda \in \mathbb{C} \) with \( |\lambda| = 1 \) and an automorphism or an antiautomorphism \( J : C(H) \rightarrow C(H) \) such that \( \phi(a) = \lambda J(a) \) for all \( a \in C(H) \).

**Remarks.** [21] A few comments must be added to this statement.

1) In Theorem 4.2.1 \( \lambda \) can be any central unitary element; however, since the center of \( C(H) \) (i.e \{ \( x \) in \( C(H) \) such that \( xa = ax \) for all \( a \) in \( C(H) \)\}) is trivial, \( \lambda \) must be a complex number of modulus 1 in our setting.

2) Further, the conclusion of Theorem 4.2.1 \( J \) is that \( J \) a Jordan isomorphism; since the algebra \( C(H) \) is prime, Theorem 4.1.19 tells us that \( J \) must be an automorphism or an antiautomorphism in our setting.

**Lemma 4.2.3.** [21] The following are equivalent:

(i) \( K \in K(H) \).

(ii) \( r_e(T + K) = 0 \) for all \( T \in B(H) \) for which \( r_e(T) = 0 \).

**Proof.** Assume that \( K \in K(H) \), and let \( T \in B(H) \) be an operator such that \( r_e(T) = 0 \). Denote by \( \pi \) the quotient map from \( B(H) \) onto \( C(H) \) where \( \pi(T) = T+K(H) \) for all \( T \in B(H) \), then \( \pi(K) = K+K(H) = K(H) = 0 \) and that

\[
\begin{align*}
    r_e(T + K) &= r_e(\pi(T + K)) \quad \text{(by remark 4.1.16)} \\
               &= r_e(\pi(T) + \pi(K)) \quad \text{(since \( \pi \) is linear)} \\
               &= r_e(\pi(T)) \\
               &= r_e(T) = 0.
\end{align*}
\]

This establishes the implication (i) \( \Rightarrow \) (ii).

Conversely, assume that \( K \in B(H) \) is an operator such that \( r_e(T + K) = 0 \) for all \( T \in B(H) \) for which \( r_e(T) = 0 \). Equivalently by remark 4.1.16, we have

\[
    r_e(\pi(T + K)) = r_e(\pi(T) + \pi(K)) = 0
\]

for all \( T \in B(H) \) for which \( r_e(\pi(T)) = 0 \).
That is in the Calkin algebra $C(H)$ we have $\pi(T + K) \in C(H)$ such that $r_e(\pi(T + K)) = r_e(\pi(T) + \pi(K)) = 0$ where $\pi(T)$ is a quasi-nilpotent in $C(H)$. Hence, by Theorem 3.1.5 we have $\pi(K) \in \text{rad}(C(H)) = \{0\}$ (since $C(H)$ is a semi simple Banach algebra), and so $K$ is a compact operator.

**Definition 4.2.4.** [21] A linear map $\phi : B(H) \to B(H)$ is said to be surjective up to compact operators if $B(H) = \text{range}(\phi) + K(H)$ (i.e.; for every $T \in B(H)$ there exists $S \in B(H)$ and $K \in K(H)$ such that $T = \phi(S) + K$).

**Remark 4.2.5.** [21] It is clear that if $\phi$ in the above definition is surjective, then it is surjective up to compact operators.

**Now, we prove the main result in this chapter.**

**Theorem 4.2.6** [21] Let $\phi : B(H) \to B(H)$ be a linear map surjective up to compact operators. Then $\phi$ preserves the essential spectral radius if and only if $\phi(K(H)) \subset K(H)$, and the induced map $\varphi : C(H) \to C(H)$ is either a continuous automorphism or a continuous antiautomorphism multiplied by a scalar of modulus 1.

**Proof.** $\Rightarrow$ Assume that $\phi$ preserves the essential spectral radius. Our first goal is to show that $\phi$ leaves $K(H)$ invariant.

So pick a compact operator $K \in K(H)$, and let us prove that $\phi(K)$ is compact as well.

Let $S \in B(H)$ be an operator for which $r_e(S) = 0$. Since $\phi$ is surjective up to compact operators, there exist $T \in B(H)$ and $K' \in K(H)$ such that $S = \phi(T) + K'$.

We have

\[
\begin{align*}
r_e(T) &= r_e(\phi(T)) \\
&= r_e(S - K') \\
&= r_e(S) \quad \text{(by lemma 4.2.3)} \\
&= 0
\end{align*}
\]

And

\[
\begin{align*}
r_e(\phi(K) + S) &= r_e(\phi(K) + \phi(T) + K) \\
&= r_e(\phi(K) + \phi(T)) \quad \text{(by remark 4.1.20)} \\
&= r_e(\phi(K + T)) \quad \text{(since $\phi$ is linear)} \\
&= r_e(K + T) \quad \text{(since $\phi$ preserves the essential spectral radius)} \\
&= r_e(T) \quad \text{(by remark 4.1.20)} \\
&= 0.
\end{align*}
\]

By Lemma 4.2.3, $\phi(K)$ is a compact operator, Thus $\phi(K(H)) \subset K(H)$.

So $\phi$ induces a linear map $\varphi : C(H) \to C(H)$ such that $\pi(T) \mapsto (\pi \circ \phi)(T)$ where $\pi$ denotes the quotient map from $B(H)$ onto $C(H) = B(H)/K(H)$.

Clearly, $\varphi$ is surjective, since $\phi$ is surjective up to compact operators and $\varphi \circ \pi = \pi \circ \phi$.

Now for every $T \in B(H)$, we have
\[ r_{\sigma}(\varphi(\pi(T))) = r_{\sigma}(\pi \circ \phi(T)) \]
\[ = r_{\sigma}(\phi(T)) \quad \text{(by remark 4.1.16)} \]
\[ = r_{\sigma}(T) \quad \text{(since } \phi \text{ preserves the essential spectral radius)} \]
\[ = r_{\sigma}(\pi(T)). \quad \text{(by remark 4.1.16)} \]

In other words, the map \( \varphi \) preserves the spectral radius.

Therefore by Lemma 4.2.2 there are \( \lambda \in \mathbb{C} \) with \( |\lambda| = 1 \) and an automorphism or an antiautomorphism \( J : C(H) \to C(H) \) such that \( \varphi(a) = \lambda J(a) \) for all \( a \in C(H) \).

Conversely, assume that \( \varphi(K(H)) \subseteq K(H) \), and the induced map \( \varphi : C(H) \to C(H) \) is either a continuous automorphism or a continuous antiautomorphism multiplied by a scalar of modulus 1.

Thus, using Theorem 4.1.9, the following equivalences hold:

- \( T \) is Fredholm in \( B(H) \) iff \( \pi(T) \) is invertible in \( C(H) \) (by Theorem 4.1.9)
  - iff \( \varphi(\pi(T)) = \pi(\phi(T)) \) is invertible in \( C(H) \)
  - iff \( \phi(T) \) is Fredholm in \( B(H) \). (by Theorem 4.1.9)

Then,
\[ r_e(T) = \sup \{ |\lambda| : \lambda \in \sigma_e(T) \} \]
\[ = \sup \{ |\lambda| : (T - \lambda I) \text{ is not Fredholm} \} \quad \text{(by definition 4.1.5)} \]
\[ = \sup \{ |\lambda| : (\phi(T) - \lambda I) \text{ is not Fredholm} \} \quad \text{(by the above equivalences)} \]
\[ = \sup \{ |\lambda| : \lambda \in \sigma_e(\phi(T)) \} \]
\[ = r_e(\phi(T)) \]

Therefore \( \phi \) preserves the essential spectral radius operators. Hence the proof of the theorem is complete.

\[ \Box \]

**Definition 4.2.7.** [6] A **unital map** on a \( C^* \)-algebra is a map \( \phi \) which preserves the identity element: \( \phi(I) = I \).

**Definition 4.2.8.** [21] A linear map \( \phi : B(H) \to B(H) \) is called **unital modulo a compact operator** if \( \phi(I) - I \) is a compact operator i.e. \( \phi(I) = I + K \) where \( K \) is compact operator.

**Corollary 4.2.9.** ([25] and [24]) Let \( \phi : B(H) \to B(H) \) be a linear map surjective up to compact operators. Then \( \phi \) preserves the essential spectrum (the semi-Fredholm spectrum) if and only if \( \phi(K(H)) \subseteq K(H) \), and the induced map \( \varphi : C(H) \to C(H) \) is either a continuous automorphism or a continuous antiautomorphism.

**Proof.** Assume that \( \phi \) preserves the essential spectrum.
We first prove that $\phi$ is unital modulo a compact operator.

Let $T \in B(H)$ such that $r_e(T) = 0$. There are two operators $S,K \in B(H)$ such that $K$ is compact and $T = \phi(S) + K$. We have

$$\sigma_e(T + \phi(I) - I) = \sigma_e(T + \phi(I)) - 1 \quad (\text{by theorem 2.1.16})$$
$$= \sigma_e(\phi(S) + K + \phi(I)) - 1 \quad (\text{by remark 4.1.20})$$
$$= \sigma_e(S + I) - 1 \quad (\text{since } \phi \text{ preserves the essential spectral radius})$$
$$= \sigma_e(S) \quad (\text{by theorem 2.1.16})$$
$$= \sigma_e(\phi(S)) \quad (\text{since } \phi \text{ preserves the essential spectral radius})$$
$$= \sigma_e(T - K) \quad (\text{by remark 4.1.20})$$
$$= \sigma_e(T) \quad (\text{by theorem 2.1.16})$$
$$= \sigma_e(\phi(S)) \quad (\text{since } \phi \text{ preserves the essential spectral radius})$$
$$= \sigma_e(T - K) \quad (\text{since } \phi \text{ preserves the essential spectral radius})$$
$$= \sigma_e(T) \quad (\text{by remark 4.1.20})$$
$$= \{0\}.$$

By Lemma 4.2.3 we see that $\phi(I) - I$ is a compact operator i.e $\phi(I) = I + K$ where $K$ is compact operator. thus $\phi$ is unital modulo a compact operator.

Note that, since $\phi$ preserves the essential spectrum (the semi-Fredholm spectrum), it preserves the essential spectral radius as well.

It follows from Theorem 4.2.6 that $\phi(K(H)) \subset K(H)$ and that there are a scalar $\lambda$ of modulus 1 and either a continuous automorphism or a continuous antiautomorphism $\psi$ such that $\phi = \lambda \psi$.

What was shown above implies that $\phi(I) = \phi(\pi(I)) = \pi(\phi(I))$
$$= \pi(I + K) \quad (\text{since } \phi \text{ is unital modulo a compact operator})$$
$$= \pi(I) + \pi(K) \quad (\text{since } \pi \text{ is linear})$$
$$= \pi(I) = I, \text{ and so } \phi \text{ is unital.}$$

As $\phi = \lambda \psi$, then $\psi$ is unital as well. Hence, we see that $\lambda = 1$ and $\phi$ is either a continuous automorphism or a continuous antiautomorphism.

Conversely, assume that $\phi(K(H)) \subset K(H)$, and the induced map $\phi : C(H) \to C(H)$ is either a continuous automorphism or a continuous antiautomorphism. By a proof similar to the proof of the “only if” part of the above theorem, we get that $\phi$ preserves the essential spectrum (the semi-Fredholm spectrum).

**Corollary 4.2.10.** [21] Let $\phi : B(H) \to B(H)$ be a surjective linear map up to compact operators. Then $\phi$ preserves the left essential spectrum (or the right essential spectrum) if and only if $\phi(K(H)) \subset K(H)$, and the induced map $\phi : C(H) \to C(H)$ is a continuous automorphism.

**Proof.** Assume that $\phi$ preserves the left essential spectrum.

Note that, since the boundary of the essential spectrum $\partial \sigma_e(T) \subset \sigma_{le}(T)$ for all $T \in B(H)$, we have $r_e(T) = \max\{|\lambda| : \lambda \in \sigma_{le}(T)\}$. Thus just as for the proof of the above corollary, one can show that $\phi$ is either a continuous automorphism or a continuous antiautomorphism. It remains to show that $\phi$ cannot be an antiautomorphism.
Now, assume by the way of contradiction that $\varphi$ is an antiautomorphism, and pick a non-essentially invertible operator $T \in B(H)$ which is left essentially invertible, i.e., $0 \in \sigma_e(T) \setminus \sigma_{le}(T)$. Therefore, there is $S \in B(H)$ such that $ST = I$ but $TS \neq I$ and so, $\pi(S)\pi(T) = \pi(I)$ but $\pi(T)\pi(S) \neq \pi(I)$. Where $\pi$ denotes the quotient map from $B(H)$ onto $C(H) = B(H)/K(H)$.

We thus have
\[
\pi(I) = \pi(\varphi(I)) = \varphi(\pi(I)) \\
= \varphi(\pi(S)\pi(T)) \\
= \varphi(\pi(T))\varphi(\pi(S)) \quad (\text{since } \varphi \text{ is an antiautomorphism})
\]
and so $\varphi(\pi(T))$ is right invertible.

Since $\pi(T)$ is left invertible, $\varphi(\pi(T))$ is left invertible as well. Thus, $\varphi(\pi(T))$ is in fact invertible and so is $\pi(T)$, which is a contradiction and shows that $\varphi$ is a continuous automorphism.

Conversely, assume that $\varphi(K(H)) \subset K(H)$, and the induced map $\varphi : C(H) \to C(H)$ is either a continuous automorphism. By a proof similar to the proof of the ‘only if’ part of the above theorem, we get that $\varphi$ preserves the left essential spectrum (or the right essential spectrum).

\[ \square \]
Bibliography


