A Generalization of 2-Absorbing Subsemimodule Over Semirings

تعميم المقاسات الجزئية الممتصة من النوع الثاني على شبه الحلقات

Ayman Mohammed Abdalnabi

Supervised by

Prof. Dr. Mohammed Al-Ashker
Dr. Arwa Ashour
Prof. of Mathematics
Associate prof. of Mathematics

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science in Pure Mathematics

December/2016
A Generalization of 2-Absorbing Subsemimodule Over Semirings

I understand the nature of plagiarism, and I am aware of the University’s policy on this.

The work provided in this thesis, unless otherwise referenced, is the researcher’s own work, and has not been submitted by others elsewhere for any other degree or qualification.
نتيجة الحكم على أطروحة ماجستير

بناءً على موافقة شؤون البحث العلمي والدراسات العليا بمجلس إدارته بأغلبية العضوين في لجنة الحكم على أطروحة الباحث/ ايمن محمد فتحي عبدالنبي لنييل درجة الماجستير في كلية العلوم قسم الرياضيات و موضوعها:

تعميم الملاحظات الجزئية المتصلة من النوع الثاني على شبه الحلقات

A Generalization of 2-absorbing sub-semimodules over semirings

وبعد المناقشة التي تمت اليوم الثلاثاء 26 ربيع الثاني 1438 هـ الموافق 24/1/2017 م، الساعة الواحدة ظهرًا في قاعة مؤتمرات مبنى طيبة. اجتمعت لجنة الحكم على الأطروحة والمكونة من:

أ. د. محمد مسعود الأشقر

د. أروى عيد عادل

أ. د. أسعد يوسف اسماعيل

د. عبد الكريم عبد القادر ناجي

وبعد المداولات أوصت اللجنة بمنح الباحث درجة الماجستير في كلية العلوم/ قسم الرياضيات والجنة إذ تمنح هذه الدرجة فإنها توصيه بتقدير الله ونوره طاعته وأن يسعد علمه في خدمة دينه ووطنه.

والمفتتح

نائب الرئيس لشؤون البحث العلمي والدراسات العليا

أ. د. عبد الرؤوف علي المناعية
Abstract

Let R be a commutative semiring with identity and M be a unitary R-semimodule. Let N be a proper subsemimodule of M. We say that N is 2-absorbing subsemimodule of M if \( abm \in N \) where \( a,b \in R \) and \( m \in M \) implies that \( ab \in (N : M) \) or \( am \in N \) or \( bm \in N \).

This concept was introduced by M. Dubey and P. Sarohe (see [18]).

In this thesis we prove a number of results concerning 2-absorbing subsemimodules. And we generalize the concept of 2-absorbing subsemimodules to weakly 2-absorbing subsemimodules.

We also investigate the relation between some subsemimodules and 2-absorbing, weakly 2-absorbing subsemimodules. And we introduce the concepts of 2-absorbing compactly packed semirings and 2-absorbing compactly packed semimodules.
Dedication

To my beloved mother
To my beloved father
To my brothers and my sisters
To my dear and supporter wife
To my eye on the future, my lovely son Mohammed
To my relatives and my in-Laws
To my friends
To all who helped me, I dedicate this work
Acknowledgements

First of all my great thanks to Allah for reconciliation of this thesis.

Of course, I am grateful to my parents for their patience and love. Without them this work would never have come in to existance.

I would like to thank Prof.Dr. Mohammed Al-Ashker and Dr. Arwa Ashour, my supervisors, for their many suggestions and constant support during this research.

Finally, thanks are also due to all the staff members of mathematics department in the Islamic University of Gaza.
Contents

Declaration i

Abstract ii

Dedication iii

Acknowledgements iv

Table of Contents vi

List of Abbreviations vii

Introduction 1

1 Basic Concepts 3

1.1 Semirings and Ideals .................................. 3

1.2 Semimodules and Subsemimodules .................... 10

2 2-Absorbing Ideals in Semirings. 17

2.1 Definition and Properties of 2-Absorbing Ideals in Semirings ........ 17

2.2 Weakly 2-Absorbing Ideals in Semirings ................ 25
2.3 Relation Between Some Ideals and 2-Absorbing, Weakly 2-Absorbing Ideals in Semiring .................................................. 33

3 2-Absorbing Subsemimodules. ........................................ 38
   3.1 Definition and Properties of 2-Absorbing Subsemimodules .......... 38
   3.2 Weakly 2-Absorbing Subsemimodules .............................. 46
   3.3 Relation Between Some Subsemimodules and 2-Absorbing
       Subsemimodules ...................................................... 55

4 2-Absorbing Compactly Packed in Semirings ...................... 61
   4.1 2-Absorbing Compactly Packed Ideals ............................. 61
   4.2 2-Absorbing Compactly Packed Semimodules .................... 64

Conclusion ................................................................. 68

Bibliography ............................................................. 69
List of Abbreviations

CP                     Compactely Packed
FCP                    Finitely Compactly Packed
2-abs.CP               2-absorbing Compactly Packed
2-abs.FCP              2-absorbing Finitely Compactly Packed
Introduction

The notion of a semiring was first introduced by H.S. Vandiver [29] in 1934. After that, various researches have been done in this area and serval applications have been found in branches of mathematics and computer science. A commutative semiring is a commutative semigroup \((R,\cdot)\) and a commutative monoid \((R,+)\) that satisfies \(a.0=0\) for all \(a \in R\) and the distributive law hold.

Let \(R\) be a commutative semiring with unity and let \(M\) be a unitary \(R\)-semimodule. A proper ideal \(I\) of a semiring \(R\) is called 2-absorbing ideal if whenever \(a,b,c \in R\) and \(abc \in I\), then \(ab \in I\) or \(ac \in I\) or \(bc \in I\). This definition was introduced by Chaudhri in [11].

The concept of 2-absorbing subsemimodule was introduced in [18] by Dubey and Sarohe as a generalization of prime subsemimodule. A proper subsemimodule \(N\) of \(M\) said to be a 2-absorbing subsemimodule if whenever \(a,b \in R\) and \(m \in M\) with \(abm \in N\), then \(ab \in (N:M)\) or \(am \in N\) or \(bm \in N\).

It is known that a proper subsemimodule \(N\) of an \(R\)-semimodule \(M\) is compactly packed if for each family \(\{P_\alpha\}_{\alpha \in \Delta}\) of prime subsemimodules of \(M\) with \(N \subseteq \bigcup_{\alpha \in \Delta} P_\alpha\), \(N \subseteq P_\beta\) for some \(\beta \in \Delta\), and a semimodule \(M\) is called compactly packed if every proper subsemimodule of \(M\) is compactly packed, see [28].

In this thesis we study these previous concepts in details and introduce the concepts of weakly 2-absorbing subsemimodule as a generalization of the concepts of 2-
absorbing subsemimodule. Also, we introduce the concept of 2-absorbing compactly
packed subsemimodules. Thus, we say that a proper subsemimodule $N$ of $M$ is 2-absorbing
compactly packed if for each family $\{P_{\alpha}\}_{\alpha \in \Delta}$ of 2-absorbing subsemimodules of $M$ with $N \subseteq \bigcup_{\alpha \in \Delta} P_{\alpha}$, $N \subseteq P_{\beta}$ for some $\beta \in \Delta$, and a semimodule $M$ is called 2-absorbing compactly
packed if every proper subsemimodule is 2-absorbing compactly packed.

The thesis consists of four Chapters.

In Chapter one, we recall some basic concepts and properties on semirings and semimod-
ules.

In the second Chapter, we recall the concepts of 2-absorbing ideals, weakly 2-absorbing
ideals and relation between some ideals and 2-absorbing, weakly 2-absorbing ideals in
semirings.

In Chapter three, we investigate the concept of 2-absorbing subsemimodules. Also, we
generalize this concept to the concept of weakly 2-absorbing subsemimodules. We, also
find the relation between some subsemimodules and 2-absorbing, weakly 2-absorbing sub-
semimodules.

Finally, In Chapter four, we introduce the concepts of 2-absorbing compactly packed
semirings. Also, we introduce the concept of 2-absorbing compactly packed semimodules.

*We assume throughout this thesis that all semirings are commutative semirings with iden-
tity and all semimodules will be unitary.*
Chapter 1

Basic Concepts

In this chapter, we give basic informations which will be needed in remainder of the thesis.

1.1 Semirings and Ideals

The following definitions and properties are known and were introduced in [1, 4, 20, 21, 24, 26].

**Definition 1.1.1.** [20] Let G be a set. A *binary operation* on G is a function that assigns to each ordered pair of elements of G an element of G.

**Definition 1.1.2.** [20] A nonempty set G together with binary operation $\ast$ is called a *group* under this operation denoted by $(G, \ast)$ if the following three properties are satisfied.

1. **Associativity**, that is, $(a \ast b) \ast c = a \ast (b \ast c)$ for all $a, b, c$ in G.
2. **Identity**, that is, there exists an element $e$ in G such that $e \ast a = a \ast e = a$ for all $a$ in G.
3. **Inverse**, that is, for each element $a$ in G there exists an element $b$ in G such that $a \ast b = b \ast a = e$.

If for each $a, b$ in G $a \ast b = b \ast a$, then G is abelian.

**Definition 1.1.3.** [24] A nonempty set R together with a binary operation $\ast$ called a
semigroup if \( \ast \) is associative in \( R \), that is, \((a\ast b)\ast c=a\ast (b\ast c)\) for all \( a,b,c \) in \( R \).

If \( R \) contains identity (zero) element then the semigroup ia called a \textit{monoid}.

So if we drop the identity and inverse properties from the definition of a group, we get only a semigroup.

A semigroup \( R \) is called commutative if \( a\ast b=b\ast a \) for all \( a,b \in R \).

\textbf{Example 1.1.4.} The set of natural numbers \( N \) under addition is a semigroup, and \( N \cup \{0\} \) is a monoid .

\textbf{Definition 1.1.5.} [24] A \textit{semiring} is a nonempty set \( R \) togther with two binary operations addition and multiplication denoted by \( +,\cdot \) respectively, satisfying:

(1) \((R,+)\) is a commutative monoid.

(2) \((R,\cdot)\) is a monoid.

(3) Distributive law holds, that is, \( a\cdot(b+c)=a\cdot b+a\cdot c \) and \( (a+b)\cdot c=a\cdot c+b\cdot c \) for all \( a,b,c \in R \).

(4) \( a\cdot 0=0 \) for all \( a \in R \).

A semiring is called commutative if \((R,\cdot)\) is a commutative semigroup.

If \( R \) contains an element \( 1 \) such that \( 1\cdot a=a\cdot 1=a \) for all \( a \in R \), then \( R \) is said to be a semiring with identity(unity).

It is clear that every ring \( R \) is a semiring. But the converse is not true.

\textbf{Example 1.1.6.} [24] (1) Let \( Z^\circ=Z^+ \cup \{0\} \). \((Z^\circ,+,\cdot)\) is a commutative semiring with unity.

(2) \( Q^\circ=Q^+ \cup \{0\} \). \((Q^\circ,+,\cdot)\) is a commutative semiring with unity.

(3) Let \( M_{2\times 2} = \{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,b,c,d \in Z^\circ \} = \) set of all \( 2 \times 2 \) matrices with entries from \( Z^\circ \).

Clearly \((M_{2\times 2},+,\cdot)\) is a semiring under matrix addition and matrix multiplication. \( M_{2\times 2} \)
is a non-commutative semiring with unity $I=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

**Definition 1.1.7.** [24] A nonempty subset $S$ of a semiring $R$ is said to be *subsemiring* of $R$ if $S$ itself is a semiring under the same operations of $R$.

**Definition 1.1.8.** [24] Let $R$ be a semiring, $I$ be a nonempty subset of $R$. Let $I$ is a left (resp. right) ideal of $R$ if the following are satisfied:

(i) $I$ is a subsemiring.

(ii) For all $a \in I$ and $r \in R$ we have $ra \in I$ (resp. $ar \in I$).

$I$ is called an *ideal* if $I$ is left and right ideal.

Since we assumed all semirings are commutative with identity, we get the following properties:

**Proposition 1.1.9.** A nonempty subset $I$ of a semiring $R$ is an ideal of $R$ if:

(i) $a+b \in I$ where $a, b \in I$;

(ii) $ra \in I$ where $a \in I$ and $r \in R$.

**Definition 1.1.10.** [22] Let $X$ be a subset of a semiring $R$. If $X=\{a_1, a_2, ..., a_n\}$ then the ideal $< X > = \{a_1 + r_2 a_2 + ... + r_n a_n : r_i \in R\}$ is called the *ideal generated* by $X$. If $X$ consists of a single element, say $a$, then $< X > = < a >$ is called a principal ideal.

**Definition 1.1.11.** [22] A principal ideal semiring is a semiring in which every ideal is principal.

**Remark 1.1.12.** If $I$ and $J$ are ideals of a semiring $R$, the product of $I$ and $J$ is the ideal defined as: $IJ=\{a_1 b_1 + a_2 b_2 + ... + a_n b_n : a_i \in I, b_i \in J, n \text{ a positive integer}\}$.

**Lemma 1.1.13.** [22] If $I, J$ are two ideals of a semiring $R$, then $I \cap J$ is an ideal.
Lemma 1.1.14. [22] Let I, J be two ideals of a semiring R, then the sum of I and J is the ideal \( I + J = \{ x = i + j : i \in I, j \in J \} \) of R.

Definition 1.1.15. [26] An ideal I of a semiring R is called \( k \)-ideal or subtractive if for any \( a \in I \) and \( b \in R \) such that \( a + b \in I \), then \( b \in I \).

Example 1.1.16. (1) In any ring R, every ideal I is subtractive ideal since for any \( a \in I \), \( x \in R \) such that \( a + x \in I \), then \( x = a + x + (-a) \in I \) and so \( x \in I \).

(2) In the semiring \( \mathbb{Z}^\circ \) under the usual addition and multiplication, the set \( m \mathbb{Z}^\circ \) is a subtractive ideal of \( \mathbb{Z}^\circ \) where \( m \in \mathbb{Z}^\circ \), because for any \( a \in m \mathbb{Z}^\circ \), \( b \in \mathbb{Z}^\circ \) such that \( a + b \in m \mathbb{Z}^\circ \), implies \( b \in m \mathbb{Z}^\circ \).

(3) In [1], every subtractive ideal of R is an ideal, but the converse is not true in general.

Let \( R = \{ 0, a, b \} \). Define addition and multiplication on R as follow:

\[
\begin{array}{ccc}
+ & 0 & a & b \\
0 & 0 & a & b \\
a & a & 0 & b \\
b & b & b & b \\
\end{array}
\]

\[
\begin{array}{ccc}
. & 0 & a & b \\
0 & 0 & 0 & 0 \\
a & 0 & 0 & 0 \\
b & 0 & 0 & b \\
\end{array}
\]

Then R is semiring under the operations. Moreover \( I = \{ 0, b \} \) is an ideal of R. But \( a + b = b \in I \) and \( a \notin I \), so I is not subtractive ideal.

Lemma 1.1.17. [1] Let R be a semiring. If A is an ideal of R such that \( A = I \cup J \) where I, J are two subtractive ideals, then A = I or A = J.

The following example shows that the condition I, J are two subtractive ideals of R in Lemma 1.1.17 is essential.
Example 1.1.18. In the semiring \( R=\{0,a,b\} \), let \( I=\{0,a\} \) and \( J=\{0,b\} \) are the ideals of \( R \). Then \( R=I \cup J \) and \( I,J \) are not subtractive ideals by Example 1.1.16(3). But \( I \neq R \) and \( J \neq R \).

Lemma 1.1.19. [15] Let \( R \) be a semiring, \( I \) be a subtractive ideal of \( R \). Then \((I : r)=\{x \in R : rx \in I \} \) is a subtractive ideal of \( R \).

Definition 1.1.20. [26] Let \( R \) be a semiring and \( I \) be a proper ideal of \( R \). The ideal \( I \) is said to be a \textit{prime ideal} if whenever \( a,b \in R \) with \( ab \in I \), then either \( a \in I \) or \( b \in I \).

Example 1.1.21. Let \( P \) be a prime number. Then in the semiring \( Z^\circ \), the ideal \( PZ^\circ \) is prime.

Definition 1.1.22. [26] Let \( R \) be a semiring and \( I \) be a proper ideal. The ideal \( I \) is said to be a \textit{weakly prime ideal} if whenever \( a,b \in R \) with \( 0 \neq ab \in I \), then either \( a \in I \) or \( b \in I \).

Clearly, every prime ideal of a semiring \( R \) is a weakly prime ideal. But the converse is not true in general. Consider \( R= Z_4 \) be a commutative semiring, \( I=\{0\} \) be an ideal of \( R \). Then \( I \) is a weakly prime ideal of \( R \) and \( 2.2 \in I \) but \( 2 \notin I \) so \( I \) is not prime ideal.

Definition 1.1.23. [26] Let \( I \) be an ideal of a semiring \( R \), the \textit{radical} of \( I \), denoted by \( \sqrt{I} \), is the ideal \( \sqrt{I}=\bigcap P \), where the intersection runs over all prime ideals of \( R \) containing \( I \).

Equivalently, \( \sqrt{I}=\{r \in R : r^n \in I, \text{ for some positive integer } n \} \).

Lemma 1.1.24. [26] Let \( R \) be a commutative semiring, If \( I \) and \( J \) are ideals of \( R \), then \( \sqrt{IJ}=\sqrt{I} \cap \sqrt{J} \).

Definition 1.1.25. [26] Let \( R \) be a semiring and \( I \) be a proper ideal of \( R \). The ideal \( I \) is said to be a \textit{primary ideal} if whenever \( a,b \in R \) with \( ab \in I \), then either \( a \in I \) or \( b \in \sqrt{I} \).
Example 1.1.26. Let $R=\mathbb{Z}$, $P\mathbb{Z}$ is primary ideal where $p$ is prime number, but $6\mathbb{Z}$, $10\mathbb{Z}$, $15\mathbb{Z}$ are not primary ideals.

Definition 1.1.27. [26] Let $R$ be a semiring and $I$ be a proper ideal of $R$. The ideal $I$ is said to be a weakly primary ideal if whenever $a,b \in R$ with $0 \neq ab \in I$, then either $a \in I$ or $b \in \sqrt{I}$.

Remark 1.1.28. (1) Every prime ideal of a semiring $R$ is a primary ideal, but the converse is not true in general. For example, let $R=\mathbb{Z}$, the ideals $4\mathbb{Z}$, $8\mathbb{Z}$ are primary but not prime.

(2) Consider $R=\mathbb{Z}_6$. Then $0$ is a weakly primary ideal of $R$, but $I=\{0\}$ is not primary, since $2,3 \in I$, but neither $2 \in I$ nor $3 \in I$.

(3) Let $R=\mathbb{Z} \times \mathbb{Z}$. The ideal $I=4\mathbb{Z} \times 0$ is a weakly primary ideal of $R$, but not weakly prime because $(0,0)\neq(2,0)(2,0)$ in $I$ and $(2,0) \notin I$.

Proposition 1.1.29. If $I$ is a primary ideal in a semiring $R$. Then $\sqrt{I}$ is a prime ideal.

Proof. Since $I$ is a proper ideal of $R$, then $1 \notin I$ and hence $1 \notin \sqrt{I}$, so $\sqrt{I}$ is a proper ideal of $R$. Let $ab \in \sqrt{I}$ for some $a,b \in R$ such that $a \notin \sqrt{I}$. Since $ab \in \sqrt{I}$, then $(ab)^n \in I$ for some positive integer $n$, and hence $(a)^n(b)^n \in I$. Since $a \notin \sqrt{I}$, $(a)^n \notin I$. Since $I$ is primary, there is a positive integer $k$ such that $(b^a)^k \in I$ implies $b \in \sqrt{I}$. Therefore $\sqrt{I}$ is a prime ideal.

Definition 1.1.30. [15] An element $a$ of a semiring $R$ is called nilpotent if there exists some positive integer $n$ such that $a^n=0$. The set $\{a \in R : a^n = 0 \text{ for some positive integer } n\}$ denoted by $\text{Nil}(R)$.

An ideal $I$ of a semiring is said to be a nilpotent ideal, if there exists a natural number $k$ such that $I^k=0$, where $I^k = \{r_1r_2...r_k : r_i \in I\}$. It is meant that the product of any $k$ elements of $I$ is 0.
Definition 1.1.31. [4] A mapping $f$ from the semiring $R$ into the semiring $S$ will be called a homomorphism if $f(a + b) = f(a) + f(b)$ and $f(ab) = f(a)f(b)$ for each $a, b \in R$. A homomorphism $f: R \rightarrow S$ is epimorphism if it maps $R$ onto $S$.

Definition 1.1.32. [26] Let $R$ and $S$ be semirings and let $f: R \rightarrow S$ be a homomorphism then $\ker(f) = \{x : x \in R \text{ and } f(x) = 0\}$ is an ideal of $R$ called the kernel of $f$.

Definition 1.1.33. [25] A nonempty subset $S$ of a semiring $R$ is said to be multiplicatively closed if $ab \in S$ whenever $a, b \in S$.

Lemma 1.1.34. [25] Let $R$ be a semiring and let $S$ be a multiplicatively closed subset of $R$. Define a relation on the set $R \times S$ by $(r, s) \sim (t, y)$ if and only if there exist $u \in S$ such that $ury = uts$. Then $\sim$ is an equivalence relation on $R \times S$.

Definition 1.1.35. [25] For $(r, s) \in R \times S$ which defined in Lemma 1.1.34, denoted the equivalence classes of $\sim$ which contains $(r, s)$ by $r/s$. The set of all equivalence classes of $R \times S$ under $\sim$ denoted by $S^{-1}R$. The addition and multiplication are defined as follows: $r/s + t/y = (ry + ts)/sy$ and $(r/s)(t/y) = rt/sy$, where $r, t \in R$ and $s, y \in S$. The semiring $S^{-1}R$ is called quotient semiring $R$ by $S$.

Lemma 1.1.36. [26] The set $S^{-1}I = \{a/b : a \in I, b \in S\}$ is an ideal of $S^{-1}R$.

Its zero element is $0/1$ and its multiplicative identity element is $1/1$.

Definition 1.1.37. [15] Let $I$ be an ideal of a semiring $R$. $I$ is called a $Q$-ideal (partitioning ideal) if there exists a subset $Q$ of $R$ such that.

1. $R = \bigcup\{q + I : q \in Q\}$ and,

2. If $q_1, q_2 \in Q$, then $(q_1 + I) \cap (q_2 + I) \neq \emptyset$ if and only if $q_1 = q_2$. 

9
Lemma 1.1.38. [15] Let I be a Q-ideal of R and \( R/I_Q = \{q + I : q \in Q\} \). Then \( R/I_Q \) forms a semiring under the binary operations "\( \oplus \)" and "\( \odot \)" define as follows: \((q_1 + I) \oplus (q_2 + I) = q_3 + I\), where \( q_3 \in Q \) is a unique element such that \( q_1 + q_2 + I \subseteq q_3 + I \). \((q_1 + I) \odot (q_2 + I) = q_4 + I\), where \( q_4 \in Q \) is a unique element such that \( q_1q_2 + I \subseteq q_4 + I \). This semiring \( R/I_Q \) is called the quotient semiring of R by I and denoted via \( (R/I_Q, \oplus, \odot) \) or just \( R/I_Q \).

Lemma 1.1.39. [11] Let I be a Q-ideal of a semiring R and \( a, b \in R \). Let \( q, q_1, q_2, q_3 \in Q \) be unique elements such that \( a + I \subseteq q + I \), \( b + I \subseteq q_1 + I \), \( q + q_1 + I \subseteq q_2 + I \), \( qq_1 + I \subseteq q_3 + I \). Then \( a + b \in q_2 + I \) and \( ab \in q_3 + I \).

Lemma 1.1.40. [11] Let R be a semiring, I a Q-ideal of R and A is a subtractive ideal of R with \( I \subseteq A \). Then I is \( Q \cap A \)-ideal of A.

Lemma 1.1.41. [12] If I is a Q-ideal of a semiring R, then I is a subtractive ideal of R.

Theorem 1.1.42. [11] Let R be a semiring, I is a Q-ideal of R. Then P is a subtractive ideal of R with \( I \subseteq P \) if and only if \( P/I_{(Q \cap P)} \) is a subtractive ideal in \( R/I_Q \).

Definition 1.1.43. [15] An ideal I of a semiring R is said to be irreducible if for ideals H and K of R, \( I = H \cap K \) implies that I=H or I=K.

1.2 Semimodules and Subsemimodules

Definition 1.2.1. [27] Let R be a semiring with identity. A left R-semimodule M is a commutative monoid \((M,+)\) which has a zero element \( 0_M \), together with a function \( R \times M \rightarrow M \) (the image of \((r, m)\) being denoted by \( rm \)) such that for all \( r, s \in R \) and \( m, m_1, m_2 \in M \):

\[(i) \ r(m_1 + m_2) = rm_1 + rm_2.\]
(ii) \((r+s)m=rm+sm\).

(iii) \(r(sm)=(rs)m\).

(iv) \(0_R \cdot m=0_M=r \cdot 0_M\).

If in addition \(1_R \cdot m=m\) for all \(m \in M\) (1 is the identity element of \(R\)), then \(M\) is said to be a \emph{unitary} \(R\)-semimodule.

A right \(R\)-semimodule is defined similarly via a function \(M \times R \rightarrow M\) denoted \((m,r) \mapsto mr\) and satisfying the obvious analogues of (i)-(iv). Since we only deal with commutative semirings in this thesis, then every left \(R\)-semimodule \(M\) can be given the structure of a right \(R\)-semimodule by defining \(mr=rm\) for all \(r \in R, \ m \in M\) (commutativity is needed for (iii)). From now on, every semimodule \(M\) is assumed to be both a left and right \(R\)-semimodule with \(rm=mr\) for all \(r \in R, \ m \in M\).

**Definition 1.2.2.** [27] Let \(R\) be a semiring, \(M\) an \(R\)-semimodule and \(N\) a nonempty subset of \(M\). \(N\) is called a \emph{subsemimodule} of \(M\) if \(N\) is closed under addition and \(rm \in N\) for all \(r \in R\) and \(m \in M\).

**Remark 1.2.3.** (1) Any semiring \(R\) is a semimodule over itself.

(2) If \(I\) is an ideal of \(R\) and \(N\) a subsemimodule of \(M\), then
\[
IN = \{r_1m_1 + r_2m_2 + \ldots + r_nm_n : r_i \in I, \ m_i \in N, \ n \text{ a positive integer} \}
\]
is a subsemimodule of \(M\).

**Definition 1.2.4.** [18] Let \(R\) be a semiring, \(M\) an \(R\)-semimodule and \(m \in M\), the \emph{cyclic} subsemimodule generated by \(m\) is a subsemimodule of \(M\) has the form \(Rm=\{rm : r \in R\}\).

**Definition 1.2.5.** [18] An \(R\)-semimodule \(M\) is said to be \emph{finitely generated} if there is a finite subset \(\{x_1, x_2, \ldots, x_n\}\) of \(M\) such that \(M= Rx_1 + Rx_2 + \ldots + Rx_n\). In this case \(M\) is called the semimodule generated by \(x_1, x_2, \ldots, x_n\).
Definition 1.2.6. [27] An R-semimodule M is called a multiplication semimodule if every subsemimodule N of M is of the form IM, for some ideal I of R.

Definition 1.2.7. [18] A proper subsemimodule N of an R-semimodule M is called subtractive if a, a+b ∈ N, b ∈ M implies b ∈ N.

Definition 1.2.8. [18] Let M be an R-semimodule and N be a proper subsemimodule of M. An residual ideal of N is defined as (N : M)=\{a ∈ R : aM ⊆ N\}.

Note that if M is a multiplication R-semimodule and N a subsemimodule of M then there exist an ideal I of R such that N=IM. Thus I ⊆ (N : M). Then we have N = IM ⊆ M(N : M) ⊆ N and therefore N=M(N : M), (see [27]).

Lemma 1.2.9. [19] Let M be a semimodule and N be a proper subtractive subsemimodule of M and let m ∈ M. Then the following hold:
(i) (N : M) is a subtractive ideal of R.
(ii) (0 : M) and (N : m) are subtractive ideals of R, where (0 : M)=\{a ∈ R : aM ⊆ \{0\}\}.

Proof. (i) (N : M) is an ideal. Next let a ∈ (N : M), b ∈ R such that a+b ∈ (N : M), then am ∈ N and (a+b)m ∈ N for all m ∈ M, then am+bm=(a+b)m ∈ N which is subtractive. Hence bm ∈ N implies that b ∈ (N : M). Thus (N : M) is a subtractive ideal of R.
(ii) The proof of this is the same way by Part (i).

Definition 1.2.10. [23] M is called a faithful R-semimodule if (0 : M)=0.

Definition 1.2.11. [3] A subsemimodule N of an R-semimodule M is called a nilpotent subsemimodule if (N : M)^kN=0 for some positive integer k, and we say that m ∈ M is nilpotent if Rm is a nilpotent subsemimodule of M.

Note that, (N : M)^kN={r_1r_2...r_ks \ , \ r_i ∈ (N : M) \ and \ s ∈ N \}
Definition 1.2.12. [18] A proper subsemimodule $N$ of $M$ is called prime if $ax \in N$, $a \in R$, $x \in M$ then either $x \in N$ or $a \in (N : M)$.

Definition 1.2.13. [10] A proper subsemimodule $N$ of an $R$-semimodule $M$ is said to be weakly prime if $0 \neq ax \in N$, $a \in R$, $x \in M$ then either $x \in N$ or $a \in (N : M)$.

It is clear that every prime subsemimodule of an $R$-semimodule is weakly prime. The following example show that the converse is not true in general (see [10]).

Example 1.2.14. Let $R=(\mathbb{Z}^\circ,+,\cdot)$ and $M=(\mathbb{Z}_6^\circ,+_6)$ and the subsemimodule $N=\{0\}$ of $M$, then $(N : M)=<6>$. Clearly $N$ is a weakly prime subsemimodule of $M$, but $N$ is not prime because $2,3 \in N$, $2 \notin N$ and $3 \notin (N : M)$.

Proposition 1.2.15. [27] If $N$ is a prime subsemimodule of an $R$-semimodule $M$, then $(N : M)$ is prime ideal of $R$.

Proof. $(N : M)$ is a proper ideal, since $1 \notin (N : M)$. Let $ab \in (N : M)$ and $b \notin (N : M)$. Then $bM \subseteq N$, then there exist $m \in M$ with $bm \notin N$. But $a(bm)=(ab)m \in N$ and $N$ is prime, hence $aM \subseteq N$. Thus $a \in (N : M)$. \hfill \square

Remark 1.2.16. [27] If $N$ is a weakly prime subsemimodule of an $R$-semimodule $M$, then $(N : M)$ is not weakly prime ideal of $R$ in general. For example, let $M=(\mathbb{Z}_6^\circ,+_6)$ and $R=(\mathbb{Z}^\circ,+,\cdot)$. Then $\{0\}$ is a weakly prime subsemimodule of an $R$-semimodule $M$, but $(\{0\} : M)=6\mathbb{Z}^\circ$ is not a weakly prime ideal of $R$ because $0 \neq 2,3 \in 6\mathbb{Z}^\circ$, but $2,3 \notin 6\mathbb{Z}^\circ$.

Proposition 1.2.17. [10] Let $R$ be a commtative semiring with unity, $M$ a faithful cyclic $R$-semimodule with unity, and $N$ a weakly prime subsemimodule of $M$. Then $(N : M)$ is a weakly prime ideal of $R$.

Proof. Assume $M=Rx$, where $x \in M$, and let $0 \neq ab \in (N : M)$ with $a \notin (N : M)$. then there exists $r \in R$ such that $rax=a(rx) \notin N$, so $ax \notin N$. As $0 \neq abM \subseteq N$, it implies that $0 \neq abx$
\[ \in N \text{ (since if } abx=0, \text{ then } ab \in (0 : x) \subseteq (0 : M)=0, \text{ a contradiction), so } 0\neq abx=bax \in N \text{ implies } b \in (N : M), \text{ since } N \text{ is a weakly prime subsemimodule of } M. \text{ Therefore } (N : M) \text{ is weakly prime ideal of } R. \]

**Definition 1.2.18.** [19] A subsemimodule \( N \) of an \( R \)-semimodule \( M \) is said to be **primary** if \( rm \in N, \ r \in R \text{ and } m \in M \), then either \( r \in \sqrt{(N : M)} \) or \( m \in N \).

**Definition 1.2.19.** [19] A subsemimodule \( N \) of an \( R \)-semimodule \( M \) is said to be **weakly primary** if \( 0 \neq rm \in N, \ r \in R \text{ and } m \in M \), then either \( r \in \sqrt{(N : M)} \) or \( m \in N \).

The following diagram shows the relation between the previous subsemimodules.

\[
\text{Summary of Relation between Subsemimodules}
\]

\[
\begin{array}{c}
\text{Prime Subsemimodule} \\
\downarrow \\
\text{Weakly Prime Subsemimodule}
\end{array}
\rightarrow
\begin{array}{c}
\text{Primary Subsemimodule} \\
\downarrow \\
\text{Weakly Primary Subsemimodule}
\end{array}
\]

**Remark 1.2.20.** From the above diagram, the converse relations are not necessarily true.

**Example 1.2.21.** [10] Let \( R=(Z^\circ,+,\cdot) \). Then

(1) \( 4Z^\circ \) is a primary subsemimodule of an \( R \)-semimodule \( (Z^\circ,+) \), which is not a prime subsemimodule, since \( 2.2 \in 4Z^\circ \) but neither \( 2 \in 4Z^\circ \) nor \( 2 \in (4Z^\circ : Z^\circ)=4Z^\circ \).

(2) \( \{0\} \) is a weakly primary subsemimodule of an \( R \)-semimodule \( (Z_6^\circ,+_6) \), which is not a primary subsemimodule, because \( 2.3 \in \{0\} \) but \( 2 \notin \{0\} \) and \( 3 \notin (\{0\} : Z_6^\circ)=6Z^\circ \).

(3) \( N=\{0,4,8\} \) is a weakly primary subsemimodule of an \( R \)-semimodule \( M=(Z_{12}^\circ,+_{12}) \),
which is not a weakly prime subsemimodule since $2.2 \in N$ but $2 \notin N$ and $2 \notin (N : M) = 4Z^°$.

**Definition 1.2.22.** [19] A subsemimodule $N$ of an $R$-semimodule $M$ is called a *partitioning subsemimodule* (= $Q$-subsemimodule) if there exists a nonempty subset $Q$ of $M$ such that.

(i) $RQ \subseteq Q$, where $RQ = \{ rq : r \in R, q \in Q \}$;

(ii) $M = \bigcup \{ q + N : q \in Q \}$;

(iii) If $q_1, q_2 \in Q$, then $(q_1 + N) \cap (q_2 + N) \neq \phi$ if and only if $q_1 = q_2$.

**Lemma 1.2.23.** [19] Let $M$ be an $R$-semimodule, and let $N$ be a $Q$-subsemimodule of $M$. Define $M/N(Q) = \{ q + N : q \in Q \}$. Then $M/N(Q)$ forms an $R$-semimodule under the binary operations “$\oplus$” and “$\odot$” define as follows: $(q_1 + N) \oplus (q_2 + N) = q_3 + N$, where $q_3 \in Q$ is a unique element such that $q_1 + q_2 + N \subseteq q_3 + N$. $r \odot (q_1 + N) = q_4 + N$, where $q_4 \in Q$ is a unique element such that $rq_1 + N \subseteq q_4 + N$. This $R$-semimodule $M/N(Q)$ is called the quotient semimodule of $M$ by $N$.

**Lemma 1.2.24.** [9] Let $N$ be a $Q$-subsemimodule of an $R$-semimodule $M$. If $P$ is a subtractive subsemimodule of $M$ such that $N \subseteq P$, then $N$ is a $Q \cap P$-subsemimodule of $P$.

**Theorem 1.2.25.** [9] Let $N$ be a $Q$-subsemimodule of an $R$-semimodule $M$. If $P$ is a subtractive subsemimodule of $M$ with $N \subseteq P$, then $P/N(Q \cap P) = \{ q+N : q \in Q \cap P \}$ is a subtractive subsemimodule of $M/N(Q)$.

**Theorem 1.2.26.** [9] Let $N$ be a $Q$-subsemimodule of an $R$-semimodule $M$ and $L$ a subtractive subsemimodule of $M/N(Q)$. Then $L = P/N(Q \cap P)$ for some subtractive subsemimodule $P$ of $M$ with $N \subseteq P$.

**Lemma 1.2.27.** [10] Let $N$ be a $Q$-subsemimodule on $R$-semimodule $M$. If $r \in R$ and $m \in M$, then there exists a unique $q \in Q$ such that $rm \in r\odot(q+N)$. 

15
Definition 1.2.28. [19] A subsemimodule $N$ of an $R$-semimodule $M$ is said to be irreducible if $N = N_1 \cap N_2$, where $N_1$ and $N_2$ are subsemimodules of $M$, then either $N = N_1$ or $N = N_2$.

Definition 1.2.29. [19] A proper subsemimodule $N$ of an $R$-semimodule $M$ is said to be a strong subsemimodule if for each $x \in N$ there exists $y \in N$ such that $x+y=0$. 
Chapter 2

2-Absorbing Ideals in Semirings.


In [11] J.N.Chaudhri, introduced the concept of 2-absorbing ideals in semirings, which is a generalization of the concept of prime ideals in semirings.

In this chapter we recall the concepts of 2-absorbing ideals, weakly 2-absorbing ideals in semirings and we investigate the relation between some ideals and 2-absorbing, weakly 2-absorbing ideals in semirings.

2.1 Definition and Properties of 2-Absorbing Ideals in Semirings

Definition 2.1.1. [11] A proper ideal I of a semiring R is called a 2-absorbing ideal if whenever a,b,c ∈ R and abc ∈ I, then ab ∈ I or ac ∈ I or bc ∈ I.

From the definitions one can see that every prime ideal of a semiring R is a 2-absorbing ideal, but the converse implication is not necessarily true. Consider the semiring \((\mathbb{Z}^\circ,+\cdot)\),
and the ideal $I = \langle 6 \rangle$. Then $I$ is 2-absorbing ideal of $Z^\circ$ but it is not prime ideal of $Z^\circ$.

**Theorem 2.1.2.** [11] If $I$ is a 2-absorbing ideal of a commutative semiring $R$, then $\sqrt{I}$ is a 2-absorbing ideal of $R$.

**Proof.** Let $I$ be a 2-absorbing ideal. If $x \in \sqrt{I}$, then $x^n \in I$ for some $n \in N$. Since $I$ is a 2-absorbing ideal, $x^2 \in I$. Let $a,b,c \in R$ and $abc \in \sqrt{I}$. Then $(abc)^2 \in I$. Since $I$ is a 2-absorbing ideal, then $a^2b^2 = (ab)^2 \in I$ or $a^2c^2 = (ac)^2 \in I$ or $b^2c^2 = (bc)^2 \in I$. Thus $ab \in \sqrt{I}$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. Hence $\sqrt{I}$ is a 2-absorbing ideal of $R$. □

**Theorem 2.1.3.** [13] Let $R$ be a commutative semiring. Then the intersection of two prime ideals of $R$ is 2-absorbing.

**Proof.** Let $I_1$ and $I_2$ be two prime ideals, we have to show that $I_1 \cap I_2$ is 2-absorbing ideal of $R$. Suppose $a,b,c \in R$ with $abc \in I_1 \cap I_2$ thus $abc \in I_i$ for $i=1,2$. Now $abc \in I_i$ implies that $a \in I_i$ or $b \in I_i$ or $c \in I_i$ for $i=1,2$ since $I_i$ is prime. If $a \in I_1$ and $a \in I_2$ then $a \in I_1 \cap I_2$ and so $ab \in I_1 \cap I_2$. If $a \in I_1$ and $b \in I_2$, then $ab \in I_1 \cap I_2$. We do the same in other cases. □

**Remark 2.1.4.** (1) The intersection of two prime ideals is not necessarily prime. For example, for the commutative semiring $(Z^\circ, +, \cdot)$. Then $2Z^\circ$, $3Z^\circ$ are prime ideal of $R$ but $2Z^\circ \cap 3Z^\circ = 6Z^\circ$ is not prime ideal of $R$, since $2.3 \in 6Z^\circ$ but $2 \notin 6Z^\circ$ and $3 \notin 6Z^\circ$.

(2) The intersection of three nonzero prime ideals is not necessarily 2-absorbing. For example, $2Z^\circ$, $3Z^\circ$ and $5Z^\circ$ are prime ideal of $R$, but $2Z^\circ \cap 3Z^\circ \cap 5Z^\circ = 30Z^\circ$ which is not 2-absorbing ideal of $R$, since $2.3.5 \in 30Z^\circ$ but neither $2.3 \in 30Z^\circ$ nor $2.5 \in 30Z^\circ$ nor $3.5 \in 30Z^\circ$.

(3) The intersection of prime ideal and a 2-absorbing ideal is not necessarily 2-absorbing. For example, $3Z^\circ \cap 4Z^\circ = 12Z^\circ$ which is not 2-absorbing ideal since $2.2.3 \in 12Z^\circ$ but $2.2$
\( \notin 12\mathbb{Z}^\circ \) and \( 2.3 \notin 12\mathbb{Z}^\circ \).

(4) The intersection of two nonzero 2-absorbing ideals is not necessarily 2-absorbing. (see the previous example)

**Corollary 2.1.5.** Let \( R \) be a commutative semiring, and if \( I_1 \) and \( I_2 \) are prime ideals then \( \sqrt{I_1I_2} \) is a 2-absorbing ideal of \( R \).

**Proof.** Since \( I_1 \) and \( I_2 \) are distinct prime ideals. Then, \( I_1 \cap I_2 \) is a 2-absorbing ideal of \( R \) and, by Theorem 2.1.2. gives that \( \sqrt{I_1 \cap I_2} \) is a 2-absorbing ideal of \( R \). Also, by Lemma 1.1.24 we get \( \sqrt{I_1I_2} \) is a 2-absorbing ideal of \( R \). \( \square \)

Now, we show that the residual of a 2-absorbing ideal is a 2-absorbing ideal, as in [15].

**Theorem 2.1.6.** [15] Let \( I \) be a 2-absorbing ideal of \( R \). Then \( (I : r) \) is a 2-absorbing ideal of \( R \) for all \( r \in R/I \).

**Proof.** Let \( r \in R/I \) and \( a,b,c \in R \) such that \( abc \in (I : r) \). Then \( ra(bc) \in I \). Since \( I \) is a 2-absorbing ideal of \( R \), we have \( ra \in I \) or \( rbc \in I \) or \( abc \in I \).

If either \( ra \), \( rbc \in I \), we are done.

If \( abc \in I \), then \( ab \in I \) or \( ac \in I \) or \( bc \in I \),

which implies that \( rab \in I \) or \( rac \in I \) or \( rbc \in I \).

Hence \( (I : r) \) is a 2-absorbing ideal of \( R \). \( \square \)

**Remark 2.1.7.** (1) The converse of the above theorem is not be true in general. For example, for the semiring \( (\mathbb{Z}^\circ, +, \cdot) \), let the ideal \( I=\langle 12 \rangle \). Since \( 2.2.3 \in I \), but neither \( 2.2 \in I \) nor \( 2.3 \in I \) then \( I \) is not 2-absorbing, but \( (I : r)=\langle <12>:2 \rangle=\langle 6 \rangle \) is a 2-absorbing.

(2) Ignoring the condition \([r \in R/I]\) must cause that \( (I : r) \) is not a proper ideal of \( R \). Because if \( r \in I \) then \( (I : r)=R \).
Now, we find the condition that makes the residual of a 2-absorbing ideal is a prime ideal as in [15].

**Theorem 2.1.8.** [15] Let I be a 2-absorbing subtractive ideal of R with \( \sqrt{I} = J \) and \( J^2 \subseteq I \). If \( I \neq J \), then for all \( r \in J/I \), \( (I : r) \) is a prime ideal of R containing I with \( J \subseteq (I : r) \).

*Proof.* Let \( ab \in (I : r) \) for some \( a, b \in R \). Then \( rab \in I \). Since I is a 2-absorbing ideal of R, therefore \( ra \in I \) or \( rb \in I \) or \( ab \in I \).

If \( ra \in I \) or \( rb \in I \), then \( a \in (I : r) \) or \( b \in (I : r) \), this means \( (I : r) \) is a prime ideal of R.

If \( ab \in I \) and also, \( r^2 \in J^2 \subseteq I \). This gives \( rb \in (I : r) \) for particular \( r \in R \). We have \( rb + ab = (r+a)b \in (I : r) \), that is, \( r(r+a)b \in I \).

Therefore \( rb \in I \) or \( r(r+a)b \in I \) or \( (r+a)b \in I \), since I is a 2-absorbing ideal of R.

If \( rb \in I \) then \( b \in (I : r) \), which is as required.

If \( (r+a)b \in I \) and \( ab \in I \), then \( rb \in I \) (as I is a subtractive ideal). This gives \( b \in (I : r) \), so \( (I : r) \) is prime.

Finally, if \( r(r+a) \in I \) and since \( r^2 \in J^2 \subseteq I \). This give \( ra \in I \) implies \( a \in (I : r) \).

Hence \( (I : r) \) is a prime ideal of R. \( \square \)

**Remark 2.1.9.** The converse of the above theorem does not necessarily hold. For example, let \((Z^\circ,+,*\)) be a commutative semiring and \( I = \langle 12 \rangle \), \( J = \sqrt{I} = \langle 6 \rangle \) and \( J^2 \subseteq I \).

Then \( (I : r) = (\langle 12 \rangle : 6) = \langle 2 \rangle \) is a prime ideal, but \( \langle 12 \rangle \) is not a 2-absorbing ideal.

Since every prime ideal is 2-absorbing, we can conclude the following result.

**Corollary 2.1.10.** [15] Let I be a 2-absorbing subtractive ideal of R with \( \sqrt{I} = J \) and \( J^2 \subseteq I \). If \( I \neq J \), then for all \( r \in J/I \), \( (I : r) \) is a 2-absorbing ideal of R with \( J \subseteq (I : r) \).

In the next theorem we find the condition that makes the converse of theorem 2.1.8 be true.

20
Theorem 2.1.11. [15] If I is a subtractive ideal of R such that $I \neq \sqrt{I}$ and $\sqrt{I}$ is a prime ideal of R with $(\sqrt{I})^2 \subseteq I$. Then I is a 2-absorbing ideal of R if and only if $(I : r)$ is prime ideal of R for each $r \in \sqrt{I}/I$.

Proof. ($\Rightarrow$) One way is straightforward by Theorem 2.1.8.

($\Leftarrow$) Let $abc \in I$ for some $a,b,c \in R$. Then we assume that $a \in \sqrt{I}$ (as $I \subseteq \sqrt{I}$ and $\sqrt{I}$ is prime ideal of R).

If $a \in I$, then $ab \in I$, which gives I is a 2-absorbing ideal of R.

If $a \in \sqrt{I}/I$. Also, $abc \in I$ implies that $bc \in (I : a)$, and by assumption $(I : a)$ is a prime ideal of R. Therefore we have either $b \in (I : a)$ or $c \in (I : a)$, so $ab \in I$ or $ac \in I$. Thus I is a 2-absorbing ideal of R.

Theorem 2.1.12. Let $f : R \rightarrow S$ be a epimorphism of commutative semirings, and I be a 2-absorbing ideal of S, then $f^{-1}(I)$ is a 2-absorbing ideal of R.

Proof. Suppose $a,b,c \in R$ with $abc \in f^{-1}(I)$. Then $f(abc) = f(a)f(b)f(c) \in I$. Since I is a 2-absorbing ideal of S, we have $f(a)f(b) \in I$ or $f(a)f(c) \in I$ or $f(b)f(c) \in I$. Hence $ab \in f^{-1}(I)$ or $ac \in f^{-1}(I)$ or $bc \in f^{-1}(I)$. Therefore $f^{-1}(I)$ is a 2-absorbing ideal of R.

Definition 2.1.13. [5] A proper ideal I of R is called strong ideal, if for each $a \in I$ there exists $b \in I$ such that $a+b=0$.

Proposition 2.1.14. [15] Let R and S be commutative semirings, $f : R \rightarrow S$ be an epimorphism such that $f(0) = 0$ and I be a subtractive strong ideal of R, and I is a 2-absorbing ideal of R such that $\ker(f) \subseteq I$, then $f(I)$ is a 2-absorbing ideal of S.

Proof. Let $a,b,c \in S$ such that $abc \in f(I)$, then there exists $n \in I$ such that $abc = f(n)$. Since f is epimorphism, therefore there exist $p,q,r \in R$ such that $f(p) = a$, $f(q) = b$, $f(r) = c$. Also, since I is a strong ideal of R and $n \in I$, then there exists $m \in I$ such that
n+m=0. This implies \( f(n+m) = 0 \), that is, \( f(pqr + m) = 0 \), implies \( pqr+m \in \ker(f) \subseteq I \). So \( pqr \in I \) (as \( I \) is subtractive). Since \( I \) is a 2-absorbing ideal of \( R \), therefore either \( pq \in I \) or \( pr \in I \) or \( qr \in I \). Thus \( ab \in f(I) \) or \( ac \in f(I) \) or \( bc \in f(I) \). Therefore \( f(I) \) is a 2-absorbing ideal of \( S \).

\[ \text{Remark 2.1.15.} \]

(1) In Theorem 2.1.12, if \( f \) is not onto may causes that \( f^{-1}(I) \) is not a proper ideal of \( R \). Consider the homomorphism \( f : \mathbb{Z}_4 \to \mathbb{Z}_4 \) defined by \( f(0) = f(2) = 0 \), \( f(1) = f(3) = 2 \) and let \( I = \{0, 2\} \), then \( f^{-1}(I) = \mathbb{Z}_4 \).

(2) In Theorem 2.1.14, if we ignore the condition \([\ker(f) \subseteq I]\) may cause that \( f(I) \) is not a proper ideal of \( S \). For example, consider the epimorphism \( f : \mathbb{Z}_{12} \to \mathbb{Z}_4 \) defined by \( f(x) = x \pmod{4} \), \( \ker(f) = \{0, 4, 8\} \) let \( I = \{0, 3, 6, 9\} \), then \( f(I) = \mathbb{Z}_4 \).

**Theorem 2.1.16.** [13] Let \( R \) be a commutative semiring, \( I \) is a \( Q \)-ideal of \( R \) and \( P \) a subtractive ideal of \( R \) with \( I \subseteq P \). Then \( P \) is a 2-absorbing ideal of \( R \) if and only if \( P/I(Q\cap P) \) is a 2-absorbing ideal of \( R/I(Q) \).

**Proof.** Let \( P \) be a 2-absorbing ideal of \( R \). Suppose that \( q_1 + I, q_2 + I, q_3 + I \in R/I(Q) \) such that \((q_1 + I) \odot (q_2 + I) \odot (q_3 + I) \in P/I(Q\cap P)\) where \( q_1, q_2, q_3 \in Q \). Then there is a unique \( q_4 \in Q \cap P \) such that \( q_1q_2q_3 + I \subseteq q_4 + I \in P/I(Q\cap P) \). So \( q_1q_2q_3 = q_4 + i \) for some \( i \in I \), then \( q_1q_2q_3 \in P \), since \( P \) is a subtractive ideal of \( R \).

Since \( P \) is 2-absorbing, then either \( q_1q_2 \in P \) or \( q_1q_3 \in P \) or \( q_2q_3 \in P \).

Assume that \( q_1q_2 \in P \). If \((q_1 + I) \odot (q_2 + I) = (q_5 + I)\), then \( q_5 \in Q \) is a unique element with \( q_1q_2 + I \subseteq q_5 + I \). Then there exist \( e, f \in I \) such that \( q_1q_2 + e = q_5 + f \in P \). So \( q_5 \in P \), since \( P \) is a subtractive ideal of \( R \), then \( q_5 \in Q \cap P \). Hence \((q_1 + I) \odot (q_2 + I) = (q_5 + I) \in P/I(Q\cap P) \). Similarly, \((q_1 + I) \odot (q_3 + I) \in P/I(Q\cap P) \) or \((q_2 + I) \odot (q_3 + I) \in P/I(Q\cap P) \). Thus \( P/I(Q\cap P) \) is a 2-absorbing ideal of \( R/I(Q) \).

Conversely, suppose that \( P/I(Q\cap P) \) is 2-absorbing ideal of \( R/I(Q) \). Let \( a, b, c \in R \) be such
that \( abc \in P \). Since \( I \) is a \( Q \)-ideal of \( R \), then there are elements \( q_1, q_2, q_3 \in Q \) such that \( a \in q_1 + I \), \( b \in q_2 + I \) and \( c \in q_3 + I \), so there exist \( c, d, e \in I \) such that \( a = q_1 + c \), \( b = q_2 + d \) and \( c = q_3 + e \).

Since \( abc = q_1q_2q_3 + q_1q_2c + q_1q_3d + q_1de + cq_2q_3 + ceq_2 + cdq_3 + cde \in P \), and \( P \) is a subtractive ideal of \( R \), we must have \( q_1q_2q_3 \in P \). Let \( q \) be the unique element in \( Q \) such that \((q_1 + I) \odot (q_2 + I) \odot (q_3 + I) = q + I\) where \( q_1q_2q_3 + I \subseteq q + I \), so \( q+f=q_1q_2q_3+g \) for some \( f, g \in I \). Then we get \( q \in P \) (as \( P \) is a subtractive ideal), we have \( q \in Q \cap P \) and \( q + I \in P/I(Q \cap P) \); hence \( P/I(Q \cap P) \) 2-absorbing gives either \((q_1 + I) \odot (q_2 + I) \in P/I(Q \cap P) \) or \((q_1 + I) \odot (q_3 + I) \in P/I(Q \cap P) \) or \((q_2 + I) \odot (q_3 + I) \in P/I(Q \cap P) \). It follows that either \( q_1q_2 \in P \) (so \( ab \in P \)) or \( q_1q_3 \in P \) (so \( ac \in P \)) or \( q_2q_3 \in P \) (so \( bc \in P \)). Thus \( P \) is a 2-absorbing ideal of \( R \).

**Theorem 2.1.17.** [16] Let \( R \) be a commutative semiring and \( S \) be a multiplicatively closed subset of \( R \). If \( I \) is a 2-absorbing ideal of \( R \) with \( I \cap S = \emptyset \). Then \( S^{-1}I \) is a 2-absorbing ideal of \( S^{-1}R \).

**Proof.** Let \( a, b, c \in R \) and \( s, t, r \in S \) with \((a/s)(b/t)(c/r) \in S^{-1}I\). Then there exist \( u \in S \) such that \((ua)bc \in I\). Then we have either \((ua)b \in I\) or \((ua)c \in I\) or \(bc \in I\), since \( I \) is a 2-absorbing ideal of \( R \).

If \((ua)b \in I\) then \((a/s)(b/t) = \frac{uab}{ust} \in S^{-1}I\). Similar if \((ua)c \in I\) then \( \frac{uac}{usr} \in S^{-1}I \).

If \(bc \in I\) then \((b/t)(c/r) = \frac{bc}{tr} \in S^{-1}I\). Thus \( S^{-1}I \) is a 2-absorbing ideal of \( S^{-1}R \). \( \square \)

**Lemma 2.1.18.** [16] Let \( I \) and \( J \) be ideals of \( R \). If \( I \) is a 2-absorbing subtractive ideal of \( R \) and \( abJ \subseteq I \) for some \( a, b \in R \), then \( ab \in I \), \( aJ \subseteq I \) or \( bJ \subseteq I \).

**Proof.** Assume on contrary that \( ab \notin I \), \( aJ \notin I \) and \( bJ \notin I \). So there exist \( j_1, j_2 \in J \) such that \( aj_1 \notin I \) and \( bj_2 \notin I \). Since \( abj_1, abj_2 \in I \), and \( I \) is a 2-absorbing ideal of \( R \). So we get \( bj_1, aj_2 \in I \).
Now since \( ab(j_1+j_2) \in I, ab \notin I \) and \( I \) is a 2-absorbing ideal of \( R \), we have \( a(j_1+j_2) \in I \) or \( b(j_1+j_2) \in I \). If \( a(j_1+j_2) \in I \), since \( I \) is a subtractive ideal and \( a j_2 \in I \), we have \( a j_1 \in I \), which is a contradiction.

Similarly, if \( b(j_1+j_2) \in I \) we get \( b j_2 \in I \), which is a contradiction. Therefore \( ab \in I \) or \( aJ \subseteq I \) or \( bJ \subseteq I \).

\[\text{Theorem 2.1.19.} \quad [16] \quad \text{Let } R \text{ be a commutative semiring and } I \text{ be a nonzero proper subtractive ideal of } R. \text{ Then the following statements are equivalent:}\]

\[(i) \quad I \text{ is a 2-absorbing ideal of } R;\]
\[(ii) \quad \text{If } I_1 I_2 I_3 \subseteq I \text{ for some ideals } I_1, I_2, I_3 \text{ of } R, \text{ then } I_1 I_2 \subseteq I \text{ or } I_1 I_3 \subseteq I \text{ or } I_2 I_3 \subseteq I.\]

\[\text{Proof.} \quad (i) \implies (ii) \quad \text{Assume that } I \text{ is a 2-absorbing ideal of } R \text{ and } I_1 I_2 I_3 \subseteq I \text{ for some ideals } I_1, I_2 \text{ and } I_3 \text{ of } R. \text{ Let } I_1 I_2 \notin I, I_1 I_3 \notin I \text{ and } I_2 I_3 \notin I. \text{ Then there exist } i_1 \in I_1 \text{ and } i_2 \in I_2 \text{ such that } i_1 i_2 I_3 \subseteq I \text{ with } i_1 I_3 \notin I \text{ and } i_2 I_3 \notin I. \]

Hence \( i_1 i_2 \in I \), by Lemma 2.1.18. Since \( I_1 I_2 \notin I \), there exist \( a \in I_1 \) and \( b \in I_2 \) such that \( ab \notin I \). By Lemma 2.1.18 and since \( ab I_3 \subseteq I \) and \( I \) is a 2-absorbing ideal of \( R \), we have \( a I_3 \subseteq I \) or \( b I_3 \subseteq I \).

Now we have three cases:

Case I: Assume that \( a I_3 \subseteq I \) but \( b I_3 \notin I \). Since \( i_1 b I_3 \subseteq I \) but \( b I_3 \notin I \) and \( i_1 I_3 \notin I \), we have \( i_1 b \in I \), by Lemma 2.1.18. Now, we have \( a I_3 \subseteq I \) but \( i_1 I_3 \notin I \), then \( (a+i_1) I_3 \notin I \). Since \( I \) is a subtractive ideal. On the other hand, \( (a+i_1) b I_3 \subseteq I \), \( b I_3 \notin I \) and \( (a+i_1) I_3 \notin I \), we get \( (a+i_1) b \in I \), by Lemma 2.1.18. Then \( ab \in I \) (as \( I \) is a subtractive ideal), which is a contradiction.

Case II: Suppose that \( a I_3 \notin I \) and \( b I_3 \subseteq I \). The proof of this case is the same way by Case I.

Case III: We assume that \( a I_3 \subseteq I \) and \( b I_3 \subseteq I \). At the first consider \( b I_3 \subseteq I \). Since \( i_2 I_3 \subseteq I \)
I and I is a subtractive ideal, we conclude that \((b+i_2)I_3 \not\subseteq I\) since \(i_1(b+i_2)I_3 \subseteq I\) but \(i_1I_3 \not\subseteq I\), and \((b+i_2)I_3 \not\subseteq I\), we have \(i_1(b+i_2) \in I\). And \(i_1i_2 \in I\), then \(i_1b \in I\), since I is a subtractive ideal.

Now we consider \(aI_3 \subseteq I\) but \(i_1I_3 \not\subseteq I\) so \((a+i_1)I_3 \not\subseteq I\). As \((a+i_1)i_2I_3 \subseteq I\) but \((a+i_1)I_3 \not\subseteq I\) and \(i_2I_3 \not\subseteq I\), we conclude that \((a+i_1)i_2 \in I\). Since \(i_1i_2 \in I\) and I is subtractive ideal, \(ai_2 \in I\).

Now as \((a+i_1)(b+i_2)I_3 \subseteq I\) but \((a+i_1)I_3 \not\subseteq I\) and \((b+i_2)I_3 \not\subseteq I\), we conclude that \((a+i_1)(b+i_2)=ab + ai_2 + bi_1 + i_1i_2 \in I\) and so \(ab \in I\) (as I is subtractive ideal), which is a contradiction.

Therefore \(I_1I_2 \subseteq I\) or \(I_1I_3 \subseteq I\) or \(I_2I_3 \subseteq I\).

(ii) \(\Rightarrow\) (i) The proof is straight forward.

\[\square\]

### 2.2 Weakly 2-Absorbing Ideals in Semirings

The concept of weakly 2-absorbing ideal was introduced in [13] by Darani as a generalization of weakly prime ideal in semiring.

**Definition 2.2.1.** [13] A proper ideal I of a commutative semiring \(R\) is said to be a **weakly 2-absorbing ideal** of \(R\) if whenever \(a,b,c \in R\) with \(0 \neq abc \in I\), then \(ab, ac \in I\) or \(bc \in I\).

**Lemma 2.2.2.** [13] Let \(R\) be a commutative semiring, and \(I\) a proper ideal of \(R\). Then

(i) \(I\) is prime ideal \(\Rightarrow\) \(I\) is 2-absorbing ideal \(\Rightarrow\) \(I\) is weakly 2-absorbing ideal;

(ii) \(I\) is weakly prime ideal \(\Rightarrow\) \(I\) is weakly 2-absorbing ideal.

**Proof.** Direct from definitions.

\[\square\]

**Remark 2.2.3.** [13] Let \(R\) be a commutative semiring, \(I\) an ideal of \(R\). By above lemma, every 2-absorbing ideal is weakly 2-absorbing ideal. But the converse does not necessarily
hold. For example, let \( R = \mathbb{Z}_8 \) and \( I = \{0\} \) be an ideal of \( R \). Then \( I \) is a weakly 2-absorbing ideal of \( R \), but it is not 2-absorbing, since 2.2.2 \( \in \) I but 2.2 \( \notin \) I.

**Theorem 2.2.4.** [16] Let \( R \) be a commutative semiring. Then the intersection of two weakly prime ideals of \( R \) is a weakly 2-absorbing ideal of \( R \).

*Proof.* The proof is similar to the proof of Theorem 2.1.3.

**Theorem 2.2.5.** Let \( f : R \rightarrow S \) be an epimorphism of commutative semirings, and \( I \) be a weakly 2-absorbing ideal of \( S \) with \( f(0) = 0 \) and \( \ker(f) = \{0\} \), then \( f^{-1}(I) \) is a weakly 2-absorbing ideal of \( R \).

*Proof.* Let \( a,b,c \in R \) such that \( 0 \neq abc \in f^{-1}(I) \). Then \( 0 \neq f(abc) = f(a)f(b)f(c) \in I \) (because if \( f(abc)=0 \), then \( abc=0 \), since \( \ker(f) = \{0\} \), a contradiction). Since \( I \) is a weakly 2-absorbing ideal of \( S \), we have \( f(a)f(b) \in I \) or \( f(a)f(c) \in I \) or \( f(b)f(c) \in I \). Hence \( ab \in f^{-1}(I) \) or \( ac \in f^{-1}(I) \) or \( bc \in f^{-1}(I) \). Therefore \( f^{-1}(I) \) is a weakly 2-absorbing ideal of \( R \).

**Theorem 2.2.6.** [3] Let \( R \) and \( S \) be commutative semirings, \( f : R \rightarrow S \) be an epimorphism such that \( f(0) = 0 \) and \( I \) be a subtractive strong ideal of \( R \). If \( I \) is a weakly 2-absorbing ideal of \( R \) such that \( \ker(f) \subseteq I \), then \( f(I) \) is a weakly 2-absorbing ideal of \( S \).

*Proof.* Suppose \( a,b,c \in S \) such that \( 0 \neq abc \in f(I) \), then there exists \( n \in I \) such that \( 0 \neq abc = f(n) \). Since \( f \) is epimorphism, therefore there exist \( p,q,r \in R \) such that \( f(p) = a \), \( f(q) = b \), \( f(r) = c \). Also, since \( I \) is a strong ideal of \( R \) and \( n \in I \), then there exists \( m \in I \) such that \( n+m=0 \). This implies \( f(n+m)=0 \), that is, \( f(pqr+m) = 0 \), implies \( pqr+m \in \ker(f) \subseteq I \). Since \( I \) is subtractive, so \( 0 \neq pqr \in I \) (because if \( 0 = pqr \), then \( f(n)=0 \) a contradiction). Since \( I \) is a weakly 2-absorbing ideal of \( R \), there fore either \( pq \in I \) or \( pr \in I \) or \( qr \in I \). Thus \( ab \in f(I) \) or \( ac \in f(I) \) or \( bc \in f(I) \). Therefore \( f(I) \) is a weakly 2-absorbing ideal of \( S \).
Theorem 2.2.7. [15] Let \( R \) be a commutative semiring, \( I \) is \( Q \)-ideal of \( R \) and \( P \) a subtractive ideal of \( R \) with \( I \subseteq P \). Then the following statements are hold:

(i) If \( P \) is a weakly 2-absorbing ideal of \( R \), then \( P/I(Q\cap P) \) is a weakly 2-absorbing ideal of \( R/I(Q) \)

(ii) If \( I \) and \( P/I(Q\cap P) \) are both weakly 2-absorbing ideals of \( R \) and \( R/I(Q) \), respectively, then \( P \) is weakly 2-absorbing ideal of \( R \).

Proof. (i) Note that if \((q_1+I) \odot (q_2+I) \odot (q_3+I) \neq 0 \) in \( P/I(Q\cap P) \), then \( q_1q_2q_3 \neq 0 \) in \( R \). Now the proof is completely similar of theorem 2.1.16.

(ii) Suppose that \( I \) and \( P/I(Q\cap P) \) are weakly 2-absorbing ideals of \( R \) and \( R/I(Q) \), respectively. Let \( 0 \neq abc \in P \) for some \( a,b,c \in R \), we have two cases:

Case I: If \( abc \in I \), then \( ab \in I \subseteq P \) or \( ac \in I \subseteq P \) or \( bc \in I \subseteq P \), since \( I \) is a weakly 2-absorbing ideal of \( R \). Then \( P \) is a weakly 2-absorbing ideal of \( R \).

Case II. If \( abc \notin I \). Since \( I \) is a \( Q \)-ideal of \( R \), then there are elements \( q_1,q_2,q_3 \in Q \) such that \( a \in q_1+I, b \in q_2+I \) and \( c \in q_3+I \), so for some \( d,e,f \in I \) we have \( a=q_1+d, b=q_2+e \) and \( c=q_3+f \). Since \( abc = q_1q_2q_3 + q_1q_2f + q_1q_3e + q_1ef + dq_2q_3 + dfq_2 + deq_3 + def \in P \), and \( P \) is a subtractive ideal of \( R \), we must have \( q_1q_2q_3 \in P \). Let \( q \) be the unique element in \( Q \) such that \((q_1+I) \odot (q_2+I) \odot (q_3+I) = q + I \) where \( q_1q_2q_3 + I \subseteq q + I \), so \( q+g=q_1q_2q_3+h \) for some \( g,h \in I \). Since \( P \) is a subtractive ideal of \( R \), we get \( q \in P \) and so \( q \in Q \cap A \). Thus \( q + I \in P/I(Q\cap P) \).

Let \( q_0 \in Q \) be the unique element such that \( q_0 + I \) is the zero element in \( R/I(Q) \). If \((q_1+I) \odot (q_2+I) \odot (q_3+I) = 0_{R/I(Q)} = q_0 + I \), then \( q_1q_2q_3 + s = q_0 + t \in I \) for some \( s,t \in I \).

Since \( I \) is a \( Q \)-ideal of \( R \) it is a subtractive ideal of \( R \) by Lemma 1.1.41. Therefore \( q_1q_2q_3 \in I \) and hence \( abc \in I \), which is a contradiction.

Hence \( 0 \neq (q_1+I) \odot (q_2+I) \odot (q_3+I) \in P/I(Q\cap P) \) and \( P/I(Q\cap P) \) weakly 2-absorbing imply that \((q_1+I) \odot (q_2+I) \in P/I(Q\cap P) \) or \((q_1+I) \odot (q_3+I) \in P/I(Q\cap P) \) or \((q_2+I) \odot (q_3+I) \in P/I(Q\cap P) \).
It follow either \( q_1q_2 \in P \) (so \( ab \in P \)) or \( q_1q_3 \in P \) (so \( ac \in P \)) or \( q_2q_3 \in P \) (so \( bc \in P \)).

Thus \( P \) is a weakly 2-absorbing ideal of \( R \). \( \square \)

**Definition 2.2.8.** [16] Let \( I \) be a weakly 2-absorbing ideal of a commutative semiring \( R \) and \( a,b,c \in R \) we say \((a,b,c)\) is a *triple-zero* of \( I \) if \( abc=0 \), implies \( ab \notin I \), \( ac \notin I \) and \( bc \notin I \).

**Theorem 2.2.9.** [16] Let \( I \) be a weakly 2-absorbing subtractive ideal of a commutative semiring \( R \) and suppose that \((a,b,c)\) is a triple-zero of \( I \) for some \( a,b,c \in R \). Then

(i) \( abI = acI = bcI = \{0\} \);

(ii) \( aI^2 = bI^2 = cI^2 = \{0\} \).

**Proof.** (i) Assume that \( abI \neq 0 \), then \( abi \neq 0 \) for some \( i \in I \). Then \( ab(c+i) \neq 0 \). Since \( ab \notin I \) and \( ab(c+i) \in I \), we get either \( a(c+i) \in I \) or \( b(c+i) \in I \) because \( I \) is weakly 2-absorbing, and hence \( ac \in I \) or \( bc \in I \) (as \( I \) is subtractive), a contradiction. Thus \( abI = \{0\} \).

Similarly \( acI = bcI = \{0\} \).

(ii) Suppose that \( aI^2 \neq 0 \), then there exist \( i_1, i_2 \in I \) such that \( ai_1i_2 \neq 0 \). Since \( abI = acI = bcI = \{0\} \) by (i) we get \( a(b+i_1)(c+i_2) = ai_1i_2 \neq 0 \). Since \( I \) is a weakly 2-absorbing ideal, then either \( a(b+i_1) \in I \) or \( a(c+i_2) \in I \) or \( b+i_1)(c+i_2) \in I \). We get \( ab \in I \) or \( ac \in I \) or \( bc \in I \), since \( I \) is subtractive ideal, this is a contradiction. Thus \( aI^2 = \{0\} \). Similarly, \( bI^2 = cI^2 = \{0\} \). \( \square \)

The following theorem provides a condition under which the concepts of 2-absorbing ideals and weakly 2-absorbing ideals of \( R \) are identical.

**Theorem 2.2.10.** [13] Let \( R \) be a commutative semiring. If \( I \) is a weakly 2-absorbing subtractive ideal, then either \( I^3=0 \) or \( I \) is a 2-absorbing.
Proof. Assume I is not a 2-absorbing ideal of R. Since I is weakly 2-absorbing ideal of R, then I has a triple-zero (a,b,c) for some a,b,c ∈ R. We claim that \( I^3 = 0 \), let \( i_1i_2i_3 \neq 0 \) for some \( i_1,i_2,i_3 \in I \). Then by Theorem 2.2.9, we have \((a + i_1)(b + i_2)(c + i_3) = i_1i_2i_3 \neq 0\). Since I is a weakly 2-absorbing ideal, we get \((a + i_1)(b + i_2) \in I \) or \((a + i_1)(c + i_3) \in I \) or \((b + i_2)(c + i_3) \in I \). Hence ab ∈ I or ac ∈ I or bc ∈ I (as I is a subtractive ideal), a contradiction. Thus \( I^3 = 0 \).

The following example shows that a converse of the previous theorem does not necessarily hold.

Example 2.2.11. Let \( R = \mathbb{Z}_{16} \) be a commutative semiring. Then \( I = \{0, 8\} \) is an ideal of \( \mathbb{Z}_{16} \), and \( I^3 = 0 \), but \( 0 \neq 2.2.2 = 8 \in I \) and \( 4 \notin I \), thus I is not weakly 2-absorbing ideal of R.

Corollary 2.2.12. [13] Let I be a weakly 2-absorbing subtractive ideal of R. If I is not 2-absorbing ideal of R, then \( I \subseteq \sqrt{0} \) and \( \sqrt{I} = \sqrt{0} \).

Proof. By theorem 2.2.10, \( I^3 = 0 \). So we get \( I \subseteq \sqrt{0} \), then \( \sqrt{I} \subseteq \sqrt{0} \), and clearly \( \sqrt{0} \subseteq \sqrt{I} \). Hence \( \sqrt{I} = \sqrt{0} \).

Theorem 2.2.13. [15] Let I be a weakly 2-absorbing subtractive ideal of R but not a 2-absorbing ideal of R. Then:

(i) If \( r \in \text{Nil}(R) \), then either \( r^2 \in I \) or \( r^2I = rI^2 = \{0\} \);

(ii) \( \text{Nil}(R)^2I^2 = \{0\} \)

Proof. (i) Let \( r \in \text{Nil}(R) \). First, we show that if \( r^2I \neq \{0\} \), then \( r^2 \in I \). Suppose that \( r^2I \neq \{0\} \). Let n be the least positive integer such that \( r^n = 0 \). Then \( n \geq 3 \) and for some \( i \in I \), we have \( 0 \neq r^2i = r^2(r^{n-2} + i) \in I \). Since I is a weakly 2-absorbing ideal of R, we have either \( r^2 \in I \) or \( r^{n-1} + ri \in I \). If \( r^2 \in I \), then we are done. Let \( r^2 \notin I \), then \( r^{n-1} + ri \in I \), which gives \( r^{n-1} \in I \), since I is subtractive ideal, \( r^{n-1} \neq 0 \), and thus \( r^2 \in I \), because I is a
weakly 2-absorbing ideal of \( R \). Hence for each \( r \in \text{Nil}(R) \) we have either \( r^2 \in I \) or \( r^2I = \{0\} \).

Now assume that \( s^2 \notin I \) for some \( s \in \text{Nil}(R) \). Then \( s^2I = \{0\} \), by previous argument. We will show that \( si_1i_2 = \{0\} \). Suppose that \( si_1i_2 \notin I \) for some \( i_1, i_2 \in I \). Let \( m \) be the least positive integer such that \( s^m = 0 \). Since \( s^2 \notin I \), \( m \geq 3 \) and \( s^2I = \{0\} \). Therefore \( s(s + i_1)(s^{m-2} + i_2) = si_1i_2 \neq 0 \). Since \( 0 \neq s(s + i_1)(s^{m-2} + i_2) \in I \) and \( I \) is a weakly 2-absorbing ideal of \( R \), we get either \( s^2 \in I \) or \( 0 \neq s^{m-1} \in I \) (as \( I \) is a subtractive ideal).

Therefore, we have \( s^2 \in I \), a contradiction. Hence \( s^2I = \{0\} \).

(ii) Let \( a, b \in \text{Nil}(R) \). We have two cases:

Case I: \( a^2 \notin I \) or \( b^2 \notin I \), then by part (i), we have \( abI^2 = \{0\} \) and hence the result.

Case II: \( a^2 \in I \) and \( b^2 \in I \), then \( ab(a + b) \in I \).

If \( 0 \neq ab(a + b) \in I \) and since \( I \) is a subtractive weakly 2-absorbing ideal of \( R \), we have \( ab \in I \). So by Theorem 2.2.10, we have \( abI^2 = \{0\} \).

If \( 0 = ab(a + b) \in I \) and \( 0 \neq abi \in I \) for some \( i \in I \), then \( 0 \neq ab(a + b + i) \in I \) implies either \( a(a + b + i) \in I \) or \( b(a + b + i) \in I \) or \( ab \in I \). In each case, we have \( ab \in I \), which is a contradiction, as \( I \) is a weakly 2-absorbing and not a 2-absorbing ideal of \( R \). Thus, we have \( abI = \{0\} \), and hence \( abI^2 = \{0\} \).

Therefore \( \text{Nil}(R)I^2 = \{0\} \).

\[ \Box \]

**Corollary 2.2.14.** Let \( I, K, L \) be weakly 2-absorbing subtractive ideals of a commutative semiring \( R \) such that none of them is a 2-absorbing ideal of \( R \). Then \( I^2KL = IK^2L = IKL^2 = I^2K^2 = I^2L^2 = K^2L^2 = \{0\} \).

**Proof.** By Theorem 2.2.13, we get \( I, K, L \subseteq \text{Nil}(R) \). Since \( I \) is a weakly 2-absorbing subtractive ideal and not a 2-absorbing ideal of \( R \), then by Theorem 2.2.13, we get \( I^2K^2 = \{0\} \), \( I^2L^2 = \{0\} \) and \( I^2KL = \{0\} \). By the same argument we get \( IK^2L = IKL^2 = K^2L^2 = \{0\} \).

Consider \( R = R_1 \times R_2 \) where each \( R_i \) is a commutative semiring with unity, \( i = 1, 2 \) with
\((a_1,a_2)(b_1,b_2)=(a_1b_1,a_2b_2)\) for all \(a_1,b_1 \in R_1\) and \(a_2,b_2 \in R_2\) (see [15]).

**Proposition 2.2.15.** [15] If \(I\) is a proper ideal of a semiring \(R_1\). Then the following statements are equivalent:

(i) \(I\) is a 2-absorbing ideal of \(R_1\);
(ii) \(I \times R_2\) is a 2-absorbing ideal of \(R=R_1 \times R_2\);
(iii) \(I \times R_2\) is a weakly 2-absorbing ideal of \(R=R_1 \times R_2\).

**Proof.** (i) \(\implies\) (ii) Let \((a_1,a_2),(b_1,b_2),(c_1,c_2) \in R\) be such that \((a_1,a_2)(b_1,b_2)(c_1,c_2) \in I \times R_2\). Then \((a_1b_1c_1,a_2b_2c_2) \in I \times R_2\). Therefore \(a_1b_1c_1 \in I\). Since \(I\) is a 2-absorbing ideal of \(R_1\), we get either \(a_1b_1 \in I\) or \(a_1c_1 \in I\) or \(b_1c_1 \in I\). If \(a_1b_1 \in I\), then \((a_1b_1,a_2b_2) \in I \times R_2\). Similarly \((a_1c_1,a_2c_2),(b_1c_1,b_2c_2) \in I \times R_2\). Hence \(I \times R_2\) is a 2-absorbing ideal of \(R\).

(ii) \(\implies\) (iii) It is obvious.

(iii) \(\implies\) (i) Let abc \(\in I\) for some \(a,b,c \in R_1\). Then for all \(0 \neq r \in R_2\), we have \((0,0) \neq (a,1)(b,1)(c,r) \in I \times R_2\). Since \(I \times R_2\) is a weakly 2-absorbing ideal of \(R\), we get either \((a,1)(b,1) \in I \times R_2\) or \((a,1)(c,r) \in I \times R_2\) or \((b,1)(c,r) \in I \times R_2\). This means that either \(ab \in I\) or \(ac \in I\) or \(bc \in I\). Hence \(I\) is a 2-absorbing ideal of \(R_1\).

The following example shows that if we replace \(R_2\) by a proper ideal of \(R_2\), the previous theorem does not necessarily hold.

**Example 2.2.16.** Let \(R = Z^\circ \times Z^\circ\). Suppose that \(I = \langle 6 \rangle\) and \(J = \langle 6 \rangle\). Clearly \(I\) is a 2-absorbing ideal of \(Z^\circ\), but \(I \times J\) is not a 2-absorbing ideal of \(Z^\circ \times Z^\circ\), since \((2,1)(1,2)(3,3) \in I \times J\), but neither \((2,2) \in I \times J\) nor \((6,3) \in I \times J\) nor \((3,6) \in I \times J\).

The following theorem gives a condition under which the concepts of the previous example must be true.

**Theorem 2.2.17.** Let \(R=R_1 \times R_2\) where \(R_1\) and \(R_2\) are commutative semirings with identity. Let \(I\) be a nonzero proper ideal of \(R_1\) and \(J\) be a nonzero ideal of \(R_2\). Then the
The following statements are equivalent:

(i) \( I \times J \) is a weakly 2-absorbing ideal of \( R \);

(ii) \( J=\mathbb{R}_2 \) and \( I \) is a 2-absorbing ideal of \( \mathbb{R}_1 \) or \( I \) and \( J \) are prime ideal of \( \mathbb{R}_1 \), \( \mathbb{R}_2 \) respectively;

(iii) \( I \times J \) is a 2-absorbing ideal of \( R \).

**Proof.**

(i) \( \implies \) (ii) Suppose that \( I \times J \) is a weakly 2-absorbing ideal of \( R \). If \( J=\mathbb{R}_2 \), then \( I \) is a 2-absorbing ideal of \( \mathbb{R}_1 \) by Proposition 2.2.15. Assume that \( \mathbb{R}_2 \neq J \). We show that \( I \) is a prime ideal of \( \mathbb{R}_1 \) and \( J \) is a prime ideal of \( \mathbb{R}_2 \). Let \( a,b \in \mathbb{R}_1 \) such that \( ab \in I \), and \( 0 \neq j \in J \). Then \( (0,0) \neq (a,1)(b,1)(1,j)=(ab,j) \in I \times J \). Since \( I \times J \) is a weakly 2-absorbing ideal of \( R \), and \( (a,1)(b,1)=(ab,1) \not\in I \times J \). We get either \( (a,1)(1,j)=(a,j) \in I \times J \) or \( (b,1)(1,j)=(b,j) \in I \times J \), and hence \( a \in I \) or \( b \in I \). Thus \( I \) is a prime ideal of \( \mathbb{R}_1 \). Similarly, let \( c,d \in \mathbb{R}_2 \) such that \( cd \in J \), and \( 0 \neq i \in I \). Then \( (0,0) \neq (1,c)(1,d)(i,1)=(i,cd) \in I \times J \). Since \( I \times J \) is a weakly 2-absorbing ideal of \( R \), and \( (1,c)(1,d)=(1,cd) \not\in I \times J \). We get either \( (1,c)(i,1)=(i,c) \in I \times J \) or \( (1,d)(i,1)=(i,d) \in I \times J \), and hence \( c \in J \) or \( d \in J \). Thus \( J \) is a prime ideal of \( \mathbb{R}_2 \).

(ii) \( \implies \) (iii) If \( J=\mathbb{R}_2 \) and \( I \) is a 2-absorbing ideal of \( \mathbb{R}_1 \), then \( I \times \mathbb{R}_2 \) is a 2-absorbing ideal of \( R \) by Proposition 2.2.15. Suppose \( I \) is a prime ideal of \( \mathbb{R}_1 \) and \( J \) is a prime ideal of \( \mathbb{R}_2 \). Let \( (m_1,n_1)(m_2,n_2)(m_3,n_3) \in I \times J \) for some \( m_1,m_2,m_3 \in \mathbb{R}_1 \) and for some \( n_1,n_2,n_3 \in \mathbb{R}_2 \). Then at least one of the \( m_i \)'s is in \( I \), say \( m_1 \), and at least one of the \( n_i \)'s is in \( J \), say \( n_2 \). Thus \( (m_1,n_1)(m_2,n_2) \in I \times J \). Hence \( I \times J \) is a 2-absorbing ideal of \( R \).

(iii) \( \implies \) (i) It is trivial. \( \square \)

The following example shows that the hypothesis that \( J \) is a nonzero ideal of \( \mathbb{R}_2 \) in Theorem 2.2.17 is essential.

**Example 2.2.18.** [7] Let \( \mathbb{R}_1=\mathbb{Z}_8 \oplus A \) and \( I=\{0\} \oplus A \), where \( A=\{0,4\} \). Let \( \mathbb{R}_2 \) be a field.
Then $I \times \{0\}$ is a weakly 2-absorbing ideal of $R_1 \times R_2$ but is not a 2-absorbing ideal of $R_1 \times R_2$. Observe that $I$ is not a prime ideal of $R_1$.

### 2.3 Relation Between Some Ideals and 2-Absorbing, Weakly 2-Absorbing Ideals in Semiring

**Definition 2.3.1.** [26] Let $R$ be a semiring and $I$ be a proper ideal. The ideal $I$ is said to be a **2-absorbing primary ideal** if whenever $a,b,c \in R$ with $abc \in I$, then either $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$.

**Definition 2.3.2.** [26] Let $R$ be a semiring and $I$ be a proper ideal. The ideal $I$ said to be a **weakly 2-absorbing primary ideal** if whenever $a,b,c \in R$ with $0 \neq abc \in I$, then either $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$.

**Lemma 2.3.3.** [26] Let $R$ be a commutative semiring. Then the following statements hold:

(i) Every primary ideal is 2-absorbing primary;

(ii) Every 2-absorbing ideal is 2-absorbing primary;

(iii) Every weakly primary ideal is weakly 2-absorbing primary;

(iv) Every 2-absorbing primary ideal is weakly 2-absorbing primary.

**Proof.** (i) Suppose that ideal $I$ of $R$ is primary. Then for $(ab)c \in I$ such that $a,b,c \in R$, we have that either $ab \in I$ or $c \in \sqrt{I}$. Since $\sqrt{I}$ is an ideal and $a,b \in R$, we get $ac \in \sqrt{I}$ and $bc \in \sqrt{I}$. Thus $I$ is a 2-absorbing primary ideal of $R$.

(ii) Let $I$ be 2-absorbing ideal of $R$. Suppose that $a,b,c \in R$ with $abc \in I$, we have either $ab \in I$ or $ac \in I$ or $bc \in I$. Since $I \subseteq \sqrt{I}$, we get $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. Therefore $I$ is a 2-absorbing primary ideal of $R$. 
(iii) Let $I$ be weakly primary ideal. Suppose $a, b, c \in R$ such that $0 \neq abc \in I$. Since $0 \neq (ab)c \in I$ and $I$ is a weakly primary ideal. We get $ab \in I$ or $c \in \sqrt{I}$, hence by (i) we get $ac \in \sqrt{I}$ and $bc \in \sqrt{I}$. Thus $I$ is a weakly 2-absorbing primary ideal of $R$.

(iv) Trivial by definition.

Remark 2.3.4. (1) Let $R$ be a commutative semiring, $I$ is an ideal of $R$. By above lemma, every primary ideal is a 2-absorbing primary ideal, but the converse does not necessarily hold. Let $R=(\mathbb{Z}^\circ, +, \cdot)$ and $I=\langle 12 \rangle$. Then $I$ is a 2-absorbing primary ideal of $R$. But $I$ is not primary ideal since $4.3 \in I$, $4 \notin I$ and $3 \notin \sqrt{I}$.

(2) By a above lemma, every 2-absorbing ideal is a 2-absorbing primary ideal, but the converse is not necessarily true. Consider the semiring $(\mathbb{Z}^\circ, +, \cdot)$. The ideal $I=\langle 8 \rangle$ is a 2-absorbing primary ideal, but not 2-absorbing ideal, because $2.2.2 \in I$, $2.2 \notin I$.

(3) In (1), $I=\langle 12 \rangle$ is a weakly 2-absorbing primary ideal, which is not weakly primary.

(4) In [26] Let $R=\mathbb{Z}_{12}$ be a commutative semiring and $I=\{0\}$ be an ideal of $R$. Then $I$ is a weakly 2-absorbing primary ideal of $R$, by definition. Now we consider $2.2.3 \in I$ but neither $2.2 \in I$ nor $2.3 \in \sqrt{I}$. So $I$ is not a 2-absorbing primary ideal of $R$.

Theorem 2.3.5. [26] Let $R$ be a commutative semiring and $I$ be an ideal of $R$. If $I$ is a 2-absorbing primary ideal of $R$, then $\sqrt{I}$ is a 2-absorbing ideal of $R$.

Proof. Let $a, b, c \in R$ such that $abc \in \sqrt{I}$ but $ac \notin \sqrt{I}$ and $bc \notin \sqrt{I}$. Then there exist a positive integer $n$ such that $(abc)^n=a^nb^n c^n \in I$. Since $I$ is a 2-absorbing ideal of $R$ and $ac$, $bc \notin \sqrt{I}$, we get $a^nb^n \in I$ and so $ab \in \sqrt{I}$. Hence $\sqrt{I}$ is a 2-absorbing ideal of $R$. □

Corollary 2.3.6. [17] Let $I$ be an ideal of a commutative semiring $R$. Then the following statements are equivalent:

(i) $I$ is a 2-absorbing primary ideal of $R$;
(ii) \( \sqrt{I} \) is a 2-absorbing ideal of \( R \) with the property that if \( abc \in I \) with \( ac, bc \notin \sqrt{I} \) then \( ab \in I \).

**Theorem 2.3.7.** Let \( R \) be a commutative semiring and \( I \) be a prime ideal of \( R \). Then the following statements are equivalent:

(i) \( I \) is a 2-absorbing ideal of \( R \);

(ii) \( I \) is a 2-absorbing primary ideal of \( R \).

**Proof.** (i) \( \implies \) (ii) by lemma 2.3.3.

(ii) \( \implies \) (i) Let \( abc \in I \) such that \( a, b, c \in R \). Since \( I \) is a 2-absorbing primary ideal of \( R \), then \( ab \in I \) or \( ac \in \sqrt{I} \) or \( bc \in \sqrt{I} \). Since \( I \) is a prime of \( R \), then \( I = \sqrt{I} \). We get, \( ac \in \sqrt{I} = I \) or \( bc \in \sqrt{I} = I \). Therefore \( I \) is a 2-absorbing ideal of \( R \).

Now the following theorem studies the relationship between irreducible subtractive and 2-absorbing ideals of a semiring \( R \).

**Theorem 2.3.8.** [15] Let \( I \) be an irreducible subtractive ideal of \( R \) and let \( J \) be an ideal of \( R \) such that \( \sqrt{I} = J \) and \( J^2 \subseteq I \). Then the following statements are equivalent:

(i) \( I \) is a 2-absorbing ideal of \( R \);

(ii) \( (I : r) = (I : r^2) \) for all \( r \in R/J \)

**Proof.** (i) \( \implies \) (ii) Let \( I \) be a 2-absorbing ideal of \( R \). For \( r \in R/J \), \( r^2 \notin I \), because if \( r^2 \in I \), then \( r \in \sqrt{I} = J \) which is a contradiction. Also, \( (I : r) \subseteq (I : r^2) \) is trivial. So, for any \( s \in (I : r^2) \) we have \( r^2s \in I \). Since \( I \) is a 2-absorbing ideal of \( R \), we have either \( rs \in I \) or \( r^2 \in I \). Since \( r^2 \notin I \), therefore \( rs \in I \), that is \( s \in (I : r) \). Thus \( (I : r) = (I : r^2) \).

(ii) \( \implies \) (i) Let \( r, s, t \in R \) with \( rst \in I \) and \( rs \notin I \). We show that either \( rt \in I \) or \( st \in I \), since \( rs \notin I \), we have \( r \notin J \) or \( s \notin J \), because if \( r \in J \) and \( s \in J \), then \( rs \in J^2 \subseteq I \) a contradiction. Now, by assumption, we have either \( (I : r) = (I : r^2) \) or \( (I : s) = (I : s^2) \).
If \((I : r) = (I : r^2)\), assume on contrary, \(rt \notin I\) and \(st \notin I\) (clearly \(I \subseteq I + rt, I \subseteq I + st\), then \(I \subseteq (I + rt) \cap (I + st)\)). Let \(p \in (I + rt) \cap (I + st)\), then there are \(p_1, p_2 \in I\) and \(r_1, r_2 \in R\) such that \(p = p_1 + r_1rt = p_2 + r_2st\). Thus \(pr = p_1r + r_1r^2t = p_2r + r_2rst \in I\), since \(rst \in I\), therefore \(r_1r^2t \in I\) (as \(I\) is a subtractive ideal of \(R\)). This implies \(r_1rt \in I\), since \((I : r) = (I : r^2)\). Hence \(p = p_1 + r_1rt \in I\). This show that \((I + rt) \cap (I + st) \subseteq I\), and thus \((I + rt) \cap (I + st) = I\), a contradiction. (because \(I\) is an irreducible and by Definition 1.1.43 we get, \(I = I + rt\) or \(I = I + st\). Then \(rt \in I\) or \(st \in I\). Thus we have \(rt \in I\) or \(st \in I\). Therefore \(I\) is 2-absorbing ideal of \(R\).

\[\text{Definition 2.3.9.}\ [22] \text{A semiring } R \text{ is said to be } \textit{regular} \text{ if for each } a \in R \text{ there exists } x \in R \text{ such that } axa = a.\]

\[\text{Lemma 2.3.10.}\ [22] \text{A semiring } R \text{ is regular if and only if } HK = H \cap K \text{ for all left ideals } K \text{ and right ideals } H \text{ of } R.\]

\[\text{Proof.}\ \text{Clearly } HK \subseteq H \cap K. \text{ Let } a \in H \cap K, \text{ then there exist } b \in R \text{ such that } aba = a, \text{ since } ab \in H. \text{ Then } aba \in HK, \text{ so } a \in HK. \text{ Therefore } HK = H \cap K.\]

\[\text{Theorem 2.3.11.}\ [15] \text{Let } R \text{ be a regular semiring. Then every irreducible ideal } I \text{ of } R \text{ is a } 2\text{-absorbing ideal of } R.\]

\[\text{Proof.}\ \text{Let } R \text{ be a regular semiring and } I \text{ be an irreducible ideal of } R. \text{ If } abc \in I \text{ and } ab \notin I, \text{ then we have to show that } ac \in I \text{ or } bc \in I. \text{ On contrary, we assume that } ac \notin I \text{ and } bc \notin I. \text{ Then } H = (I + (ac)) \text{ and } K = (I + (bc)) \text{ are two ideals of } R \text{ properly contain } I. \text{ Since } I \text{ is irreducible, therefore } I \neq H \cap K. \text{ Thus, there exists } p \in R \text{ such that } p \in (I + (ac)) \cap (I + (bc))/I. \text{ Also by regularity of } R, \text{ we have } HK = H \cap K, \text{ therefore } p \in (I + (ac))(I + (bc))/I. \text{ Then there exist } p_1, p_2 \in I \text{ and } r_1r_2 \in R \text{ such that } p = (p_1 + r_1ac)(p_2 + r_2bc) = p_1p_2 + p_1r_2bc + p_2r_1ac + r_1r_2abc^2 \in I. \text{ This implies that } p \in I, \text{ which is a contradiction. Hence } I \text{ is a } 2\text{-absorbing ideal of } R.\]
The following diagram shows the relation between some ideals and 2-absorbing, weakly 2-absorbing ideals in semiring.

Figure 1: Relation between ideals.
Chapter 3

2-Absorbing Subsemimodules.

In this chapter, we investigate the concept of 2-absorbing subsemimodules, weakly 2-absorbing subsemimodules and the relation between some subsemimodules and 2-absorbing, weakly 2-absorbing subsemimodules in semirings.

3.1 Definition and Properties of 2-Absorbing Subsemimodules

In this section, we introduce the concept of 2-absorbing subsemimodules which is a generalization to concept of 2-absorbing ideals. Let \( R \) be a commutative semiring with identity and let \( M \) be a unitary \( R \)-semimodule.

Definition 3.1.1. [18] Let \( R \) be a commutative semiring. Let \( M \) be an \( R \)-semimodule and \( N \) be a proper subsemimodule of \( M \). Then \( N \) is called a 2-absorbing subsemimodule of \( M \), if for \( a,b \in R \) and \( x \in M \), \( abx \in N \) implies that \( ab \in (N : M) \) or \( ax \in N \) or \( bx \in N \).

Clearly, every prime subsemimodule of \( M \) is a 2-absorbing subsemimodule of \( M \) and the following example shows that the converse need not be true.
**Example 3.1.2.** [18] Let $R = \mathbb{Z}^\circ$ where $\mathbb{Z}^\circ = \mathbb{Z}^+ \cup \{0\}$. Then $M = \mathbb{Z}^\circ \times \mathbb{Z}^\circ$ is an $R$-semimodule. If we take the subsemimodule $N = \{0\} \times 4 \mathbb{Z}^\circ$ of $M$, then the residual ideal of $N$ is $\{0\}$. Clearly, $N$ is a 2-absorbing subsemimodule of $M$ but $N$ is not prime subsemimodule of $M$ because $2 \cdot (0,2) \notin N$ but $2 \notin (N : M) = \{0\}$ and $(0,2) \notin N$.

**Proposition 3.1.3.** [18] If $N$ is a 2-absorbing subsemimodule of $M$ and $K$ is any subsemimodule of $M$, then $K \cap N$ is a 2-absorbing subsemimodule of $K$.

*Proof.* Since $N$ is a proper subsemimodule of $M$, then $K \cap N$ is a proper subsemimodule of $K$. Let $a, b \in R$ and $k \in K$ such that $abk \in K \cap N$. Since $abk \in N$ and $N$ is a 2-absorbing subsemimodule of $M$, therefore $ak \in N$ or $bk \in N$ or $ab \in (N : M)$. If $ak \in N$ or $bk \in N$, then $ak \in K \cap N$ or $bk \in K \cap N$. If $ab \in (N : M)$, then $abM \subseteq N$. In particular, $abK \subseteq N$ which implies $abK \subseteq K \cap N$. Thus $ab \in (K \cap N : K)$. Hence $K \cap N$ is a 2-absorbing subsemimodule of $K$. \hfill \Box

**Theorem 3.1.4.** [18] If $N$ is an intersection of two prime subsemimodule of an $R$-semimodule $M$, then $N$ is 2-absorbing.

*Proof.* Let $N_1$ and $N_2$ be two prime subsemimodule of $M$. Then to show that $N_1 \cap N_2$ is a 2-absorbing subsemimodule of $M$. Let $a, b \in R$ and $m \in M$ with $abm \in N_1 \cap N_2$. Then $abm \in N_1$ and $abm \in N_2$. Now, $abm \in N_1$ implies $a \in (N_1 : M)$ or $b \in (N_1 : M)$ or $m \in N_1$, since $(N_1 : M)$ is prime ideal of $R$, see Proposition 2.1.15. Similarly, $abm \in N_2$ implies $a \in (N_2 : M)$ or $b \in (N_2 : M)$ or $m \in N_2$, because $(N_2 : M)$ is prime ideal of $R$. If $a \in (N_1 : M)$ and $a \in (N_2 : M)$ then $aM \subseteq N_1$ and $aM \subseteq N_2$, and so $aM \subseteq N_1 \cap N_2$ implies $a \in (N_1 \cap N_2 : M)$, then $ab \in (N_1 \cap N_2 : M)$. Again if $a \in (N_1 : M)$ and $m \in N_2$, then $am \in N_1 \cap N_2$. Similarly, we can prove the other cases. \hfill \Box

**Remark 3.1.5.** [18] It is easy to see that the intersection of two distinct nonzero 2-absorbing subsemimodules need not be a 2-absorbing subsemimodule of $M$. Consider $N_1 = \{0\} \times \mathbb{Z}^\circ$. 

39
4Z^\circ$ and $N_2 = \{0\} \times 3Z^\circ$ are 2-absorbing subsemimodules of $Z^\circ \times Z^\circ$ where $Z^\circ = Z^+ \cup \{0\}$, and their intersection $N_1 \cap N_2 = (\{0\} \times 4Z^\circ) \cap (\{0\} \times 3Z^\circ) = (\{0\} \times 12Z^\circ)$, clearly $(N_1 \cap N_2 : Z^\circ \times Z^\circ) = \{0\}$. Then $N_1 \cap N_2$ is not a 2-absorbing subsemimodule of $Z^\circ \times Z^\circ$, since $2.2.2(0,3) \in N_1 \cap N_2$ but neither $(0,6) \in N_1 \cap N_2$ nor $2.2 \in \{0\}$.

Now, we show that the residual ideal of a 2-absorbing subtractive subsemimodule is a 2-absorbing ideal as in [18].

**Proposition 3.1.6.** [18] Let $M$ be an $R$-semimodule and let $N$ be a 2-absorbing subtractive subsemimodule of $M$. Then $(N : M)$ is a 2-absorbing ideal of $R$.

**Proof.** Clearly, since $N$ is subtractive subsemimodule of $M$, then $(N : M)$ is a subtractive ideal of $R$. Now, we show that $(N : M)$ is a 2-absorbing ideal of $R$. Let $a,b,c \in R$ be such that $abc \in (N : M)$. Assume $ac,bc \notin (N : M)$, then there exist $x,y \in M - N$ such that $acx,bcy \notin N$. Since $abc \in (N : M)$, then $abcM \subseteq N$ implies $ab(c(x+y)) \in N$. Since $N$ is a 2-absorbing subsemimodule of $M$, we get $ac(x+y) \in N$ or $bc(x+y) \in N$ or $ab \in (N : M)$. If $ac(x+y) \in N$ and since $acx \notin N$, then we have $acy \notin N$ (as $N$ is a subtractive subsemimodule of $M$). Since $ab(cy) \in N$ and $N$ is a 2-absorbing subsemimodule of $M$, therefore either $ab \in (N : M)$ or $acy \in N$ or $bcy \in N$. Thus $ab \in (N : M)$. Similarly, if $bc(x+y) \in N$, then $ab \in (N : M)$. Hence $(N : M)$ is a 2-absorbing ideal of $R$. 

**Corollary 3.1.7.** [18] Let $N$ be a subtractive 2-absorbing subsemimodule of an $R$-semimodule $M$. Then $\sqrt{(N : M)}$ is a 2-absorbing ideal of $R$.

**Proof.** Let $N$ be a 2-absorbing subsemimodule of an $R$-semimodule $M$. Then by Proposition 3.1.6, we have $(N : M)$ is a 2-absorbing ideal of $R$. By Theorem 2.1.2, we have $\sqrt{(N : M)}$ is a 2-absorbing ideal of $R$. 

**Remark 3.1.8.** [18] The converse of the above proposition is not true in general. For example, Let $R$ be $Z^\circ = Z^+ \cup \{0\}$, $M = Z^\circ \times Z^\circ$ is an $R$-semimodule. Consider the
subsemimodule $N = \{0\} \times 8Z^\circ$ of $M$. Then the residual ideal of $N$ is $\{0\}$. Clearly, $(N : M)$ is a 2-absorbing ideal of $R$. But $N$ is not 2-absorbing subsemimodule of $M$ because $2.2.(0,2) \in N$ and neither $2.2 \in (N : M)$ nor $2.(0,2) \in N$.

Now, we find the condition that makes the converse of Proposition 3.1.6 is true.

**Proposition 3.1.9.** [18] Let $M$ be a cyclic $R$-semimodule. Then $N$ is a 2-absorbing subtractive subsemimodule of $M$ if and only if $(N : M)$ is a 2-absorbing ideal of $R$.

**Proof.** $(\implies)$ By Proposition 3.1.6 as required.

$(\impliedby)$ Let $(N : M)$ be a 2-absorbing ideal of $R$. Let $M=Rm$, for some $m \in M$. Assume that $a,b \in R$ and $x \in M$ such that $abx \in N$. Then there exists $c \in R$ such that $x=cm$, we get $abcx \in N$, therefore $abc \in (N : m)=(N : M)$. Since $(N : M)$ is a 2-absorbing ideal of $R$. Then either $ab \in (N : M)$ or $ac \in (N : M)$ or $bc \in (N : M)$. Therefore $ab \in (N : M)$ or $ax \in N$ or $bx \in N$. Thus $N$ is a 2-absorbing subsemimodule of $M$. $\square$

**Theorem 3.1.10.** Let $R$ be a commutative semiring and $f:M \longrightarrow M'$ be an epimorphism of a $R$-semimodules $M$ and $M'$. If $N$ is a 2-absorbing subsemimodule of $M'$, then $f^{-1}(N)$ is also a 2-absorbing subsemimodule of $M$.

**Proof.** Let $a,b \in R$ and $m \in M$ such that $abm \in f^{-1}(N)$. Then $f(abm) = abf(m) \in N$, but $N$ is a 2-absorbing subsemimodule of $M'$, so $ab \in (N : M')$ or $af(m) \in N$ or $bf(m) \in N$. If $ab \in (N : M')$ then $abM' \subseteq N \implies abM = abf^{-1}(M') = f^{-1}(abM') \subseteq f^{-1}(N)$. So $ab \in (f^{-1}(N) : M)$. If $af(m) \in N$ then $f(am) \in N \implies am \in f^{-1}(N)$. Similarly if $bf(m) \in N$, then $bm \in f^{-1}(N)$. Thus $ab \in (f^{-1}(N) : M)$ or $am \in f^{-1}(N)$ or $bm \in f^{-1}(N)$, and hence $f^{-1}(N)$ is a 2-absorbing subsemimodule of $M$. $\square$

**Theorem 3.1.11.** Let $f:M \longrightarrow M'$ be an epimorphism of a $R$-semimodule $M$ and $M'$ such that $f(0)=0$ and $N$ be a subtractive strong subsemimodule of $M$. If $N$ is a 2-absorbing subsemimodule of $M$ with $\ker(f) \subseteq N$, then $f(N)$ is a 2-absorbing subsemimodule of $M'$. 41
Proof. Let \( abx \in f(N) \) for some \( a,b \in R \) and \( x \in M' \), then there exists \( n \in N \) such that \( abx=f(n) \). Since \( f \) is epimorphism and \( x \in M' \), then there exist \( y \in M \) such that \( f(y)=x \).

As \( n \in N \) and \( N \) is a strong subsemimodule of \( M \), therefore there exist \( n' \in N \) such that \( n+n'=0 \) which gives \( f(n+n')=0 \). Therefore, \( abx+f(n') = f(aby)+f(n') = f(aby+n')=0 \) implies \( aby+n' \in ker(f) \subseteq N \). So \( aby \in N \), since \( N \) is a subtractive subsemimodule of \( M \).

Since \( N \) is 2-absorbing, we conclude that \( ab \in (N : M) \) or \( ay \in N \) or \( by \in N \).

If \( ab \in (N : M) \), then \( abM \subseteq N = f(N) \). Thus \( ab \in (f(N) : M') \). If \( ay \in N \), then \( f(ay)=af(y)=ax \in f(N) \). If \( by \in N \), then as before \( bx \in f(N) \). Thus \( ab \in (f(N) : M') \) or \( ax \in f(N) \) or \( bx \in f(N) \). Hence \( f(N) \) is a 2-absorbing subsemimodule of \( M' \).

Remark 3.1.12. (1) In Theorem 3.1.10, if \( f \) is not onto may causes that \( f^{-1}(N) \) is not a proper subsemimodule of \( M \). Consider the homomorphism \( f : \mathbb{Z}_4 \longrightarrow \mathbb{Z}_4 \) defined by \( f(0)=f(2)=0, f(1)=f(3)=2 \) and let \( N=\{0,2\} \), then \( f^{-1}(N)=\mathbb{Z}_4^2 \).

(2) In Theorem 3.1.11, if we ignore the condition \([ker(f) \subseteq N]\) may cause that \( f(N) \) is not a proper subsemimodule of \( M' \). For example, consider the epimorphism \( f : \mathbb{Z}_{12}^n \longrightarrow \mathbb{Z}_4^n \) defined by \( f(x) = x(mod 4) \), \( ker(f) = \{0,4,8\} \) let \( N= \{0,3,6,9\} \), then \( f(N)=\mathbb{Z}_4^2 \).

Theorem 3.1.13. Let \( M \) be an \( R \)-semimodule, \( N \) be a \( Q \)-subsemimodule of \( M \) and \( P \) be a subtractive subsemimodule of \( M \) such that \( N \subseteq P \). Then \( P \) is a 2-absorbing subsemimodule of \( M \) if and only if \( P/N_{Q\cap P} \) is a 2-absorbing subsemimodule of \( M/N_{Q} \).

Proof. Let \( P \) be a 2-absorbing subsemimodule of \( M \). Suppose \( a,b \in R \) and \( q+N \in M/N_{Q} \) be such that \( ab\otimes q+N=q_1+N \in P/N_{Q\cap P} \) where \( q_1 \in Q\cap P \) is a unique element such that \( abq+N \subseteq q_1+N \). So \( abq=q_1+y_1 \), for some \( y_1 \in N \subseteq P \). Then \( abq \in P \), and since \( P \) is a 2-absorbing subsemimodule of \( M \) implies that either \( ab \in (P : M) \) or \( aq \in P \) or \( bq \in P \). First, let \( ab \in (P : M) \). Consider \( ab\otimes q_2+N=q_3+N \) where \( q_2+N \in M/N_{Q} \) and \( q_3 \in Q \)
is a unique element such that \( abq_2+N \subseteq q_3+N \). Then there exist \( y_2 \in N \subseteq P \) such that \( abq_2=q_3+y_2 \). Since \( ab \in (P : M) \), and \( abM \subseteq P \), we have \( abq_2 \in P \). Since \( P \) is a subtractive subsemimodule of \( M \), so \( q_3 \in P \). Thus we have \( q_3 \in Q \cap P \) which gives \( ab \circ q_2+N = q_3+N \in P/N(Q \cap P) \) and hence \( ab \in (P/N(Q \cap P) : M/N(Q)) \).

If \( aq \in P \), consider \( a \circ q+N = q_4+N \) where \( q_4 \in Q \) is a unique element such that \( aq+N \subseteq q_4+N \). Then there exist \( y_3 \in N \subseteq P \) such that \( aq = q_4+y_3 \). We have \( q_4 \in P \) (as \( P \) is a subtractive) and hence \( q_4 \in Q \cap P \). Therefore \( a \circ q+N = q_4+N \in P/N(Q \cap P) \). Consequently, \( P/N(Q \cap P) \) is a 2-absorbing subsemimodule of \( M/N(Q) \).

Conversely, let \( P/N(Q \cap P) \) be a 2-absorbing subsemimodule of \( M/N(Q) \). Let \( a,b \in R \) and \( x \in M \) such that \( abx \in P \). Since \( N \) is a \( Q \)-subsemimodule of \( M \) and \( x \in M \), we have \( x \in q+N \) where \( q \in Q \). So \( abx \in \text{abq}+N \).

Now, let \( ab \circ (q+N) = q_5+N \) where \( q_5 \in Q \) is a unique element such that \( \text{abq}+N \subseteq q_5+N \). Then there exist \( x_1 \in N \subseteq P \) such that \( abq = q_5+x_1 \). Therefore, we have \( q_5 \in P \), since \( P \) is subtractive. Thus \( q_5 \in Q \cap P \) and hence \( ab \circ (q+N) = q_5+N \in P/N(Q \cap P) \). Since \( P/N(Q \cap P) \) is a 2-absorbing subsemimodule of \( M/N(Q) \), we have \( ab \in (P/N(Q \cap P) : M/N(Q)) \) or \( a \circ (q+N) \in P/N(Q \cap P) \) or \( b \circ (q+N) \in P/N(Q \cap P) \).

If \( ab \in (P/N(Q \cap P) : M/N(Q)) \). Let \( y \in M \), then there exists a unique element \( q_6 \in Q \) such that \( y \in q_6+N \in M/N(Q) \).

Now \( ab \circ (q_6+N) \in P/N(Q \cap P) \). Therefore there exists a unique element \( q_7 \in Q \cap P \) such that \( abq_6+N \subseteq q_7+N \) this gives that \( aby \in abq_6+N \subseteq q_7+N \). Thus, \( aby = q_7+x_2 \in P \) for some \( x_2 \in N \subseteq P \). Hence \( aby \in P \) implies that \( abM \subseteq P \) since \( y \in M \) is arbitrary. Thus \( ab \in (P : M) \).

Let \( a \circ (q+1) \in P/N(Q \cap P) \). Then, there exists a unique element \( q_8 \in Q \cap P \) such that \( aq+N \subseteq q_8+N \). We have, \( x \in q+N \) implies \( ax \in aq+N \subseteq q_8+N \). Therefore \( ax = q_8+x_3 \) for
some \( x_3 \in N \subseteq P \). Hence \( ax \in P \). Similarly, \( bx \in P \). Thus \( ab \in (P : M) \) or \( ax \in P \) or \( bx \in P \). Therefore \( P \) is a 2-absorbing subsemimodule of \( M \).

**Theorem 3.1.14.** Let \( S \) be a multiplicatively closed subset of \( R \), and \( M \) be an \( R \)-semimodule and let \( S^{-1} N \) be a subsemimodule of the \( S^{-1} R \)-semimodule \( S^{-1} M \). If \( N \) is a 2-absorbing subsemimodule of \( M \) and \( S^{-1} N \neq S^{-1} M \), then \( S^{-1} N \) is a 2-absorbing subsemimodule of \( S^{-1} M \).

**Proof.** Let \( a_1, a_2 \in R \), \( m \in M \) and \( b_1, b_2, b \in S \) such that \( \frac{a_1 a_2 m}{b_1 b_2 b} \in S^{-1} N \). Then there exist \( u \in S \) such that \( u a_1 a_2 m \in N \). Since \( N \) is a 2-absorbing subsemimodule of \( M \), then we have either \( u a_1 m \in N \) or \( a_2 m \in N \) or \( u a_1 a_2 \in (N : M) \).

If \( u a_1 m \in N \), then \( \frac{a_1}{b_1} m = \frac{u a_1 m}{u b_1 b} \in S^{-1} N \). Similarly, if \( a_2 m \in N \) then \( \frac{a_2 m}{b_2 b} \in S^{-1} N \). Now, if \( u a_1 a_2 \in (N : M) \), then \( \frac{u a_1 a_2}{u b_1 b_2} = \frac{a_1 a_2}{b_1 b_2} \in S^{-1} (N : R M) \subseteq (S^{-1} N : S^{-1} R S^{-1} M) \).

Thus \( S^{-1} N \) is a 2-absorbing subsemimodule of \( S^{-1} M \). \( \square \)

**Lemma 3.1.15.** Let \( N \) and \( K \) be subsemimodules of \( M \). If \( N \) is a subtractive 2-absorbing subsemimodule of an \( R \)-semimodule \( M \) and \( abK \subseteq N \) for some \( a, b \in R \), then \( ab \in (N : M) \) or \( aK \subseteq N \) or \( bK \subseteq N \).

**Proof.** Suppose that \( ab \notin (N : M) \) and \( aK \notin N \) and \( bK \notin N \). So there exist \( k_1, k_2 \in K \) such that \( ak_1 \notin N \) and \( bk_2 \notin N \). Since \( abk_1, abk_2 \in N \) and \( N \) is a 2-absorbing subsemimodule of \( M \), we get \( bk_1 \in N \) and \( ak_2 \in N \). Now since \( ab(k_1 + k_2) \in N \), \( ab \notin (N : M) \) and \( N \) is a 2-absorbing subsemimodule of \( M \), we have \( a(k_1 + k_2) \in N \) or \( b(k_1 + k_2) \in N \). If \( a(k_1 + k_2) \in N \), since \( I \) is subtractive and \( ak_2 \in N \), we have \( ak_1 \in N \), which a contradiction. Similar, if \( b(k_1 + k_2) \in N \), we have \( bk_2 \in N \), which is a contradiction.

Thus \( ab \in (N : M) \) or \( aK \subseteq N \) or \( bK \subseteq N \). \( \square \)

**Theorem 3.1.16.** Let \( N \) be a subtractive subsemimodule of \( M \) and \( (N : M) \) be a subtractive ideal of \( R \). If \( N \) is a 2-absorbing subsemimodule of \( M \), then whenever \( IJK \subseteq N \) for some
ideals $I, J, K$ of $R$ and a subsemimodule $K$ of $M$, then $IJ \subseteq (N : M)$ or $IK \subseteq N$ or $JK \subseteq N$.

**Proof.** Let $N$ be a 2-absorbing subsemimodule of $M$ and let $IJK \subseteq N$ for some ideals $I, J, K$ of $R$ and a subsemimodule $K$ of $M$, and let $IJ \not\subseteq (N : M)$. We show that $IK \subseteq N$ or $JK \subseteq N$. Suppose on contrary, $IK \not\subseteq N$ and $JK \not\subseteq N$. Then there exist $i_1 \in I$ and $j_1 \in J$ such that $i_1K \not\subseteq N$ and $j_1K \not\subseteq N$. Since $i_1j_1K \subseteq N$ and $i_1K \not\subseteq N$ and $j_1K \not\subseteq N$, we have $i_1, j_1 \in (N : M)$ by Lemma 3.1.15. But $IJ \not\subseteq (N : M)$, then there exist $i_2 \in I$ and $j_2 \in J$ such that $i_2j_2 \not\subseteq (N : M)$. By previous lemma and since $i_2j_2K \subseteq N$ and $i_2j_2 \not\subseteq (N : M)$, we have $i_2K \subseteq N$ or $j_2K \subseteq N$. Now we have three cases:-

Case I: Assume $i_2K \subseteq N$ but $j_2K \not\subseteq N$. Since $i_2j_2K \subseteq N$ and $j_2K \not\subseteq N$ and $i_1K \not\subseteq N$, by Lemma 3.1.15 we have $i_1j_2 \in (N : M)$. Since $i_2K \subseteq N$ but $i_1K \not\subseteq N$, therefore $(i_1+i_2)K \not\subseteq N$ (as $N$ is subtractive). On the other hand, $(i_1+i_2)j_2K \subseteq N$ and $j_2K \not\subseteq N$ and $(i_1+i_2)K \not\subseteq N$ implies $(i_1+i_2)j_2 \in (N : M)$ by previous lemma. Since $(i_1+i_2)j_2 \in (N : M)$ and $i_1j_2 \in (N : M)$, we have $i_2j_2 \in (N : M)$ (as $(N : M)$ is subtractive), a contradiction.

Case II: when $j_2K \subseteq N$ and $i_2K \not\subseteq N$. Since $i_2j_1K \subseteq N$ and $i_2K \not\subseteq N$ and $j_1K \not\subseteq N$, then by Lemma 3.1.15 we have $i_2j_1 \in (N : M)$. Since $j_2K \subseteq N$ and $j_1K \not\subseteq N$, we have $(j_1+j_2)K \not\subseteq N$. Since $i_2(j_1+j_2)K \subseteq N$ and $i_2K \not\subseteq N$ and $(j_1+j_2)K \not\subseteq N$, we have $i_2(j_1+j_2) \in (N : M)$ by previous lemma. Since $i_2(j_1+j_2) \in (N : M)$ and $i_2j_1 \in (N : M)$, we get $i_2j_2 \in (N : M)$, since $(N : M)$ is subtractive. Which is a contradiction.

Case III: We assume that $i_2K \subseteq N$ and $j_2K \subseteq N$. At the first consider $j_2K \subseteq N$ and $j_1K \not\subseteq N$, and $N$ is subtractive subsemimodule of $M$, we conclude that $(j_1+j_2)K \not\subseteq N$. Since $i_1(j_1+j_2)K \subseteq N$ and $(j_1+j_2)K \not\subseteq N$ and $i_1K \not\subseteq N$, we have $i_1(j_1+j_2) \in (N : M)$ by Lemma 3.1.15. Since $i_1j_1 \in (N : M)$ and $i_1(j_1+j_2) \in (N : M)$, we get $i_1j_2 \in (N : M)$, as $(N : M)$ is subtractive. Again, $i_2K \subseteq N$ and $i_1K \not\subseteq N$ implies $(i_1+i_2)K \not\subseteq N$. Since $(i_1+i_2)j_1K \subseteq N$ and $(i_1+i_2)K \not\subseteq N$ and $j_1K \not\subseteq N$, therefore $(i_1+i_2)j_1 \in (N : M)$, and $i_1j_1 \in (N : M)$.
implies $i_2j_1 \in (N : M)$.

Now as $(i_1 + i_2)(j_1 + j_2)k \subseteq N$ and $(i_1 + i_2)k \not\subseteq N$ and $(j_1 + j_2)k \not\subseteq N$, then by Lemma 3.1.15 we conclude that $(i_1 + i_2)(j_1 + j_2) \in (N : M)$. Since $i_2j_1, i_1j_2, i_1j_1 \in (N : M)$, we have $i_2j_2 \in (N : M)$ (as $(N : M)$ is subtractive), a contradiction. Hence $IK \subseteq N$ or $JK \subseteq N$.

Therefore $IJ \subseteq (N : M)$ or $IK \subseteq N$ or $JK \subseteq N$. \hfill $\square$

### 3.2 Weakly 2-Absorbing Subsemimodules

The concept of weakly 2-absorbing submodules was introduced by A.Y.Darani and F.Soheilnia in [14]. We study this concept and we generalize some results of weakly prime subsemimodule that was proved in [10] to weakly 2-absorbing subsemimodules. Also, we generalize the concept of weakly 2-absorbing ideals in semirings in [13] to weakly 2-absorbing subsemimodule.

In this section, we define and study the concept of weakly 2-absorbing subsemimodules.

**Definition 3.2.1.** Let $R$ be a commutative semiring and $M$ be an $R$-semimodule. A proper subsemimodule $N$ of $M$ is said to be *weakly 2-absorbing subsemimodule* if whenever $a, b \in R$, $x \in M$ with $0 \neq abx \in N$ then $ab \in (N : M)$ or $ax \in N$ or $bx \in N$.

**Proposition 3.2.2.** Let $R$ be a commutative semiring, $M$ be an $R$-semimodule and $N$ a subsemimodule of $M$.

(i) $N$ is prime subsemimodule $\implies$ $N$ is 2-absorbing subsemimodule $\implies$ $N$ is weakly 2-absorbing subsemimodule.

(ii) $N$ is weakly prime subsemimodule $\implies$ $N$ is weakly 2-absorbing subsemimodule.

*Proof.* Direct from the definitions. \hfill $\square$

**Remark 3.2.3.** Let $R$ be a commutative semiring and $M$ an $R$-semimodule. By above
proposition, every 2-absorbing subsemimodules is weakly 2-absorbing, but the converse
does not necessarily hold. Consider \( R=\langle \mathbb{Z}_{12}, +_{12} \rangle \) where \( \mathbb{Z}_{12} = \{0, 1, 2, ..., 11\} \), \( M=\mathbb{Z}_{12} \times \mathbb{Z}_{12} \),
\( N=\{(0,0)\} \), then \( (N : M)=\langle 12 \rangle \). Clearly \( N \) is a weakly 2-absorbing subsemimodule of
\( M \) but \( N \) not 2-absorbing subsemimodule, because \( 2.2.(3,3) \in N \), \( 2.2 \notin (N : M) \) and \( (6,6) \notin N \).

**Proposition 3.2.4.** Let \( R \) be a commutative semiring and \( M \) be an \( R \)-semimodule. If
\( N \) is a weakly 2-absorbing subsemimodule of \( M \) and \( K \) is any subsemimodule of \( M \), then
\( K \cap N \) is a weakly 2-absorbing subsemimodule of \( K \).

*Proof.* The proof is similar to that of Proposition 3.1.3. \( \square \)

**Theorem 3.2.5.** Let \( R \) be a commutative semiring and \( M \) be an \( R \)-semimodule, then the
intersection of two weakly prime subsemimodule of \( M \) is a weakly 2-absorbing.

*Proof.* The proof is similar to that of Theorem 3.1.4. \( \square \)

**Remark 3.2.6.** If \( N \) is a weakly 2-absorbing subsemimodule of an \( R \)-semimodule \( M \), then
\( (N : M) \) is not weakly 2-absorbing ideal of \( R \) in general. For example, let \( R=(\mathbb{Z}^{\circ}, +, \cdot) \)
and \( M=\mathbb{Z}_{8}^{\circ} \times \mathbb{Z}_{8}^{\circ} \) and \( N=\{(0,0)\} \), then \( (N : M)=8\mathbb{Z}^{\circ} \). Clearly \( N \) is weakly 2-absorbing
subsemimodule, but \( (N : M) \) is not weakly 2-absorbing ideal of \( R \), because \( 0 \neq 2.2.2 \in (N : M) \) but \( 2.2 \notin (N : M) \).

Now, we find the condition that makes the residual of a weakly 2-absorbing subsemimodule a weakly 2-absorbing ideal.

**Proposition 3.2.7.** Let \( R \) be commutative semiring, \( M \) be an \( R \)-semimodule and let \( N \) be a subsemimodule of \( M \). If in addition \( M \) is faithful and \( N \) is a subtractive weakly 2-absorbing subsemimodule of \( M \), then \( (N : M) \) is a weakly 2-absorbing ideal of \( R \).
Proof. Clearly \((N : M)\) is a subtractive ideal of \(R\). Let \(a, b, c \in R\) with \(0 \neq abc \in (N : M)\).
Suppose \(ac, bc \notin (N : M)\). Then there exist \(m_1, m_2 \in M - N\) such that \(acm_1, bcm_2 \notin N\).
Since \(abc \in (N : M)\), \(0 \neq abcM \subseteq N\) implies that \(0 \neq ab(c(m_1 + m_2)) \in N\), for other wise \(abc(m_1 + m_2) = 0\) then \(abc \in (0 : M) = 0\), a contradiction. Then \(0 \neq ab(c(m_1 + m_2)) \in N\) since \(N\) is a weakly 2-absorbing subsemimodule of \(M\), we have either \(ab \in (N : M)\) or \(ac(m_1+m_2) \in N\) or \(bc(m_1+m_2) \in N\).
If \(ac(m_1+m_2) \in N\) and \(acm_1 \notin N\), then we get \(acm_2 \notin N\) (as \(N\) is a subtractive). Since \(ab(cm_2) \in N\) and \(N\) is a weakly 2-absorbing subsemimodule of \(M\), therefore \(ab \in (N : M)\) or \(a(cm_2) \in N\) or \(b(cm_2) \in N\). Thus \(ab \in (N : M)\). Similarly, if \(bc(m_1+m_2) \in N\), then as before we get, \(ab \in (N : M)\).
Hence \((N : M)\) is a weakly 2-absorbing ideal of \(R\). \(\Box\)

**Proposition 3.2.8.** Let \(M\) be a cyclic \(R\)-semimodule and let \(N\) be a subtractive subsemimodule of \(M\). If in addition \(M\) is faithful, then \(N\) is a weakly 2-absorbing subsemimodule of \(M\) if and only if \((N : M)\) is a weakly 2-absorbing ideal of \(R\).

**Proof.** (\(\implies\)) We prove this direction in previous proposition.

(\(\iff\)) Let \((N : M)\) be a weakly 2-absorbing ideal of \(R\). Let \(a, b \in R\) and \(m \in M\) such that \(0 \neq abm \in N\). Let \(M = Rx\), then there exists \(c \in R\) such that \(m = cx\), we get \(0 \neq abcx \in N\) other wise if \(abcx = 0\), then \(abc \in (0 : M) = 0\), a contradiction. Since \(0 \neq abcx \in N\), therefore \(abc \in (N : m) = (N : M)\). Since \((N : M)\) is a 2-absorbing ideal of \(R\). Then we have either \(ab \in (N : M)\) or \(ac \in (N : M)\) or \(bc \in (N : M)\). Hence \(ab \in (N : M)\) or \(ax \in N\) or \(bx \in N\). Thus \(N\) is a weakly 2-absorbing subsemimodule of \(M\). \(\Box\)

**Theorem 3.2.9.** Let \(R\) be a commutative semiring and \(f: M \rightarrow M'\) be an epimorphism of an \(R\)-semimodules \(M\) and \(M'\) with \(f(0) = 0\) and \(\ker(f) = \{0\}\). If \(N\) is a weakly 2-absorbing subsemimodule of \(M'\), then \(f^{-1}(N)\) is a 2-absorbing subsemimodule of \(M\).
Proof. Let 0\neq abm \in f^{-1}(N) such that a,b \in R and m \in M. Then 0\neq f(abm)=abf(m) \in N, because if f(abm)=0, then abm=0, since ker(f)={0}, this is a contradiction. Now, since 0\neq abf(m) \in N, N is a 2-absorbing subsemimodule of M'. Then either ab \in (N : M') or af(m) \in N or bf(m) \in N. Then by the same method of proving Theorem 3.1.10 we get ab \in (f^{-1}(N) : M) or am \in f^{-1}(N) or bm \in f^{-1}(N). Thus f^{-1}(N) is a weakly 2-absorbing subsemimodule of M. 

Theorem 3.2.10. Let f:M \longrightarrow M' be an isomorphism of a R-semimodule M and M' such that f(0)=0 and N be a subtractive strong subsemimodule of M. If N is a weakly 2-absorbing subsemimodule of M with ker(f) \subseteq N, then f(N) is a weakly 2-absorbing subsemimodule of M'.

Proof. Let a,b \in R and x \in M' such that 0\neq abx \in f(N) then there exist n \in N such that 0\neq abx=f(n). Since f is epimorphism and x \in M', then there exist y \in M such that f(y)=x. For n \in N and N is a strong subsemimodule of M, therefore there exist n' \in N such that n+n'=0 which gives f(n+n')=0. Therefore, abx+f(n') = f(aby)+f(n') = f(aby+n')=0 implies aby+n'=0. As aby+n' \in N, then 0\neq aby \in N (as N is subtractive), other wise if aby=0 then f(aby)=abf(y)=abx=0, a contradiction.

Now 0\neq aby \in N, N is a weakly 2-absorbing subsemimodule of M, we conclude that ab \in (N : M) or ay \in N or by \in N. By the same method of proving Theorem 3.1.11 we get ab \in (f(N) : M') or ax \in f(N) or bx \in f(N). Thus f(N) is a weakly 2-absorbing subsemimodule of M'. 

Theorem 3.2.11. Let R be a commutative semiring, N be a Q-subsemimodule of an R-semimodule M and P a subtractive subsemimodule of M with N \subseteq P. Then.

(i) If P is a weakly 2-absorbing subsemimodule of M, then P/N_{(Q\cap P)} is a weakly 2-absorbing subsemimodule of M/N_{(Q)}.
(ii) If $N$, $P/N_{(Q∩P)}$ are weakly 2-absorbing subsemimodule of $M$, $M/N_{(Q)}$ respectively, then $P$ is a weakly 2-absorbing subsemimodule of $M$.

Proof. (i) Let $P$ be a weakly 2-absorbing subsemimodule of $M$. Let $q_o$ be the unique element of $Q$ such that $q_o+N$ is the zero element of $M/N_{(Q)}$. Let $a,b ∈ R$ and $q+N ∈ M/N_{(Q)}$ such that $q_o+N≠ab⊙(q+N) ∈ P/N_{(Q∩P)}$. Then there exists a unique $q_1 ∈ Q∩P$ such that $ab⊙(q+N)=q_1+N$ where $abq+N ⊆ q_1+N$. So for some $y_1 ∈ N ⊆ P$ we get $abq=q_1+y_1$. Then $0≠abq ∈ P$(as $P$ is subtractive), other wise if $abq=0$, then $0=abq ∈ (q_o+N) ∩ (q_1+N)$, since $0 ∈ (q_o+N)$. So $q_o=q_1$ and hence $q_o+N=q_1+N$ a contradiction. Thus $0≠abq ∈ P$, as $P$ is a weakly 2-absorbing subsemimodule of $M$, then either $ab ∈ (P : M)$ or $aq ∈ P$ or $bq ∈ P$.

First, Let $ab ∈ (P : M)$. Consider $ab⊙q_2+N=q_3+N$ where $q_2+N ∈ M/N_{(Q)}$ and $q_3 ∈ Q$ is a unique element such that $abq_2+N ⊆ q_3+N$. Then there exist $y_2 ∈ N ⊆ P$ such that $abq_2=q_3+y_2$. Since $ab ∈ (P : M)$ and $abM ⊆ P$, we have $abq_2 ∈ P$. Since $P$ is a subtractive of $M$, so $q_3 ∈ P$. Thus $q_3 ∈ Q∩P$. Now $ab⊙q_2+N=q_3+N ∈ P/N_{(Q∩P)}$, and hence $ab⊙M/N_{(Q)} ⊆ P/N_{(Q∩P)}$. Therefore $ab ∈ (P/N_{(Q∩P)} : M/N_{(Q)})$.

If $aq ∈ P$ or $bq ∈ P$, then by similar method of Theorem 3.1.13, we get either $aq ∈ P/N_{(Q∩P)}$ or $bq ∈ P/N_{(Q∩P)}$. Thus $P/N_{(Q∩P)}$ is a weakly 2-absorbing subsemimodule of $M/N_{(Q)}$.

(ii) Suppose that $N$, $P/N_{(Q∩P)}$ are weakly 2-absorbing subsemimodule of $M$, $M/N_{(Q)}$ respectively. Let $r,s ∈ R$, $m ∈ M$ such that $0≠rsm ∈ P$. We have two cases:-

Case I: If $0≠rsm ∈ N$, since $N$ is a weakly 2-absorbing subsemimodule of $M$. we have $rs ∈ (N : M)$ or $rm ∈ N$ or $sm ∈ N$. Thus $rs ∈ (P : M)$ or $rm ∈ N ⊆ P$ or $sm ∈ N ⊆ P$, since $rsM ⊆ N ⊆ P$ implies $rs ∈ (P : M)$. So $P$ is a weakly 2-absorbing subsemimodule of $M$.

Case II: Suppose $rsm ∈ P-N$. By using Lemma 1.2.27, then there exist a unique $q_1 ∈ Q$ such that $m ∈ q_1+N$ and $rsm ∈ rs⊙q_1+N=q_2+N$ such that $q_2$ is a unique element of $Q$.
such that $rsq_1+N \subseteq q_2+N$. Now $rsm \in P$, $rsm \in q_2+N$ implies $q_2 \in P$, as $P$ is a subtractive subsemimodule and $N \subseteq P$. Hence $q_0+N \neq rs\circ(q_1+N) = q_2+N \in P/N(Q \cap P)$. As $P/N(Q \cap P)$ is a weakly 2-absorbing subsemimodule $M/N(Q)$, then either $rs \in (P/N(Q \cap P) : M/N(Q))$ or $r\circ(q_1+N) \in P/N(Q \cap P)$ or $s\circ(q_1+N) \in P/N(Q \cap P)$.

If $r\circ(q_1+N) \in P/N(Q \cap P)$, then $rq_1 \in P$. Hence $rm \in r.(q_1+N) \subseteq P/N(Q \cap P)$. Similar if $s\circ(q_1+N) \in P/N(Q \cap P)$, then $sm \in P/N(Q \cap P)$.

Now assume that $rs \in (P/N(Q \cap P) : M/N(Q))$, then $rs\circ M/N(Q) \subseteq P/N(Q \cap P)$. Let $x \in M$ and by Lemma 1.2.27, there exist a unique $q_3 \in Q$ such that $x \in q_3+N$ and $rx \in rs\circ(q_3+N) = (q_4+N)$ where $q_4$ is a unique element of $Q$ such that $rsq_3+N \subseteq q_4+N$. Then there exist $y_3 \in N \subseteq P$ such that $r^s_3=q_4+y_3$, then $rsq_3 \in P$ and $P$ is a subtractive implies that $q_4 \in P$. Since $rsx \in q_4+N$ and $N \subseteq P$, then $rsx \in P$ so $rs \in (P : M)$. Thus $P$ is a weakly 2-absorbing subsemimodule of $M$.

**Theorem 3.2.12.** Let $R$ be a commutative semiring and $S$ be a multiplicatively closed subset of $R$ and $S^{-1}N$ be a subsemimodule of $S^{-1}R$-semimodule $S^{-1}M$. If $N$ is a weakly 2-absorbing subsemimodule of $M$ with $S^{-1}N \neq S^{-1}M$, then $S^{-1}N$ is a weakly 2-absorbing subsemimodule of $S^{-1}M$.

**Proof.** Assume that $a,b \in R$, $m \in M$ and $r,s,t \in S$ such that $0 \neq abm \in S^{-1}N$. Then there exists $s_1 \in S$ such that $0 \neq s_1abm \in N$. Since $N$ is a weakly 2-absorbing subsemimodule of $M$, then we have either $s_1ab \in (N : M)$ or $am \in N$ or $s_1bm \in N$. If $s_1ab \in (N : M)$, then $ab \in (S^{-1}N : S^{-1}_R S^{-1}M)$.

Again, if $am \in N$, then $abm = am \in S^{-1}N$. Similarly if $s_1bm \in N$, then $b \in S^{-1}_N M$. Therefore $S^{-1}N$ is a weakly 2-absorbing subsemimodule of $S^{-1}M$.

**Definition 3.2.13.** Let $N$ be a proper subsemimodule of $M$ and $a,b \in R$, $m \in M$. If $N$ is a weakly 2-absorbing subsemimodule and $abm=0$, $ab \notin (N : M)$, $am \notin N$ and $bm \notin N$, then $S^{-1}N$ is a weakly 2-absorbing subsemimodule of $S^{-1}M$. 

51
then $(a,b,m)$ is called a *triple-zero* of $N$.

**Theorem 3.2.14.** Let $N$ be a subtractive weakly 2-absorbing subsemimodule of $M$ and suppose that $(a,b,m)$ is a triple-zero of $N$ for some $a,b \in R$, $m \in N$. Then.

(i) $abN = 0$;

(ii) $a(N : M)m = b(N : M)m = 0$;

(iii) $(N : M)^2m = 0$;

(iv) $a(N : M)N = b(N : M)N = 0$.

**Proof.** (i) Assume that $abn \neq 0$ for some $n \in N$. Then $0 \neq abn + abm = ab(n+m) \in N$. Since $ab \notin (N : M)$, we have either $a(n+m) \in N$ or $b(n+m) \in N$. Since $N$ is a subtractive and $an \in N$ or $bn \in N$, then $am \in N$ or $bm \in N$, a contradiction. Thus $abN = 0$.

(ii) Let $axm \neq 0$ for some $x \in (N : M)$. Then $0 \neq a(b+x)m \in N$, $am \notin N$, we get either $a(b+x) \in (N : M)$ or $(b+x)m \in N$. Since $N$ is a subtractive with $ax \in (N : M)$ and $xm \in N$, then $ab \in (N : M)$ or $bm \in N$, a contradiction. Thus $a(N : M)m = 0$.

Similarly, $b(N : M)m = 0$.

(iii) To show that $(N : M)^2m = 0$, assume that on contrary $(N : M)^2m \neq 0$. Then there exist $r,s \in (N : M)$ such that $rsm \neq 0$. Then by part(ii) we get $(a+r)(b+s)m = rsm \neq 0$, clearly $0 \neq (a+r)(b+s)m \in N$. Then $(a+r)(b+s) \in (N : M)$ or $(a+r)m \in N$ or $(b+s)m \in N$. Hence by subtractively of $N$, we get either $ab \in (N : M)$ or $am \in N$ or $bm \in N$, which is a contradiction. Therefore $(N : M)^2m = 0$.

(iv) Assume that $axn \neq 0$ for some $x \in (N : M)$ and $n \in N$, therefore by part (i) and (ii) we conclude that $0 \neq a(b+x)(m+n) = axn \in N$. Hence $a(b+x) \in (N : M)$ or $a(m+n) \in N$ or $(b+x)(m+n) \in N$. Thus $ab \in (N : M)$ or $am \in N$ or $bm \in N$ (as $N$, $(N : M)$ are subtractive). Therefore $a(N : M)N = 0$.

Similarly, $b(N : M)N = 0$. \qed
Now, the following theorem provides a condition under which the concept of 2-absorbing subsemimodules and weakly 2-absorbing subsemimodules of $M$ are identical.

**Theorem 3.2.15.** Let $R$ be a commutative semiring, $M$ be an $R$-semimodule and $N$ be a weakly 2-absorbing subtractive subsemimodule of $M$, Then either $(N : M)^2 N = 0$ or $N$ is a 2-absorbing subsemimodule of $M$.

*Proof.* Assume that $N$ is not a 2-absorbing subsemimodule of $M$, then $N$ has a triple-zero of $N$ $(a,b,m)$ for some $a,b \in R$, $m \in M$. We claim that $(N : M)^2 N = 0$, let $rst \neq 0$ for some $r,s \in (N : M)$ and $t \in N$. Then by Theorem 3.2.14, we have $(a+r)(b+s)(m+t)=rst \neq 0$, since $N$ is a weakly 2-absorbing subsemimodule of $M$. Therefore, $(a+r)(b+s) \in (N : M)$ or $(a+r)(m+t) \in N$ or $(b+s)(m+t) \in N$. Hence $ab \in (N : M)$ or $am \in N$ or $bm \in N$ (as $N$, $(N : M)$ are subtractive), a contradiction. Thus $(N : M)^2 N = 0$. \qed

The following example shows that the converse of the previous theorem does not necessarily hold.

**Example 3.2.16.** Let $R=(\mathbb{Z}^0, +, \cdot)$ and $M=\mathbb{Z}^0 \times \mathbb{Z}^0$ be an $R$-semimodule. Consider the subsemimodule $N=\{0\} \times 12\mathbb{Z}^0$ of $M$. Then $(N : M)=0$ and $(N : M)^2 N = 0$. Now $2.2.(0,3) \in N$ but $2.2 \notin (N : M)$ and $(0,6) \notin N$. Thus $N$ is not weakly 2-absorbing subsemimodule of $M$.

**Corollary 3.2.17.** If $N$ is a subtractive weakly 2-absorbing subsemimodule of an $R$-semimodule $M$ but it is not a 2-absorbing subsemimodule of $M$, then.

(i) $N$ is nilpotent; (ii) $\sqrt{(N : M)}=\sqrt{\text{Ann}_R(M)}$.

*Proof.* (i) By Theorem 3.2.15 and definition of nilpotent subsemimodule, we get $N$ is nilpotent.

(ii) Since $N$ is not 2-absorbing subsemimodule, then by Theorem 3.2.15 $(N : M)^2 N = 0$.  

53
Clearly $\sqrt{(0 : M)} \subseteq \sqrt{(N : M)}$. Now we show the reverse enclosure $(N : M)^3=(N : M)^2 (N : M) \subseteq ((N : M)N : M)=(0 : M)$ and so $(N : M) \subseteq \sqrt{(0 : M)}$. Therefore $\sqrt{(N : M)}=\sqrt{(0 : M)}=\sqrt{\text{Ann}_R(M)}$.

**Corollary 3.2.18.** Let $R$ be a commutative semiring and $M$ a faithful multiplication $R$-semimodule. Let $N$ be a subtractive weakly 2-absorbing subsemimodule of $M$. If $N$ is not 2-absorbing, then $N \subseteq \sqrt{0} M$.

**Proof.** By part (ii) in previous corollary, $(N : M)^3 \subseteq (0 : M)=0$. Since $M$ is faithful, so that $(N : M)^3=0$. If $x \in (N : M)$, then $x^2=0$ and hence $x \in \sqrt{0}$. So we have $(N : M) \subseteq \sqrt{0}$ and therefore $N=(N : M)M \subseteq \sqrt{0}M$.

Consider $R=R_1 \times R_2$ where each $R_i$ is a commutative semiring with identity, let $M_i$ be an $R_i$-semimodule where $i=1,2$ and let $M=M_1 \times M_2$ be the $R$-semimodule with $(r_1,r_2)(m_1,m_2)=(r_1m_1,r_2m_2)$ for all $r_i \in R_i$, $m_i \in M_i$, $i=1,2$.

**Theorem 3.2.19.** If $N$ is a proper $R_1$-subsemimodule of $M_1$ and $M_1 \neq 0$. Then the following statements are equivalent.

(i) $N$ is a 2-absorbing $R_1$-subsemimodule of $M_1$;

(ii) $N \times M_2$ is a 2-absorbing $R$-subsemimodule of $M=M_1 \times M_2$;

(iii) $N \times M_2$ is a weakly 2-absorbing $R$-subsemimodule of $M=M_1 \times M_2$.

**Proof.** (i) $\implies$ (ii) Let $(r_1,r_2)(s_1,s_2)(m_1,m_2) \in N \times M_2$ such that $(r_1,r_2),(s_1,s_2) \in R$ and $(m_1,m_2) \in M$. Then $(r_1s_1m_1,r_2s_2m_2) \in N \times M_2$, therefore $r_1s_1m_1 \in N$ . Since $N$ is a 2-absorbing subsemimodule of $M_1$, then either $r_1s_1 \in (N : M_1)$ or $r_1m_1 \in N$ or $s_1m_1 \in N$. If $r_1m_1 \in N$, then $(r_1,r_2)(m_1,m_2) \in N \times M_2$. Similarly, if $s_1m_1 \in N$, then $(s_1,s_2)(m_1,m_2) \in N \times M_2$.

Again, if $r_1s_1 \in (N : M_1)$, then $(r_1,r_2)(s_1,s_2) \in (N \times M_2 : M)$ implies $N \times M_2$ is a
2-absorbing R-subsemimodule of M.

(ii) \( \implies \) (iii) It is obvious.

(iii) \( \implies \) (i) Let \( a, b \in R_1, x \in M_1 \) such that \( abx \in N \). Let \( 0 \neq m \in M_2 \) and \( (0,0) \neq (a,1)(b,1)(x,m) \in N \times M_2 \) but \( N \times M_2 \) is a weakly 2-absorbing subsemimodule of \( M \), we get either \( (a,1)(b,1) \in (N \times M_2 : M) \) or \( (a,1)(x,m) \in N \times M_2 \) or \( (b,1)(x,m) \in N \times M_2 \). Then \( (ab,1) \in (N \times M_2 : M) \) or \( (ax,m) \in N \times M_2 \) or \( (bx,m) \in N \times M_2 \). If \( (ab,1) \in (N \times M_2 : M) \), then \( (ab,1)M \subseteq N \times M_2 \implies abM_1 \subseteq N \implies ab \in (N : M_1) \). Also, if \( (ax,m) \in N \times M_2 \), therefore \( ax \in N \). Similarly, for \( (bx,m) \in N \times M_2 \) we have \( bx \in N \).

Thus \( N \) is a 2-absorbing subsemimodule of \( M_1 \).

\[ \Box \]

3.3 Relation Between Some Subsemimodules and 2-Absorbing, Weakly 2-Absorbing Subsemimodules

In this section, we study the concept of relation between some subsemimodules and 2-absorbing, weakly 2-absorbing subsemimodules.

**Definition 3.3.1.** [19] Let \( M \) be a semimodule over a commutative semiring \( R \) and \( N \) be a proper subsemimodule of \( M \). Then \( N \) is said to be a **2-absorbing primary subsemimodule** of \( M \) if whenever \( abm \in N \) where \( a, b \in R, m \in M \), then \( ab \in \sqrt{(N : M)} \) or \( am \in N \) or \( bm \in N \).

**Definition 3.3.2.** Let \( M \) be a semimodule over a commutative semiring \( R \) and \( N \) be a proper subsemimodule of \( M \). Then \( N \) is said to be a **weakly 2-absorbing primary subsemimodule** of \( M \) if whenever \( 0 \neq abm \in N \) where \( a, b \in R, m \in M \), then \( ab \in \sqrt{(N : M)} \) or \( am \in N \) or \( bm \in N \).

Clearly every 2-absorbing primary subsemimodule is a weakly 2-absorbing primary
subsemimodule, the following example show that a 2-absorbing primary subsemimodule need not be a weakly 2-absorbing primary subsemimodule.

**Example 3.3.3.** Let $R=(\mathbb{Z}_{12}, +_{12})$ and $M=\mathbb{Z}_{12} \times \mathbb{Z}_{12}$ and take $N=(0,0)$ is a subsemimodule of $M$. Then $(N : M)=12\mathbb{Z}^\circ$ and $\sqrt{(N : M)}=6\mathbb{Z}^\circ$. Clearly $N$ is a weakly 2-absorbing primary subsemimodule of $M$ but not 2-absorbing primary subsemimodule, since $2.(3,3) \in N$, $2.2 \notin \sqrt{(N : M)}$ and $2.(3,3) \notin N$.

**Lemma 3.3.4.** Let $R$ be a commutative semiring. $M$ be an $R$-semimodule and $N$ be a subsemimodule of $M$.

(i) Every primary subsemimodule is 2-absorbing primary ;

(ii) Every 2-absorbing subsemimodule is 2-absorbing primary ;

(iii) Every weakly primary subsemimodule is weakly 2-absorbing primary ;

(iv) Every 2-absorbing primary subsemimodule is weakly 2-absorbing primary.

**Proof.** (i) Suppose that $N$ is primary subsemimodule of $M$. Let $a,b \in R$, $m \in M$ with $(ab)m \in N$. Since $N$ is primary, then $ab \in \sqrt{(N : M)}$ or $m \in N$. If $m \in N$, $a,b \in R$, we have $am \in N$ or $bm \in N$. Therefore $N$ is a 2-absorbing primary subsemimodule of $M$.

(ii) Let $N$ be a 2-absorbing subsemimodule of $M$. Consider $a,b \in R$, $x \in M$ such that $abx \in N$. Since $N$ is a 2-absorbing, we have either $ab \in (N : M)$ or $am \in N$ or $bm \in N$. Now, since $(N : M) \subseteq \sqrt{(N : M)}$, we get $ab \in \sqrt{(N : M)}$. Thus $N$ is a 2-absorbing primary subsemimodule of $M$.

(iii) Similar to part(ii).

(iv) Trivial by definition. 

**Remark 3.3.5.** [19] (1) Let $R$ be a commutative semiring, $N$ be a 2-absorbing subsemimodule of an $R$-semimodule $M$. By above lemma, every 2-absorbing subsemimodule is a 2-absorbing primary subsemimodule, but the converse need not be true. Consider
R=(\mathbb{Z}^\circ,+,\cdot) \text{, } M=\mathbb{Z}^\circ_{16}=\{0,1,2,...,11\}. \text{ Let a subsemimodule } N=(0,8) \text{ generated by } 8. \text{ Then } (N : M)=\langle 8 \rangle \text{ and } \sqrt{(N : M)}=\langle 2 \rangle. \text{ Now, } 2.2.2 \in N \text{ but } 2.2 \notin N \text{ and } 2.2 \notin (N : M). \text{ Therefore } N \text{ is not a } 2\text{-absorbing subsemimodule of } M, \text{ but it is a } 2\text{-absorbing primary subsemimodule of } M.

(2) Every primary subsemimodule of } M \text{ is a } 2\text{-absorbing primary subsemimodule but the converse need not be true. For example, Let } R=(\mathbb{Z}^\circ,+,\cdot) \text{ is a commutative semiring and } M=\mathbb{Z}^\circ \times \mathbb{Z}^\circ \text{ be a semimodule over } R. \text{ Take } N=\{0\} \times 4\mathbb{Z}^\circ \text{ be a subsemimodule of } M, \text{ then } (N : M)=\{0\} \text{ and } \sqrt{(N : M)}=\{0\}. \text{ Clearly } N \text{ is a } 2\text{-absorbing primary subsemimodule of } M \text{ but } N \text{ is not primary subsemimodule of } M, \text{ since } 2.(0,2) \in N \text{ but neither } 2 \in \sqrt{(N : M)} \text{ nor } (0,2) \in N.

**Theorem 3.3.6.** Let } R \text{ be a commutative semiring and } M \text{ be an } R\text{-semimodule and } N \text{ is a prime subsemimodule of } M. \text{ Then the following statements are equivalent.}

(i) } N \text{ is a } 2\text{-absorbing subsemimodule of } M; \text{} 

(ii) } N \text{ is a } 2\text{-absorbing primary subsemimodule of } M.

**Proof.** (i) \implies (ii) \text{ By Lemma 3.3.4 as required.} 

(ii) \implies (i) \text{ Since } N \text{ is prime subsemimodule, then } (N : M) \text{ is prime ideal of } R \text{ by Proposition 1.2.15. Let } N \text{ be a } 2\text{-absorbing primary subsemimodule of } M. \text{ Suppose } abm \in N \text{ such that } a,b \in R \text{ and } m \in M. \text{ Since } N \text{ is } 2\text{-absorbing primary, we have } ab \in \sqrt{(N : M)} \text{ or } am \in N \text{ or } bm \in N. \text{ Since } (N : M) \text{ is prime ideal, then } (N : M)=\sqrt{(N : M)}. \text{ Thus } ab \in (N : M) \text{ or } am \in N \text{ or } bm \in N. \text{ Therefore, } N \text{ is a } 2\text{-absorbing subsemimodule of } M. \quad \Box

**Corollary 3.3.7.** Let } R \text{ be a commutative semiring and } M \text{ be a faithful cyclic } R\text{-semimodule and } N \text{ be a prime subsemimodule of } M. \text{ Then } N \text{ is a weakly } 2\text{-absorbing subsemimodule of } M \text{ if and only if } N \text{ is a weakly } 2\text{-absorbing primary subsemimodule of } M.

**Proof.** (\implies) \text{ By Lemma 3.3.4 as required.}
Clearly, By Proposition 1.2.15, we have $(N : M)$ is a prime ideal of $R$. Let $N$ be a weakly 2-absorbing primary subsemimodule of $M$. Suppose $0 \neq abm \in N$ such that $a, b \in R$ and $m \in M$. Since $N$ is a weakly 2-absorbing primary, we have $ab \in \sqrt{(N : M)}$ or $am \in N$ or $bm \in N$. Since $(N : M)$ is a prime ideal, then $(N : M) = \sqrt{(N : M)}$. Thus $ab \in (N : M)$ or $am \in N$ or $bm \in N$. Thus $N$ is a weakly 2-absorbing subsemimodule of $M$. \qed

The following result study the conditions that make the following concepts similar: 2-absorbing, weakly 2-absorbing, 2-absorbing primary and weakly 2-absorbing primary subsemimodules of an $R$-semimodules for a semiring $R$.

**Theorem 3.3.8.** Let $R$ be a commutative semiring and $M$ be a faithful cyclic $R$-semimodule and $N$ be a prime and subtractive subsemimodule of $M$ with $(N : M)^2 N \neq 0$. Then the following statements are equivalent.

(i) $N$ is a 2-absorbing subsemimodule of $M$;
(ii) $N$ is a weakly 2-absorbing subsemimodule of $M$;
(iii) $N$ is a 2-absorbing primary subsemimodule of $M$;
(iv) $N$ is a weakly 2-absorbing primary subsemimodule of $M$.

**Proof.** (i) $\iff$ (iii) By Theorem 3.3.6 and (ii) $\iff$ (iv) By Corollary 3.3.7.

Clearly, (i) $\implies$ (ii) As Proposition 3.2.2.

Now we want to show the reverse inclusion. Since $(N : M)^2 N \neq 0$, then by Theorem 3.2.15 and $N$ is a weakly 2-absorbing subsemimodule of $M$, then $N$ is a 2-absorbing subsemimodule of $M$.

Thus (i) $\iff$ (ii) $\iff$ (iii) $\iff$ (iv). \qed

Now the following theorem study the relationship between irreducible and 2-absorbing subtractive subsemimodule of $M$. 58
Theorem 3.3.9. Let $N$ be a proper subtractive subsemimodule of an $R$-semimodule $M$, $\left(\sqrt{(N : M)}\right)^2 \subseteq (N : M)$. If $N$ is an irreducible subsemimodule of $M$, then $N$ is a 2-absorbing subsemimodule of $M$ if and only if $(N : r) = (N : r^2)$ for all $r \in R - \sqrt{(N : M)}$.

Proof. ($\implies$) Let $N$ be a 2-absorbing subsemimodule of $M$ and let $r \in R - \sqrt{(N : M)}$. We will show that $(N : r) = (N : r^2)$. Clearly $(N : r) \subseteq (N : r^2)$. Let $a \in (N : r^2)$, so $r^2a \in N$. Then either $r^2 \in (N : M)$ or $ra \in N$. Now, $r^2 \notin (N : M)$ because if $r^2 \in (N : M)$, then $r \in \sqrt{(N : M)}$, a contradiction. Then $ra \in N$ and hence $a \in (N : r)$. Thus $(N : r) = (N : r^2)$ for all $r \in R - \sqrt{(N : M)}$.

($\impliedby$) Let $r_1, r_2 \in R$ and $m \in M$ such that $r_1r_2m \in N$ and $r_1r_2 \notin (N : M)$. We show that either $r_1m \in N$ or $r_2m \in N$. Since $r_1r_2 \notin (N : M)$ we have, $r_1 \notin \sqrt{(N : M)}$ or $r_2 \notin \sqrt{(N : M)}$, because if $r_1 \in \sqrt{(N : M)}$ and $r_2 \in \sqrt{(N : M)}$, then $r_1r_2 \in (\sqrt{(N : M)})^2 \subseteq (N : M)$, a contradiction. Now by assumption, we get either $(N : r_1) = (N : r_2^2)$ or $(N : r_2) = (N : r_1^2)$. Assume on contrary, $r_1m \notin N$ and $r_2m \notin N$, then clearly $N \subseteq (N + Rr_1m) \cap (N + Rr_2m)$. Let $y \in (N + Rr_1m) \cap (N + Rr_2m)$. Then there exist $n_1, n_2 \in N$ and $t_1, t_2 \in R$ such that $y = n_1 + t_1r_1m = n_2 + t_2r_2m$. Thus $r_1y = r_1n_1 + t_1r_1^2m = r_1n_2 + t_2r_1r_2m \in N$, since $r_1r_2m \in N$, therefore $t_1r_1^2m \in N$ (as $N$ is subtractive). Hence $t_1m \in (N : r_1^2)$ and since $(N : r_1) = (N : r_1^2)$, $t_1m \in (N : r_1)$ this implies that $t_1r_1m \in N$. Therefore $y = n_1 + t_1r_1m \in N$. This show that $(N + Rr_1m) \cap (N + Rr_2m) \subseteq N$. Consequently, $(N + Rr_1m) \cap (N + Rr_2m) = N$, a contradiction, since $N$ is irreducible subsemimodule. Thus $N$ is a 2-absorbing subsemimodule of $M$.

\par

Theorem 3.3.10. Let $R$ be a regular semiring and $M$ be an $R$-semimodule and $N$ is a subsemimodule of $M$. If $N$ is an irreducible subsemimodule of $M$, then $N$ is a 2-absorbing subsemimodule of $M$.

Proof. Let $R$ be a regular semiring and $N$ be an irreducible subsemimodule of $R$. Let $a, b$
∈ R, x∈ M such that abx ∈ N and ab ∉ (N : M), then we have to show that ax ∈ N or bx ∈ N. Suppose that ax ∉ N and bx ∉ N. Then K=N+Rax and L=N+Rbx are two subsemimodules of the R-semimodule M properly contain N. Since N is irreducible, then N≠K∩L. Thus there exist y ∈ R such that y ∈ (N+Rax)∩(N+Rbx) and y ∉ N. Also by regularity of R, we have K∩L=KL, we have y ∈ (N+Rax)∩(N+Rbx). Then there exist \( n_1,n_2 \in N \) and \( r_1,r_2 \in R \) such that 
\[
y = (n_1+r_1ax)(n_2+r_2bx) = n_1n_2 + n_1r_2bx + n_2r_1ax + r_1r_2abx^2 \in N.
\]
This means that y ∈ N, a contradiction. Therefore N is a 2-absorbing subsemimodule of M.

\( \Box \)
Chapter 4

2-Absorbing Compactly Packed in Semirings

Let $R$ be a commutative semiring with identity and $M$ be a unitary $R$-semimodule. A proper ideal $I$ of $R$ is compactly packed if for each family $\{I_\alpha\}_{\alpha \in \Delta}$ of prime ideals of $R$ with $I \subseteq \bigcup_{\alpha \in \Delta} I_\alpha$, $I \subseteq I_\beta$ for some $\beta \in \Delta$.

A semiring is called compactly packed (CP) if every proper ideal of $R$ is compactly packed (CP). This concept was introduced for rings by Pakala and Shores (see [28]). Also Abu Oda in [2] introduced the concept of 2-absorbing compactly packed modules in rings. In this chapter, we introduce the concepts of 2-absorbing compactly packed ideals and 2-absorbing compactly packed semimodules in semirings.

4.1 2-Absorbing Compactly Packed Ideals

In this section, we define and investigate the concept of 2-absorbing compactly packed ideals in semirings.
**Definition 4.1.1.** Let $R$ be a commutative semiring. A proper ideal $I$ of $R$ is called *2-absorbing compactly packed* (briefly: 2-abs.CP) if for each family $\{I_\alpha\}_{\alpha \in \Delta}$ of 2-absorbing ideals of $R$ with $I \subseteq \bigcup_{\alpha \in \Delta} I_\alpha$, $I \subseteq I_\beta$ for some $\beta \in \Delta$. A semiring $R$ is called (2-abs.CP) semiring if every proper ideal of $R$ is (2-abs.CP).

**Definition 4.1.2.** Let $R$ be a commutative semiring. A proper ideal $I$ of $R$ is called *2-absorbing finitely compactly packed* (briefly: 2-abs.FCP) if for each family $\{I_\alpha\}_{\alpha \in \Delta}$ of 2-absorbing ideals of $R$ with $I \subseteq \bigcup_{\alpha \in \Delta} I_\alpha$, there exist $\alpha_1, \alpha_2, \ldots, \alpha_n$ in $\Delta$ such that $I \subseteq \bigcup_{i=1}^{n} I_{\alpha_i}$. A semiring $R$ is called (2-abs.FCP) semiring if every proper ideal of $R$ is (2-abs.FCP).

**Proposition 4.1.3.** every 2-abs.CP (resp. 2-abs.FCP) is CP (resp. FCP).

**Proof.** Since every prime ideal is 2-absorbing ideal, then the proof is trivial by definitions. \qed

**Remark 4.1.4.** Let $R$ be a commutative semiring. Clearly from the definition, every 2-abs.CP is 2-abs.FCP, but the converse is not true. For example, let $R=\mathbb{Z}^o_2 \times \mathbb{Z}^o_2 \times \mathbb{Z}^o_2$ where $\mathbb{Z}^o_2=\{0, 1\}$ and let $e_1=\{(1, 0, 0)\}$, $e_2=\{(0, 1, 0)\}$. Define $I_1=\{0, e_1\}$, $I_2=\{0, e_2\}$ and $I_3=\{0, e_1+e_2\}$. Consider $I=I_1+I_2=\{0, e_1, e_2, e_1+e_2\}$. Clearly $I$ is a proper ideal of $R$ such that $I=I_1 \cup I_2 \cup I_3$. Also, $I_1, I_2$ and $I_3$ are 2-absorbing ideals of $R$ with $I \subseteq \bigcup_{j=1}^{3} I_j$, but $I \nsubseteq I_j$ for all $j \in \{1, 2, 3\}$. Therefore I is 2-abs.FCP, but I is not 2-abs.CP.

**Proposition 4.1.5.** Let $R$ be a commutative semiring. Then every cyclic ideal of $R$ is a 2-abs.CP. ideal of $R$.

**Proof.** Let $I$ be a proper ideal of $R$ such that $I \subseteq \bigcup_{\alpha \in \Delta} I_\alpha$ where $\{I_\alpha\}_{\alpha \in \Delta}$ is a family of 2-absorbing ideals of $R$. Also, $I=\langle a \rangle$ for some $a \in I$, since I is cyclic. Since $a \in I \subseteq \bigcup_{\alpha \in \Delta} I_\alpha$, therefore $a \in I_\beta$ for some $\beta \in \Delta$. Thus $I=\langle a \rangle \subseteq I_\beta$ for some $\beta \in \Delta$ and hence $R$ is 2-abs.CP. \qed
Definition 4.1.6. A semiring $R$ is a principal ideal semiring if every ideal of $R$ is cyclic.

Corollary 4.1.7. If $R$ is a principal ideal of commutative semiring, then $R$ is 2-abs.CP semiring.

Theorem 4.1.8. Let $f: R \rightarrow S$ be an epimorphism of a commutative semiring. If $R$ is 2-abs.CP (resp. 2-abs.FCP) then so is $S$.

Proof. Let $R$ be a 2-abs.CP and let $I$ be a proper ideal of $S$ such that $I \subseteq \bigcup_{\alpha \in \Delta} I_\alpha$, where $I_\alpha$ is a 2-absorbing ideal of $S$ for each $\alpha \in \Delta$. Since $f$ is an epimorphism, therefore $f^{-1}(I) \subseteq f^{-1}\left(\bigcup_{\alpha \in \Delta} I_\alpha\right)$ and thus $f^{-1}(I) \subseteq \bigcup_{\alpha \in \Delta} f^{-1}(I_\alpha)$. Since $I_\alpha$ is a 2-absorbing ideal of $S$ for each $\alpha \in \Delta$, then by Theorem 2.1.12 we have, $f^{-1}(I_\alpha)$ is a 2-absorbing ideal of $R$ for each $\alpha \in \Delta$. Since $R$ is 2-abs.CP semiring, then there exists $\beta \in \Delta$ such that $f^{-1}(I_\alpha) \subseteq f^{-1}(I_\beta)$. Hence $I \subseteq I_\beta$ for some $\beta \in \Delta$, and therefore $I$ is 2-abs.CP ideal. Thus $S$ is 2-abs.CP semiring. \qed

Theorem 4.1.9. Let $f: R \rightarrow S$ be an epimorphism of a commutative semirings $R$ and $S$ such that $f(0)=0$. If any 2-absorbing ideal of $R$ is a subtractive strong ideal that contains $\ker(f)$, then $S$ is 2-abs.CP if and only if $R$ is 2-abs.CP.

Proof. $\Leftarrow$ By Theorem 4.1.8 the direction holds.

$\Rightarrow$ Let $S$ be a 2-abs.CP semiring and $I$ be a proper ideal of $R$ such that $I \subseteq \bigcup_{\beta \in \lambda} I_\beta$ where $I_\beta$ is a 2-absorbing ideal of $R$ for each $\beta \in \lambda$. Therefore $f(I) \subseteq f\left(\bigcup_{\beta \in \lambda} I_\beta\right)$ and thus $f(I) \subseteq \bigcup_{\beta \in \lambda} f(I_\beta)$. Since for all $\beta \in \lambda$, $I_\beta$ is a subtractive strong ideal that contains $\ker(f)$, then by Theorem 2.1.14 we get $f(I_\beta)$ is a 2-absorbing ideal of $S$ for each $\beta \in \lambda$. Now, since $S$ is a 2-abs.CP semiring, then there exists $\gamma \in \lambda$ such that $f(I) \subseteq f(I_\gamma)$ we prove that $I \subseteq I_\gamma$ for some $\gamma \in \lambda$ to show that $R$ is 2-abs.CP. Now let $i \in I$ then $f(i) \in f(I) \subseteq f(I_\gamma)$ so there exists $j_1 \in I_\gamma$ such that $f(i) = f(j_1)$. Since $I_\gamma$ is a strong ideal, then there exists $j_2 \in I_\gamma$ such that...
\( j_1 + j_2 = 0 \) which gives \( f(j_1 + j_2) = 0 \). Therefore \( f(i + j_2) = f(i) + f(j_2) = f(j_1) + f(j_2) = f(j_1 + j_2) = 0 \), this implies that \( i + j_2 \in \ker(f) \subseteq I_\gamma \), since \( j_2 \in I_\gamma \) and \( I_\gamma \) is a subtractive ideal, we have \( i \in I_\gamma \). Hence \( I \subseteq I_\gamma \). Thus \( I \) is 2-abs.CP ideal and therefore \( R \) is 2-abs.CP semiring. \( \square \)

### 4.2 2-Absorbing Compactly Packed Semimodules

In this section, we investigate the concept of 2-absorbing compactly packed semimodule.

Now, we define the following definitions.

**Definition 4.2.1.** Let \( M \) be an \( R \)-semimodule. A proper subsemimodule \( N \) of \( M \) is called *compactly packed* (CP) if for each family \( \{P_\alpha\}_{\alpha \in \Delta} \) of prime subsemimodules of \( M \) with \( N \subseteq \bigcup_{\alpha \in \Delta} P_\alpha \), \( N \subseteq P_\beta \) for some \( \beta \in \Delta \). A semimodule \( M \) is called (CP) if every proper subsemimodule of \( M \) is (CP).

**Definition 4.2.2.** Let \( M \) be an \( R \)-semimodule. A proper subsemimodule \( N \) of \( M \) is called *2-absorbing compactly packed* (2-abs.CP) if for each family \( \{P_\alpha\}_{\alpha \in \Delta} \) of 2-absorbing subsemimodules of \( M \) with \( N \subseteq \bigcup_{\alpha \in \Delta} P_\alpha \), \( N \subseteq P_\beta \) for some \( \beta \in \Delta \). A semimodule \( M \) is called (2-abs.CP) if every proper subsemimodule of \( M \) is (2-abs.CP).

**Definition 4.2.3.** Let \( M \) be an \( R \)-semimodule. A proper subsemimodule \( N \) of \( M \) is called *2-absorbing finitely compactly packed* (2-abs.FCP) if for each family \( \{P_\alpha\}_{\alpha \in \Delta} \) of 2-absorbing subsemimodule of \( M \) with \( N \subseteq \bigcup_{\alpha \in \Delta} P_\alpha \), there exist \( \alpha_1, \alpha_2, \ldots, \alpha_n \) in \( \Delta \) such that \( N \subseteq \bigcup_{i=1}^n P_{\alpha_i} \). A semimodule \( M \) is called (2-abs.FCP) if every proper subsemimodule of \( M \) is (2-abs.FCP).

**Proposition 4.2.4.** Every 2-abs.CP (resp. 2-abs.FCP) semimodule is CP (resp. FCP)

**Proof.** Directly by the previous two definitions and Proposition 3.2.2. \( \square \)
Remark 4.2.5. It is clear from the definitions that every 2-abs.CP semimodule is 2-abs.FCP semimodule but the converse is not true in general see Remark 4.1.4

**Proposition 4.2.6.** Let $M$ be an $R$-semimodule. If every subsemimodule of $M$ is cyclic, then $M$ is CP.

*Proof.* Let $N$ be a proper subsemimodule of $M$ and \( \{N_\alpha\}_{\alpha \in \Delta} \) be a family of prime subsemimodule of $M$ such that $N \subseteq \bigcup_{\alpha \in \Delta} N_\alpha$. Since $N$ is cyclic, then $N=Ra$ for some $a \in N$. Since $a \in N \subseteq \bigcup_{\alpha \in \Delta} N_\alpha$, therefore $a \in N_\beta$ for some $\beta \in \Delta$. Thus $N=Ra \subseteq N_\beta$ and hence $M$ is CP. \( \square \)

**Corollary 4.2.7.** Let $M$ be an $R$-semimodule. If every subsemimodule of $M$ is cyclic, then $M$ is 2-abs.CP.

**Theorem 4.2.8.** Let $f \colon M \longrightarrow M'$ be an epimorphism $R$-semimodule. If $M$ is 2-abs.CP (resp. 2-abs.FCP) then so is $M'$.

*Proof.* Let $M$ be a 2-abs.CP semimodule and $N'$ be a proper subsemimodule of $M'$. Let $N' \subseteq \bigcup_{\alpha \in \Delta} P_\alpha$, where $P_\alpha$ is a 2-absorbing subsemimodule of $M'$ for each $\alpha \in \Delta$. Therefore $f^{-1}(N) \subseteq f^{-1}(\bigcup_{\alpha \in \Delta} P_\alpha)$. Since $f$ is an epimorphism, then $f^{-1}(N) \subseteq \bigcup_{\alpha \in \Delta} f^{-1}(P_\alpha)$. Since $P_\alpha$ is a 2-absorbing subsemimodule of $M'$ for each $\alpha \in \Delta$, by Theorem 3.1.10 we have, $f^{-1}(P_\alpha)$ is a 2-absorbing subsemimodule of $M$ for each $\alpha \in \Delta$. Then there exists $\beta \in \Delta$ such that $f^{-1}(N') \subseteq f^{-1}(P_\beta)$. Since $M$ is a 2-abs.CP semimodule and $f^{-1}(N') \subseteq f^{-1}(P_\beta)$, therefore $N' \subseteq P_\beta$ for some $\beta \in \Delta$. Thus $N'$ is 2-abs.CP. Therefore, $M'$ is 2-abs.CP. \( \square \)

**Theorem 4.2.9.** Let $f \colon M \longrightarrow M'$ be an epimorphism $R$-semimodule such that any 2-absorbing subsemimodule of $M$ is a subtractive strong subsemimodule that contains $\ker(f)$, and let $f(0)=0$. If $M'$ is 2-abs.CP, then so is $M$. 

65
Proof. Suppose that $M'$ is a 2-abs.CP. Let $N$ be a subsemimodule of $M$ such that $N \subseteq \bigcup_{\alpha \in \Delta} P_{\alpha}$ where $P_{\alpha}$ is a 2-absorbing subsemimodules of $M$ for each $\alpha \in \Delta$. So $f(N) \subseteq f(\bigcup_{\alpha \in \Delta} P_{\alpha})$ and then we have, $f(N) \subseteq \bigcup_{\alpha \in \Delta} f(P_{\alpha})$. Since for all $\alpha \in \Delta$, $P_{\alpha}$ is a subtractive strong subsemimodule that contains $\text{ker}(f)$, then by Theorem 3.1.11 we have $f(P_{\alpha})$ is a 2-absorbing subsemimodule of $M'$ for each $\alpha \in \Delta$. Therefore there exists $\beta \in \Delta$ such that $f(N) \subseteq f(P_{\beta})$, since $M'$ is 2-abs.CP. Now, we want to show that $N \subseteq P_{\beta}$ for some $\beta \in \Delta$. Let $n \in N$, then $f(n) \in f(N) \subseteq f(P_{\beta})$. So there exists $k \in P_{\beta}$ such that $f(n)=f(k)$. Since $P_{\beta}$ is a strong subsemimodule, then there exist $k' \in P_{\beta}$ such that $k+k'=0$ which gives $f(k+k')=0$. Thus $f(n+k')=f(k+k')=0$, and hence $n+k' \in \text{ker}(f) \subseteq P_{\beta}$. Since $k' \in P_{\beta}$ and $P_{\beta}$ is a subtractive subsemimodule, we have $n \in P_{\beta}$. This implies that $N \subseteq P_{\beta}$. Therefore $N$ is 2-abs.CP and thus $M$ is 2-abs.CP.

In [8] Sharma and Bhatwedekar give a definition of S-component of a module. This leads us to give a definition of S-component of a semimodule.

Definition 4.2.10. Let $M$ be an $R$-semimodule, and let $S$ be a multiplicatively closed subset of $R$. An S-component of $M$ is denoted by $M_s$ and defined as $M_s=\{a : a \in R \text{ and } as \in M \text{ for some } s \in S\}$.

Proposition 4.2.11. Let $M$ be an $R$-semimodule and $S$ be a multiplicatively closed subset of $R$. If $M$ is 2-abs.CP (resp. 2-abs.FCP) then so is $M_s$.

Proof. Let $N$ be a proper subsemimodule of $M_s$ such that $N \subseteq \bigcup_{\alpha \in \Delta} P_{\alpha}$ where $P_{\alpha}$ is a 2-absorbing subsemimodule of $M_s$ for each $\alpha \in \Delta$. Define $\phi: M \rightarrow M_s$ as follow $\phi(m)=\frac{m}{1}$ for any $m \in M$. Thus $\phi$ is an epimorphism, so $\phi^{-1}(N) \subseteq \bigcup_{\alpha \in \Delta} \phi^{-1}(P_{\alpha})$ for each $\alpha \in \Delta$. Since $P_{\alpha}$ is a 2-absorbing subsemimodule of $M_s$ and $\phi$ is an epimorphism, by Theorem 3.1.10 we have $\phi^{-1}(P_{\alpha})$ is a 2-absorbing subsemimodule of $M$ for each $\alpha \in \Delta$. Also, since $M$ is 2-abs.CP, then $\phi^{-1}(N) \subseteq \phi^{-1}(P_{\beta})$ for some $\beta \in \Delta$. Thus $(\phi^{-1}(N))_s \subseteq (\phi^{-1}(P_{\beta}))_s$.
Now, we want to show that \((\phi^{-1}(K))_s = K\) for any subsemimodule \(K\) of \(M_s\). Let \(\frac{x}{s} \in (\phi^{-1}(K))_s\) such that \(x \in \phi^{-1}(K)\) and \(s \in S\) then \(\phi(x) \in K\). Therefore \(\frac{x}{1} \in K\), hence \(\frac{x}{s} = \frac{1}{s} \cdot \frac{x}{1} \in K\).

Thus \((\phi^{-1}(K))_s \subseteq K\) ...... (1)

On the other hand, let \(\frac{x}{s} \in K\) then \(\frac{1}{s} \cdot \frac{x}{1} \in K\) and \(\frac{x}{1} \in K\), \(\phi(x) \in K\) and so \(x \in \phi^{-1}(K)\) and \(\frac{x}{s} \in (\phi^{-1}(K))_s\).

Thus \(K \subseteq (\phi^{-1}(K))_s\) ...... (2)

Hence from (1) and (2) we get \((\phi^{-1}(K))_s = K\) for any subsemimodule \(K\) of \(M_s\). Now, for some \(\beta \in \Delta\), \((\phi^{-1}(N))_s \subseteq (\phi^{-1}(P_\beta))_s\) implies that \(N \subseteq P_\beta\) for some \(\beta \in \Delta\). Thus \(N\) is 2-abs.CP and hence \(M_s\) is 2-abs.CP .
Conclusion

In this thesis, we introduced the concept of 2-absorbing subsemimodules and we generalize the concept weakly prime subsemimodules to weakly 2-absorbing subsemimodules. Also, we study the relation between some subsemimodules and 2-absorbing, weakly 2-absorbing subsemimodules. We also investigate the concept of 2-absorbing compactly packed ideals in semirings and 2-absorbing compactly packed semimodules in semirings. For future study, we recommend to study the 2-absorbing subsemimodule in non-commutative semirings.

At the end of this thesis, the following diagram shows the relation between some subsemimodules and 2-absorbing, weakly 2-absorbing subsemimodules.
Bibliography


