Applications of Linear Algebra Methods in Solution of Electromagnetic Waves in Multilayer Waveguides

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نتيجة الحكم على أطروحة ماجستير

بناءً على موافقة شئون البحث العلمي والدراسات العليا بالجامعة الإسلامية بغزة على تشكيل لجنة الحكم على أطروحة الباحث/ أحمد نصير الشهابي ليلح درجة الماجستير في كلية العلوم قسم الرياضيات موضوعًها:

تطبيقات طرق الجبر الخطي في حل الموجات الكهرومغناطيسية في آلة موجة متعددة الطبقات

Applications of Linear Algebra methods in Solution of electromagnetic waves multilayer waveguides

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Abstract

In this thesis, we have implemented a case of polarization of the electromagnetic field (hybrid modes) which includes both transverse electric field and transverse magnetic field. The purpose of applying this case is to find a transfer matrix, which depends on analytical and mathematical bases.

Solar cells provide a renewable and clean energy by converting the sunlight to electricity and it is considered as a dependable energy source.

Also, a new waveguide solar cell structure has been designed and investigated as a solar cell model. The designed waveguide structure contains four layers, the first layer is Air, the second is Aluminium oxynitride (AlON), the third is Iron- indium Gallium arsenide phosphide (Fe-InGaAsP), and the last layer is Silicon (Si) is a Substrate, has been used in order to improve the absorption of light.

In this thesis, theoretical and analytical investigations of electromagnetic wave propagation through the proposed structure have been achieved. Transverse electric polarized plane waves are normally and obliquely incident upon the proposed structure.

The electromagnetic transmitted, reflected and by absorbed versus of the incident frequency has been computed numerically using formulated reflection and transmission coefficients. In computations, the effect of changing the values of absorbing factor, thicknesses of slabs, and angles of incidence has been studied. We used MAPLE software to run the simulation of the proposed waveguide structure.

This work has shown excellent results for high transmission of the incident waves for certain frequency bands. Hence, the studied structure would be promising to be utilized in the field of solar cells.

Keywords: Linear Algebra, Electromagnetic waves, Multilayer waveguides, Hybrid mode, Transfer Matrix, Solar Cells
ملخص البحث

في هذه الأطروحة قمنا بتطبيق حالة من حالات استقطاب المجال الكهرومغناطيسي وهي حالة الهجين المستعرض والتي تم تطبيق هذه الحالة في إيجاد مصفوفة التحويل التي يعتمد في إيجادها على أساس رياضي تحليلي. و يشمل البحث أيضا على تحليل أحد الأبحاث العلمية ذات صلة بالموضوع من خلال تطبيق النتائج التي تم الحصول عليها في هذا البحث.

قمنا أيضا بدراسة نموذج لتركيب خلية شمسية يحتوي على أربع طبقات. الطبقات الأولى هو و الثالثة تحتوي على مادة أوكسي نيترات الألومنيوم و الثانية هي خليط من مادة الحديد و الفوسفور والانديوم و الزرنيخ و الطبقة الأخيرة تحتوي على مادة السيلكون وذلك لتحسين امتصاص الضوء.

حيث، يتناول هذا البحث دراسة نظرية و تحليلية لاختراق الأمواج الكهرومغناطيسية عبر التركيب المقترح، و يركز على دراسة قدرات الانتشار و الانتشار بالإضافة إلى الامتصاص عبر التركيب لموجة مستوية و كهربية مستعرضة تكلف بشكل عمودي و حتى مائل. حيث تم حساب قدرات الانتشار، الانتشار و الامتصاص عند قترات في التردد الساقط من خلال إيجاد صيغ رياضية لمعاملات الانتشار و الانتشار. وأثناء العملية الحسابية تم دراسة تأثير تغيير القيم العددية لكل من معامل الانتشار وسمك الطبقة وزوايا السقوط لموجة. كذلك قمنا باستخدام برامج حاسوبية لمحاكاة النموذج المدرس ورسم العلاقات بين المتغيرات المدرستة.

كما و تميز الخلايا الشمسية بقدرها على توفير طاقة متجددة و نظيفة. يعتمد مبدأ عملها على تحويل الطاقة الضوئية إلى طاقة كهربائية. تواجه الخلايا الشمسية العديد من التحديات منها الكتلية العالية نسبا عند التصنيع و مدى الفعالية وذلك مقارنة بغيرها من البديلات. مواجهة هذه التحديات مهم جدا لجعل الخلايا الشمسية مصدرا يعتمد عليه للحصول على الطاقة.

في هذا العمل تم الحصول على نتائج ممتازة للفالسكتة عبر التركيب لموجات ساقطة عليه بترددات معينة الأمر الذي يجعل منه تركيبا واعدا في المستقبل لتم توظيفه في حق الخلايا الشمسية.

كلمات مفتاحية: الجبر الخطي في الفيزياء، الموجات الكهرومغناطيسية، الموجة الموجي، الهجين المستعرض، مصفوفة التحويل، الخلايا الشمسية.
Dedication

To whom he strives to bless comfort and welfare and never stints what he owns to push me in the success way, to the big heart my dear father.

to the Spring that never stops giving, who weaves my happiness with strings from her merciful heart, to my dear mom.

to whose love flows in my veins, and my heart always remembers them, to my brothers and sisters.

To those who reworded to us their knowledge simply and from their thoughts made a lighthouse guides us through the knowledge and success path, to our honored teachers and professors.

To my

homeland" Palestine"
capital city "Jerusalem"
city "Gaza"

Ahmed N. Kh. AlShembari
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<td>Layer hybrid matrix</td>
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<tr>
<td>$T^f$</td>
<td>Transfer matrix</td>
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<tr>
<td>AlON</td>
<td>Aluminium Oxynitride</td>
</tr>
<tr>
<td>Fe-InGaAsP</td>
<td>Iron- indium Gallium arsenide phosphide</td>
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Preface

In this work, we will study the applications of linear algebra methods in providing solution for electromagnetic waves-multilayer waveguides. In addition to the theoretical application of these waves, they are designed to build a solar cell consisting of four layers. The first layer is Air the second is Aluminium oxynitride (AlON), the third is Fe-InGaAsP, and the last layer is (Si) as a Substrate. The goal of this structure in the cell construction is to improve the performance of solar cells, in terms of getting the maximum amount of electrical energy and at the lowest possible cost.

The Transfer Matrix method will be used to find reflection, transmission and absorption using MAPLE program. The extracted numerical results will be discussed and analyzed.

Study Framework:

In Chapter 1: We will introduce some physical definitions to be consider along the study. We will also introduce analytical and numerical methods, linear algebra, the Transfer Matrix method, and planner waveguide. Additionally, we will derive the required equations from Maxwell's equations, and finally we will introduce both boundary conditions and refractive index, absorption.

In Chapter 2: The Hybrid Modes method will be derived to find the Transfer Matrix and then find the Transfer Matrix for both transverse electric (TE) and transverse magnetic (TM). This chapter contains some theoretical ideas needed for this study such as deriving transmittance and reflectance of transverse magnetic (TM) and transverse electric (TE) for 3 and 4 layers structure using the Transfer Matrix method.

In Chapter 3: "Simple and stable analysis of multilayered anisotropic materials for design of absorbers and shields” has been analyzed and discussed.
In Chapter 4: We will discuss and analyze the extracted numerical results of (TM) and (TE) and plotted it.

In Chapter 5: The conclusion is presented.
Chapter 1

Introduction
Chapter 1

Introduction

This chapter deals with the fundamental concepts of the required physical definitions in this thesis. Analytical and numerical methods in electromagnetic, transfer matrix method, linear algebra physics, it displays waveguides, Electromagnetic waves, Maxwell’s equations, Boundary Conditions, Refractive Index (n), Absorption, are also presented in some details.

1.1 Analytical and Numerical Methods

The analytical methods give the most direct satisfactory solution for the Maxwell’s equations, however, many of problems which can be solved using the analytical methods are very much limited (Sadiku, M. N. 2011). The reasons for this restriction are irregular shape of the structure, dielectric in-homogeneity, and/or in-homogeneous boundary conditions. Therefore, approximation methods and numerical methods or any other methods are employed in such situations.

The analytical solutions although limited they are useful in evaluating the results of the numerical methods. Also, it is able to appreciate the need for another numerical methods better after seeing the limitations of other methods. The general methods for solving the Maxwell’s equations may be classified to two main methods, namely, (Qatanani, N. 2012).

- Analytical methods
- Numerical methods.

The most commonly used analytical methods in solving EM-related problems include:

- **Separation of Variables**: the Partial Differential Equation is split into ordinary differential equations that may be solved easily. This analytical method does not always work, but it is often the simplest method when it does work.

- **Eigen-Function Expansion Method**: We use it to solve non-homogeneous
problems which could not be solved by the previous method.

- **Green’s Function**: This analytical method produces a solution in the form of an integral.

- **Conformal Mapping**: This method is limited to solving the Laplace equation in two dimensions.

- **Integral Transforms**: May be used to solve the Maxwell's equations. This involves Laplace and Fourier transforms methods.

Many problems in Electromagnetic cannot be solved analytically and require a numerical solution such as the transfer matrix that we will discuss in the next section. The governing equations in these problems (Maxwell’s equations or related equations) can take the form of either a differential equation or an integral equation.

### 1.2 Transfer Matrix Method (TMM)

There are many methods that employed for calculation of the reflection coefficients such as the Recursive method (Shabat and Ubeid, 2014; Kong, 2002; Cory and Zach, 2004; Gerardin and Lakhtakia, 2002) and the transfer matrix method (TMM) (Hamouche, Shabat, 2016; El-Amassi, El-Khozondar, Shabat, 2015; Macleod, 2001; Withayachumnankul, Fischer and Abbott, 2008). The transfer matrix can be defined as a matrix relate between the amplitudes of the waves on both sides of a film. The derivation displayed here is taken from (Pedrotti, 1993; Markos and Soukoulis, 2008; Al-Turk, 2011).

**The Transfer Matrix is a factor in so many topics, most importantly**

- **geometric optics**:

The transfer-matrix method is the fundamental electromagnetic responses of materials and introduce a versatile technique in modeling the behavior of nanoscale heterostructures. It is a method used in optics and acoustics to analyze the
propagation of electromagnetic or acoustic waves through a stratified (layered) medium (Bom M., Wolf E., 1980).

The transfer-matrix method is a fruitful object widely used in the treatment of layered systems, like superlattices (Tsu and Esaki, 1973), or photonic crystals (Bendickson J. M., 1996) and used extensively by industries that make highly reflecting mirrors called distributed Bragg reflectors and developed for optical calculations of non-interacting graphene layers.

The transfer matrix method is suitable for simulations going through systems of multiple layers, and have also been successfully applied in simulating metamaterials (Soukoulis and Marko’s, 2003) in studying Anderson metal-insulator transitions (MacKinnon, A., 2003) and in fields other than optics.

- **Ray analysis:**

  Is also known as ABCD matrix analysis, is a type of ray tracing technique used in the design of some optical systems, particularly lasers. It includes the construction of a ray transfer matrix which describes the optical system. Tracing of a light path through the system can then be performed by multiplying this matrix with a vector representing the light ray. The same analysis is also used in accelerator physics to track particles through the magnet installations of a particle accelerator.

- **Statistical physics:**

  In physics and mathematics, the transfer-matrix method is a general technique for solving problems in statistical mechanics.

  The transfer matrix method is considered as one of the effective ways of solution in linear algebra that we will discuss in the next section.
1.3 Linear Algebra Physics

1.3.1 Introduction to Linear Algebra in Physics

Linear algebra in physics is widely used in many activities and other important subjects of physics such as the vectors and the mappings. However, the main role of linear algebra in physics is the vector calculus. As we know that the beauty of physics lies in its numerical questions and thus linear algebra plays an important role in physics.

1.3.2 Importance of Linear Algebra in Physics

The mathematical idea of vectors plays an important role in many areas of physics, some of them are:

- When a particle travels through space, its velocity and direction are influenced by a vector. A vector comes in three dimensions and can be represented in space. When referring to its path, then we say that it is a varying line, and the variation changes through time.

- The vector calculus, which represents only the linear algebra of physics, has a great role in calculating the structure of the bridge at several points of its constructions. This is because these vectors provide us with direction and magnitude of the force acting at several isolated points.

- In electromagnetic theory, the Maxwell’s equations are the main ones; they deal with the vector fields that are changing through time according to the requirements. Thus, it is also a linear algebra of physics.

- The theory of relativity is another example of the linear algebra in physics because it uses the transformations of distance and time from one frame to another by means of a linear mapping of the vector spaces.

- One of the most important branches of physics is the quantum mechanics, which uses the vector spaces and their mapping in its various results. Thus, it is also a linear algebra of physics.
1.3.3 Conclusion for Linear Algebra in Physics

Linear algebra in physics has various roles in vector analysis, matrices, solving expressions etc. Physics is a branch of science that deals with measurements, and linear algebra is a boon to it as it helps in simplifying many equations easily.

1.4 Introduction of Waveguide

Waveguides, like transmission lines are structures that confine and directs wave propagation, such as electromagnetic waves or sound waves, where the electromagnetic waves can be transmitted from point to point in space (Hayt, W. & Buck, J. A. 2001). The waveguides are considered as essential parts of millimeter and submillimeter-wave devices and system, as they enable a signal to propagate with minimal loss of energy by restricting expansion to one dimension or two. This is a similar effect to waves of water constrained within a canal.

A waveguide is also defined as electromagnetic feed line used in microwave, communications, broadcasting, and radar installations and it consists of many different forms that depend on the aim of the guide, and the frequency of the waves to be transmitted. The simplest form is the parallel plate guide as shown in figure (1.1). Other formats also are the hollow-pipe guide, including the rectangular waveguide as shown in figure (1.2), and the cylindrical waveguide as shown in figure (1.3), (Hayt, W. & Buck, J. A. 2001).

Waveguides can be generally classified as either metal waveguides or dielectric waveguides,

- **Metallic waveguides**: normally take the form of an enclosed conducting metal pipe where the waves propagating inside the metal waveguide may be characterized by the reflections from the conducting walls.

- **The dielectric waveguide**: consists of dielectrics only and employs reflections from dielectric interfaces to propagate the electromagnetic wave along the waveguide, used at optical frequencies and contains the slab waveguide as shown in figure (1.4), and the optical fiber as shown in figure (1.5).
In this thesis, we will consider the simple four-layer planar waveguide structure as shown in figure (1.6). The layers are all assumed to be infinite in extent in the y and z directions, and layers 1 and 4 are also assumed to be semi-infinite in the x-direction, (Hunsperger, R. G. 2009).
For planner waveguide, the modes is hybrid which will be discussed in section (1.5.3).

\[ \text{Fig (1.6): Diagram of the Four-layer planar waveguide structure.} \]

1.5 Electromagnetic waves

During the early stages of studies of electric and magnetic phenomena, electric and magnetic fields were considered unrelated fields. In 1865 James Clark Maxwell provided a mathematical theory that showed a close relation between electric and magnetic phenomena.

An electromagnetic wave consists of mutually perpendicular electric and magnetic fields. The electric field and the magnetic field are known to generate each other and behave perpendicular in nature. Both the vectors and propagation direction are strictly perpendicular in nature, see figure (1.7)

These electromagnetic waves must be solved with boundary conditions that we will discuss in section (1.8) at the edges of the waveguides using Maxwell or variants thereof to obtain solutions. The boundary conditions are typically represented by the material composition as well as their interfaces. The solutions (also called modes) to these equations would yield eigenvalues that are direct representations of axial wave velocities in the waveguides.
Electromagnetic waves do not require a medium to propagate, they can not only travel through solid materials but also travel through vacuum, at the same speed of light $c \ (c=3.00 \times 10^8 \text{ m/s})$. Electric field and magnetic field's strength of an electromagnetic wave vanishes as $z \to \infty$, electromagnetic waves are classified into several different types, but they are all part of a single continuous spectrum, (White, F. W. G. 1950).

![Diagram of electromagnetic wave propagation](image)

**Figure (1.7):** plane electromagnetic wave propagation, the wave is traveling in the z-direction (Kerle, N., Janssen, L. L., & Huurneman, G. C. 2004).

There are four types of modes in terms of the wave propagation. Each propagation mode has an associated propagation constant and cutoff frequency. The dominant mode of operation is the lowest mode possible. It is the mode with the lowest cutoff frequency.

Any electromagnetic wave can be expressed as a linear combination of TE, and TM waves. There are also two other modes, namely the TEM (transverse electromagnetic) and the HE (hybrid) mode.

**The four different mode categories as described above are:**

a. Transverse electromagnetic (TEM).

b. Transverse electric (TE).

c. Transverse magnetic (TM).

d. Hybrid (EH or HE).
1.5.1 TE- waves

In the case of the TE polarization the electric field vector $E$ is parallel to the interface, it is incident on the interface between two media, the $xz$-plane defines the plane of incidence and $xy$-plane defines the plane of the boundary interface [see figure (1.8)], and therefore, all $E$-components are transverse to the direction of propagation, so it has only one component, and vector $H$ has two components as,

$$E = (0, E_y, 0), \quad H = (H_x, 0, H_z), \quad (1.1)$$

The $y$–direction corresponds to the perpendicular to the plane of incidence. So, the electric field vector is in the $y$-plane direction, but there is no component of $H$ in this direction, because the magnetic field strength $H$ is totally transverse, and it also called the (s-polarized), (Markoš and Soukoulis, 2008).

![Diagram of TE polarization](image)

Figure (1.8): TE polarized electromagnetic wave incident on the interface, where $i, r$ and $t$ refers to the incident, reflection and transition waves, respectively.

1.5.2 TM- waves

In the case of the TM polarization the magnetic field is parallel to the interface and it is incident on the interface between the two media. the $xz$-plane defines the plane of incidence and the $xy$-plane defines the plane of the boundary interface as in figure (1.9). The magnetic field vector $H$ has only one component in the y-direction, and
the vector E has two components in another different direction, and it is also called (p- polarized), (Markoš nd Soukoulis, 2008).

\[ H = (0, H_y, 0), \quad E = (E_x, 0, E_z) \] (1.2)

**Figure (1.9):** TM polarized electromagnetic wave incident on the interface.

1.5.3 Hybrid mode

In the Hybrid modes (EH or HE) both the electric and magnetic fields have longitudinal components \([H_z \neq 0, E_z \neq 0]\). The longitudinal electric field is dominant in the EH mode while the longitudinal magnetic field is dominant in the HE mode.

Hybrid modes are commonly found in waveguides with in-homogeneous dielectrics as optical fibers, and cylindrical waveguide. (Ciumac, M., & Mihalache, D. 1995).

The tangential component \((t)\) of the hybrid mode are both of field vectors \(E_t\) and \(H_t\) in \(x\) and \(y\) direction. Also, the normal component \((n)\) are both of field vectors \(D_n\) and \(B_n\) in \(z\)-direction as shown in Figure (1.10),

\[ E_t = \left( E_x \hat{x} + E_y \hat{y} \right), \quad H_t = \left( H_x \hat{x} + H_y \hat{y} \right) \]
\[ D_n = D_z \hat{z}, \quad B_n = B_z \hat{z} \] (1.3)
The Analysis of electromagnetic waves propagation depends on Maxwell's equations, which govern the time dependence of the intensity of the electric and magnetic fields $E$ and $H$ respectively, (Akhmediev, N. N. 1998).

\[
\nabla \times E = i\omega \mu H, \quad (1.4) \\
\nabla \times H = -i\omega \varepsilon E, \quad (1.5)
\]

Also,

\[
D = \varepsilon E, \quad (1.6) \\
B = \mu H, \quad (1.7)
\]

where the permittivity $\varepsilon$ and permeability $\mu$ are defined as

\[
\varepsilon = \varepsilon_0 \varepsilon_r, \quad (1.8) \\
\mu = \mu_0 \mu_r \quad (1.9)
\]

where, $\varepsilon_0$ and $\mu_0$ are the permittivity and permeability of a vacuum, and $\varepsilon_r$ and $\mu_r$ are the permittivity and permeability of the material, (Kenji K., 2001). Also, D: is the electric flux density or dielectric displacement ($C/m^2$).

B: is the magnetic flux density ($Wb/m^2$ or $T$).

**Figure (1.10):** EM polarized electromagnetic wave incident on the interface, the upper and lower bounding interfaces of two media are denoted by $z_0^\infty$, $z_1^\infty$ respectively.

1.6 **Maxwell's equation**

![Diagram of EM waves propagating through two media, showing polarization and bounding interfaces.](image-url)
E: is the electric field \((V/m)\).

H: is the magnetic field \((A/m)\).

The electric field and magnetic field for the Hybrid waves propagating along the -axis with angular frequency \(\omega\), and waves vector \(k\) can be expressed as

\[
E = (E_x, E_y, E_z) = E_0 e^{i(k_xx + k_yy + k_zz)},
\]

\[
H = (H_x, H_y, H_z) = H_0 e^{i(k_xx + k_yy + k_zz)} \tag{1.10}
\]

Such that \(E_0\) and \(H_0\) are their amplitudes. Proceeding from the Maxwell curl equations:

\[
\nabla \times E = i\omega \mu H
\]

where, \(\epsilon\) and \(\mu\) are 3×3 matrices in anisotropic.

Then,

\[
\begin{bmatrix}
\hat{x} & \hat{y} & \hat{z} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
E_x & E_y & E_z
\end{bmatrix} = i\omega \begin{pmatrix}
\mu_{xx} & \mu_{xy} & \mu_{xz} \\ 
\mu_{yx} & \mu_{yy} & \mu_{yz} \\ 
\mu_{zx} & \mu_{zy} & \mu_{zz}
\end{pmatrix} \begin{pmatrix}
H_x \\ H_y \\ H_z
\end{pmatrix} \tag{1.11a}
\]

or

\[
\hat{x}: \quad \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = i\omega (\mu_{xx}H_x + \mu_{xy}H_y + \mu_{xz}H_z)
\]

\[
\hat{y}: \quad -\frac{\partial E_z}{\partial x} + \frac{\partial E_x}{\partial z} = i\omega (\mu_{yx}H_x + \mu_{yy}H_y + \mu_{yz}H_z)
\]

\[
\hat{z}: \quad \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i\omega (\mu_{zx}H_x + \mu_{zy}H_y + \mu_{zz}H_z) \tag{1.11b}
\]

However, the spatial variation in \(x, y, z\) is known so that, from Eq. (1.10):

\[
\frac{\partial E}{\partial x} = ik_xE \quad \text{and} \quad \frac{\partial H}{\partial x} = ik_xH
\]

\[
\frac{\partial E}{\partial y} = ik_yE \quad \text{and} \quad \frac{\partial H}{\partial y} = ik_yH
\]
\[
\frac{\partial E}{\partial z} = ik_zE \quad \text{and} \quad \frac{\partial H}{\partial z} = ik_zH \quad (1.12)
\]

Consequently, substitute equation (1.12) in equation (1.11) then these curl equations simplify to

\[
\frac{\partial E_y}{\partial z} - ik_yE_z = -i\omega \left( \mu_{xx}H_x + \mu_{xy}H_y + \mu_{xz}H_z \right), \quad (1.13)
\]

\[
\frac{\partial E_x}{\partial z} - ik_xE_z = i\omega \left( \mu_{yx}H_x + \mu_{yy}H_y + \mu_{yz}H_z \right), \quad (1.14)
\]

\[
ik_xE_y - ik_yE_x = i\omega \left( \mu_{zx}H_x + \mu_{zy}H_y + \mu_{zz}H_z \right). \quad (1.15)
\]

We can perform a similar expansion of Ampere’s equation:

\[
\nabla \times H = -i\omega \varepsilon E \quad (1.16)
\]

It is expanded as:

\[
\begin{vmatrix}
\hat{x} & \hat{y} & \hat{z} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
H_x & H_y & H_z
\end{vmatrix} = -i\omega \varepsilon (E_x\hat{x} + E_y\hat{y} + E_z\hat{z})
\]

Such that

\[
\varepsilon = \begin{pmatrix}
\varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\
\varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\
\varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz}
\end{pmatrix}
\]

to obtain

\[
\frac{\partial H_y}{\partial x} - ik_yH_z = i\omega (\varepsilon_{xx}E_x + \varepsilon_{xy}E_y + \varepsilon_{xz}E_z) \quad (1.17)
\]

\[
\frac{\partial H_x}{\partial z} - ik_xH_z = -i\omega (\varepsilon_{yx}E_x + \varepsilon_{yy}E_y + \varepsilon_{yz}E_z) \quad (1.18)
\]

\[
k_xH_y - ik_yH_x = -i\omega (\varepsilon_{zx}E_x + \varepsilon_{zy}E_y + \varepsilon_{zz}E_z) \quad (1.19)
\]

Now, (1.15) and (1.19) can be manipulated to produce simple algebraic equation for the transverse z- component of E an H as.

\[
E_z = \frac{1}{\omega \varepsilon_{zz}} \left( -k_xH_y + k_yH_x - \omega \varepsilon_{zx}E_x - \omega \varepsilon_{zy}E_y \right), \quad (1.20)
\]

\[
H_z = \frac{1}{\omega \mu_{zz}} \left( k_xE_y - k_yE_x - \omega \mu_{zx}H_x - \omega \mu_{zy}H_y \right). \quad (1.21)
\]

We will use this result in chapter 3.
1.7 Refractive Index

The refraction of light occurs when light passes from any medium to another and is determined from the refractive index \( n \). The refractive index is defined as the ratio between the speed of light in a vacuum and that in the medium as (Pedrotti, 1993; Serway and Jewett, 2008).

\[
n = \frac{\text{speed of light in vacuum}}{\text{speed of light in a medium}} = \frac{c}{v}, \tag{1.22}
\]

where,

\[
c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}, \quad v = \frac{1}{\sqrt{\varepsilon_{\text{med}} \mu_{\text{med}}}}
\]

such that, \( \varepsilon_{\text{med}} \) and \( \mu_{\text{med}} \) are the permittivity and permeability of the medium, respectively.

The refractive index is usually a complex number as:

\[
N = n - ik,
\]

where \( n \) is the real refractive index (often referred to simply as the refractive index), \( k \) is the extinction coefficient (also called the damping constant) and \( i \) is the imaginary unit \( (i^2 = -1) \).

The real refractive index indicates phase speed while the extinction coefficient is a measure of the absorption in a material. Both parts are wavelength-dependent, a phenomenon known as dispersion, (Al-Turk, S. 2011).

From equation (1.22) we can obtain another definition of the refractive index can be made established (Markoš and Soukoulis, 2008).

\[
n = \sqrt{\frac{\varepsilon_{\text{med}} \mu_{\text{med}}}{\varepsilon_0 \mu_0}} = \sqrt{\frac{\varepsilon_{\text{med}}}{\varepsilon_0}} \sqrt{\frac{\mu_{\text{med}}}{\mu_0}} = \sqrt{\varepsilon} \sqrt{\mu} = \sqrt{\varepsilon \mu}, \tag{1.23}
\]

Where,

\[
\varepsilon = \frac{\varepsilon_{\text{med}}}{\varepsilon_0}, \quad \mu = \frac{\mu_{\text{med}}}{\mu_0}.
\]

The wave number of the medium is \( k = \frac{2\pi}{\lambda} \) and \( \lambda = \frac{v}{u} \) then \( k = \frac{2\pi u}{v} = \frac{\omega}{v} \).

From equation (1.22) we obtain,
\[ k = \frac{\omega}{c} n = \frac{2\pi u}{\lambda_0} = \frac{2\pi}{\lambda_0} n, \quad (1.24) \]

where,

- \( \lambda \) is the wavelength of light,
- \( \lambda_0 \) is the wavelength of light in vacuum,
- \( u \) is the frequency of light,
- \( \omega \) is the angular frequency.

The refractive index (n) can also be used to determine the change in the direction of the wave as it passes through a medium interface, that is the refraction index (Frenkel, Y. 2008).

If we change simultaneously the signs of \( \varepsilon \) and \( \mu \), the ratio \( n^2 = \varepsilon \mu \) will not change. If both \( \varepsilon \) and \( \mu \) are positive, this means that \( n = \sqrt{\varepsilon \mu} \). If \( \varepsilon \) and \( \mu \) are negative in a given wavelength range, this means that \( n = -\sqrt{\varepsilon \mu} \) (Golovkina, M. 2010).

### 1.8 Boundary Conditions

The boundary conditions required for the electromagnetic fields cross a given boundary between two different media as shown in Figure (1.11) must be satisfied.

**Figure (1.11):** Field directions at boundary. The tangential and normal components are denoted by \( \hat{t} \) and \( \hat{n} \) respectively.
The continuous boundary conditions at the interface between the two media are presented as, (Kenji, K., & Tsutomu, K. 2001).

- The tangential component of the electric field are continuous across the surface, such that
  \[ \hat{n} \times (E_1 - E_2) = 0 \quad \text{implies} \quad E_{1t} = E_{2t} \quad (1.25) \]

- The tangential component of the magnetic field is continuous, when no current flows on the surface, such that
  \[ \hat{n} \times (H_1 - H_2) = 0 \quad \text{implies} \quad H_{1t} = H_{2t} \quad (1.26) \]

- The normal component of the electric flux density are continuous across the surface, such that.
  \[ \hat{n} \cdot (\bar{D}_1 - \bar{D}_2) = 0 \quad \text{implies} \quad D_{1n} = D_{2n} \quad (1.27) \]

- The normal component of the magnetic flux density are continuous across the surface, such that.
  \[ \hat{n} \cdot (\bar{B}_1 - \bar{B}_2) = 0 \quad \text{implies} \quad B_{1n} = B_{2n} \quad (1.28) \]

where the subscript \( t \) and \( n \) denotes the tangential and normal components to the boundary, respectively. And superscripts (1) and (2) indicate the medium, respectively. An example shows how we can get the boundary condition in the hybrid modes method and relying on the Figure (1.10) as:

- Tangential component:
  \[ \hat{n} \times (\bar{E}(z_0^\gamma) - \bar{E}(z_1^\gamma)) = 0 \quad \text{implies} \]
  \[ \hat{k} \times ([E_x(z_0^\gamma)\hat{i} + E_y(z_0^\gamma)\hat{j}] + E_z(z_0^\gamma)\hat{k}] - [E_x(z_1^\gamma)\hat{i} + E_y(z_1^\gamma)\hat{j}] + E_z(z_1^\gamma)\hat{k}] = 0 \]
\[
\hat{k} \times ([E_x(z_0^\ast) - E_x(z_1^\ast)]\hat{i} + [E_y(z_0^\ast) - E_y(z_1^\ast)]\hat{j} + [E_z(z_0^\ast) - E_z(z_1^\ast)]\hat{k}) = 0
\]

\[
[E_x(z_0^\ast) - E_x(z_1^\ast)]\hat{i} + [E_y(z_0^\ast) - E_y(z_1^\ast)]\hat{j} = 0 \implies E_x(z_0^\ast) = E_x(z_1^\ast)
\]

\[
[E_y(z_0^\ast) - E_y(z_1^\ast)]\hat{j} = 0 \implies E_y(z_0^\ast) = E_y(z_1^\ast)
\]

implies \( E_t(z_0^\ast) = E_t(z_1^\ast) \) (1.29)

We can do a similar steps for the magnetic field (H) and get on

\[
\implies H_t(z_0^\ast) = H_t(z_1^\ast)
\]

(1.30)

- Normal component:

\[
\hat{n} \cdot (\bar{D}(z_0^\ast) - \bar{D}(z_1^\ast)) = 0 \implies \hat{n} \cdot (D_x(z_0^\ast)\hat{i} + D_y(z_0^\ast)\hat{j} + D_z(z_0^\ast)\hat{k}) = 0
\]

\[
\hat{k} \cdot ([D_x(z_0^\ast)\hat{i} + D_y(z_0^\ast)\hat{j} + D_z(z_0^\ast)\hat{k}] - [D_x(z_1^\ast)\hat{i} + D_y(z_1^\ast)\hat{j} + D_z(z_1^\ast)\hat{k}]) = 0
\]

\[
\hat{k} \cdot ([D_x(z_0^\ast) - D_x(z_1^\ast)]\hat{i} + [D_y(z_0^\ast) - D_y(z_1^\ast)]\hat{j} + [D_z(z_0^\ast) - D_z(z_1^\ast)]\hat{k}) = 0
\]

\[
0 + 0 + [D_z(z_0^\ast) - D_z(z_1^\ast)] = 0
\]

\[
[D_z(z_0^\ast) - D_z(z_1^\ast)] = 0 \implies D_z(z_0^\ast) = D_z(z_1^\ast)
\]

implies \( D_n(z_0^\ast) = D_n(z_1^\ast) \) (1.31)

We can do a similar steps for the magnetic flux density field (H) and get on

\[
B_n(z_0^\ast) = B_n(z_1^\ast)
\]

(1.32)

where, \( z \)- direction is the normal components, so \( \hat{n} = \hat{k} \).

This boundary condition is for one interface out of two mediums. We can also make similar steps when we have more than one interface by getting boundary condition for each interface separately. We will use this results "equations from (1.25) to (1.28) in next chapters."
1.9 Absorption

The amplitude of an electromagnetic wave incident on an optically dense material decreases exponentially with distance through the material. This is called the Beer-Lambert law and can be expressed as the following (Yamamoto M. 2002. Al –Turk, 2011):

\[ I(x) = I_0 e^{-\alpha x} \]

Where,

- \( I(x) \): is the irradiance for a given wavelength at a distance \( x \) into the material.
- \( I_0 \): is the irradiance at the surface.
- \( \alpha \): is the absorption coefficient.

The amount absorbed is:

\[ a(x) = \frac{I_0 - I(x)}{I_0} = 1 - e^{-\alpha x} \] (1.33)

The absorption coefficient is directly related to the extinction coefficient by the relation:

\[ \alpha = \frac{4\pi k}{\lambda} \]

Figure(1.12): shows the behavior of the beam of light on a material.
Chapter 2

Theoretical Survey
Chapter 2

Theoretical Survey

2.1 Introduction

In this chapter, electromagnetic wave propagation through a structure consisting of a many layered inserted in vacuum is investigated theoretically and numerically. A Hybrid modes plane polarized wave plane (EH or HE) incidence on the structure is considered. In we use, Maxwell's equations to determine the electric and magnetic fields of the incident waves at each layer of the structure. Also, the boundary conditions are imposed at each interface to obtain a number of equations with unknown parameters. Then, the equations are solved for the unknown parameters to obtain the reflection and the transmission coefficients. These coefficients are used to determine the reflectance, transmittance and absorption of the whole structure.

2.2 Transfer Matrix Method

In this section, we discuss the method of finding the Transfer Matrix using hybrid modes. There are several methods to find The Transfer Matrix (Heavens, O. S. 1991, F. L. Pedrotti, 1993, Macleod, H. A. 1986), one of them is the method mentioned in (Markos, P., & Soukoulis, C. M. 2008) which addresses on finding the Transfer Matrix for both transverse electric TE and transverse magnetic TM separately, where TE matrix and TM matrix are alike and equal to 2x2.

We apply the hybrid modes (EH) on the foresaid method, and that generates a new form combining between (TE) and (TM) in joint solution steps and joint matrix that equal to 4x4. This matrix is called the transfer matrix for both (TE) and (TM), as illustrated below. Then, we apply the Transfer Matrix method obtained to derive the characteristic matrix of the structure of Tri-layered waveguide. Hence, we have two interfaces. The tangential components of the electric field E and magnetic field H are continuous across each interface, see figure (2.1).
2.2.1 Transfer Matrix for Hybrid mode

Now, consider a single thin film, homogenous and isotropic, with thickness \( l \), on a substrate as depicted in Figure 2.1. The light is incident from the left on the boundary between medium 0 and medium 1. A beam of light with its associated electric and magnetic fields undergoes an external reflection at \( z_0^\text{r} \) and the transmitted portion undergoes another reflection at \( z_1^\text{r} \). Multiple beams in the interference are accounted for by ensuring that \( E_{r1}, H_{r1} \) represents the sum of all reflected beams at \( z_0^\text{r} \) emerging from the film, \( E_{i2}, H_{i2} \) represents the sum of all beams incident on the film/substrate boundary at \( z_1^\text{r} \), etc.

As in the derivation of the Fresnel equations (Lvovsky A. I. 2013), the boundary conditions of the electric and magnetic fields are considered at each interface. The requirement that the components of the fields parallel to the interfaced be continuous
still holds. So, we have the boundary conditions as in equations (1.3) and from (1.29) to (1.32), but here we have two interface so, we get on:

- **Tangential component:**
  
  \[ E_t(z_0^\gamma) = E_t(z_1^\gamma), \quad H_t(z_0^\gamma) = H_t(z_1^\gamma), \]
  \[ E_t(z_1^\gamma) = E_t(z_2^\gamma), \quad H_t(z_1^\gamma) = H_t(z_2^\gamma) \]

  where \((t)\) denotes the tangential components as in Eq.(1.3). The last two equations, giving this analysis are:

  \[ E_y(z_1^\gamma) = E_0 + E_r1 = E_{t1} + E_{i1} \quad (2.1) \]
  \[ H_y(z_1^\gamma) = H_0 + H_r1 = H_{t1} + H_{i1} \quad (2.2) \]
  \[ E_y(z_1^\gamma) = E_{r2} + E_{r2} = E_{t2} \quad (2.3) \]
  \[ H_y(z_1^\gamma) = H_{i2} + H_{r2} = H_{t2} \quad (2.4) \]
  \[ E_x(z_1^\gamma) = -E_0x + E_{r1}x = -E_{t1}x + E_{i1}x \quad (2.5) \]
  \[ H_x(z_1^\gamma) = H_0x - H_{r1}x = H_{t1}x - H_{i1}x \quad (2.6) \]
  \[ E_x(z_1^\gamma) = -E_{i2}x + E_{r2}x = -E_{t2}x \quad (2.7) \]
  \[ H_x(z_1^\gamma) = H_{i2}x - H_{r2}x = H_{t2}x \quad (2.8) \]

- **Normal component:**

  \[ D_n(z_0^\gamma) = D_n(z_1^\gamma), \quad B_n(z_0^\gamma) = B_n(z_1^\gamma), \]
  \[ D_n(z_1^\gamma) = D_n(z_2^\gamma), \quad B_n(z_1^\gamma) = B_n(z_2^\gamma) \]

  where \((n)\) denote the normal components. The last two equations, giving this analysis as:

  \[ D_z(z_1^\gamma) = D_0z - D_{r1}z = D_{t1}z - D_{i1}z \quad (2.9) \]
  \[ B_z(z_1^\gamma) = B_0z - B_{r1}z = B_{t1}z - B_{i1}z \quad (2.10) \]
\[ E_z(\hat{z}_1^+) = D_{i2}z - D_{r2}z = D_{t2}z \]  
\[ B_z(\hat{z}_1^+) = B_{i2}z - B_{r2}z = B_{t2}z \] (2.11) (2.12)

Now, we will use the tangential component to get the transfer matrix as:

By using Maxwell’s Equations that link electric fields with magnetic fields and magnetic fields with electric fields we can get:

\[ H = \left( \frac{k_z}{\mu \mu_0 \omega} \right) E, \quad E = -\left( \frac{k_z}{\epsilon \epsilon_0 \omega} \right) H \] (2.13)

Compensation for the equation (2.13), equation (2.5), (2.6), (2.7) and (2.8) can be rewritten as:

\[ E_x(\hat{z}_1^+) = \gamma_i (H_0 - H_{r1}) = \gamma_i (H_{t1} - H_{i1}) \] (2.14)
\[ H_x(\hat{z}_1^+) = \beta_i (E_0 - E_{r1}) = \beta_i (E_{t1} - E_{i1}) \] (2.15)
\[ E_x(\hat{z}_1^-) = \gamma_i (H_{i2} - H_{r2}) = \gamma_s H_{t2} \] (2.16)
\[ H_x(\hat{z}_1^-) = \beta_i (E_{i2} - E_{r2}) = \beta_s E_{t2} \] (2.17)

where the following substitutions have been made:

\[ \gamma_i = \frac{k_{ix}}{\epsilon_i} = \frac{k_i}{\epsilon_i} \cos(\theta_i) = a_i \cos(\theta_i), \quad \beta_i = \frac{k_{ix}}{\mu_i} = \frac{k_i}{\mu_i} \cos(\theta_i) = b_i \cos(\theta_i) \]
\[ i = 0, 1, s \] (2.18)

\( E_{i2}, E_{t1}, H_{i2} \) and \( H_{t1} \) differ from the phase difference that develops from a single traversal of the film. The optical path length associated with one traversal of film is simply one-half that expression (Pedrotti, 1993):

\[ \Delta_1 = n_1 l_1 \cos(\theta_1), \quad n_1 = \frac{k_1 c}{\omega}, \quad k_0 = \left( \frac{2\pi}{\lambda_0} \right) \]

where, \( l_1 \) and \( n_1 \) is a thickness and refractive index, respectively. The phase difference is the product of the optical path length with the wave number,
\[ \delta_1 = k_0 \Delta_1 = k_0 \frac{k_1 c}{\omega} l_1 \cos(\theta_1) = k_0 \frac{k_1}{k_0} l_1 \cos(\theta_1) = k_1 l_1 \cos(\theta_1) = k_1 \Delta_1. \]

Thus:

\[ E_{i2} = E_{t1} e^{-i \delta_1}, \quad E_{i1} = E_{r2} e^{-i \delta_1}, \quad H_{i2} = H_{t1} e^{-i \delta_1} \quad \text{and} \quad H_{i1} = H_{r2} e^{-i \delta_1} \quad (2.19) \]

\( E_{i2}, E_{r2}, H_{i2} \) and \( H_{r2} \) in equations (2.3), (2.4), (2.16) and (2.17) can then be eliminated using equation (2.19):

\[ E_x(z_1^\gamma) = \gamma_1 (H_{t1} e^{-i \delta_1} - H_{t1} e^{i \delta_1}) = \gamma_s H_{t2} \quad (2.20) \]

\[ H_y(z_1^\gamma) = H_{t1} e^{-i \delta_1} + H_{t1} e^{i \delta_1} = H_{t2} \quad (2.21) \]

\[ E_y(z_1^\gamma) = E_{t1} e^{-i \delta_1} + E_{t1} e^{i \delta_1} = E_{t2} \quad (2.22) \]

\[ H_x(z_1^\gamma) = \beta_1 (E_{t1} e^{-i \delta_1} - E_{t1} e^{i \delta_1}) = \beta_s E_{t2} \quad (2.23) \]

Solving for \( E_{t1}, E_{i1}, H_{t1} \) and \( H_{i1} \) in terms of \( E_x(z_1^\gamma), H_y(z_1^\gamma), E_y(z_1^\gamma) \) and \( H_x(z_1^\gamma) \) yields:

\[ E_{t1} = \left( \frac{H_x(z_1^\gamma) + \beta_1 E_y(z_1^\gamma)}{2 \beta_1} \right) e^{i \delta_1} \quad (2.24) \]

\[ E_{i1} = \left( \frac{\beta_1 E_y(z_1^\gamma) - H_x(z_1^\gamma)}{2 \beta_1} \right) e^{-i \delta_1} \quad (2.25) \]

\[ H_{t1} = \left( \frac{\gamma_1 H_y(z_1^\gamma) + E_y(z_1^\gamma)}{2 \gamma_1} \right) e^{i \delta_1} \quad (2.26) \]

\[ H_{i1} = \left( \frac{\gamma_1 H_y(z_1^\gamma) - E_y(z_1^\gamma)}{2 \gamma_1} \right) e^{-i \delta_1} \quad (2.27) \]

With the help of the Euler identities for sine and cosine, equations (2.24) until (2.27) can be substituted into (2.1), (2.2), (2.14) and (2.15) as:

\[ E_x(z_1^\gamma) = \cos(\delta_1) E_x(z_1^\gamma) + i \gamma_1 \sin(\delta_1) H_y(z_1^\gamma) \quad (2.28) \]

\[ E_y(z_1^\gamma) = \cos(\delta_1) E_y(z_1^\gamma) + \frac{i \sin(\delta_1)}{\beta_1} H_x(z_1^\gamma) \quad (2.29) \]
\[ H_x(z_1^\ge) = i\beta_1 \sin(\delta_1)E_y(z_1^\ge) + \cos(\delta_1)H_x(z_1^\ge) \]
\[ H_y(z_1^\ge) = \frac{i\sin(\delta_1)}{\gamma_1}E_x(z_1^\ge) + \cos(\delta_1)H_y(z_1^\ge) \]

The last four equations can be written in matrix form:

\[
\begin{bmatrix}
E_x(z_1^1) \\
E_y(z_1^1) \\
H_x(z_1^1) \\
H_y(z_1^1)
\end{bmatrix} = \begin{bmatrix}
\cos(\delta_1) & 0 & 0 & i\gamma_1 \sin(\delta_1) \\
0 & \cos(\delta_1) & \frac{i\sin(\delta_1)}{\beta_1} & 0 \\
i\sin(\delta_1) & 0 & \cos(\delta_1) & 0 \\
\frac{\gamma_1}{\gamma_1} & 0 & 0 & \cos(\delta_1)
\end{bmatrix}
\begin{bmatrix}
E_x(z_1^1) \\
E_y(z_1^1) \\
H_x(z_1^1) \\
H_y(z_1^1)
\end{bmatrix}
\]

The 4x4 matrix in (2.32) is called the transfer matrix for three layers in hybrid modes, which is generally represented by

\[
M_1 = \begin{bmatrix}
m_{11} & 0 & 0 & m_{14} \\
0 & m_{22} & m_{23} & 0 \\
0 & m_{32} & m_{33} & 0 \\
m_{41} & 0 & 0 & m_{44}
\end{bmatrix}
\]

For a multilayer stack of N thin films, each layer is associated with its own transfer matrix. Equation (2.32) can then be generalized as:

\[
\begin{bmatrix}
E_x(z_1^1) \\
E_y(z_1^1) \\
H_x(z_1^1) \\
H_y(z_1^1)
\end{bmatrix} = M_1 M_2 M_3 \ldots M_N
\begin{bmatrix}
E_x(z_N^1) \\
E_y(z_N^1) \\
H_x(z_N^1) \\
H_y(z_N^1)
\end{bmatrix}
\]

The product of the individual transfer matrices is an overall transfer matrix representing the entire stack in the order in which light rays encounter them:

\[
M_T = M_1 M_2 M_3 \ldots M_N
\]
2.2.2 Transfer Matrix for TE and TM modes (special case)

To find the transfer matrix for TE and TM polarizations in three layers, compensate separately about equations (1.1) and (1.2) in the equations from (2.28) to (2.31), we get on:

- For TE mode as in section (1.5.1), we have $E_x(z_1^{<}) = 0$, $H_y(z_1^{<}) = 0$, after substituting it in Equations from (2.28) to (2.31) we will get two equations, these equations can be written in the form of a matrix as follows,

$$
\begin{bmatrix}
E_y(z_1^{<}) \\
H_x(z_1^{<})
\end{bmatrix}
= \begin{bmatrix}
cos(\delta_1) & \frac{i\sin(\delta_1)}{\beta_1} \\
i\beta_1\sin(\delta_1) & \cos(\delta_1)
\end{bmatrix}
\begin{bmatrix}
E_y(z_1^{<}) \\
H_x(z_1^{<})
\end{bmatrix}
$$

(2.36)

and equation (2.33) becomes:

$$M_1 = \begin{bmatrix} m_{22} & m_{23} \\ m_{32} & m_{33} \end{bmatrix}$$

- For TM mode as in section (1.5.2), we have $E_y(z_1^{<}) = 0$, $H_x(z_1^{<}) = 0$, after substituting it in Equations from (2.28) to (2.31) we will get two equations, these equations can be written in the form of a matrix as follows,

$$
\begin{bmatrix}
E_x(z_1^{<}) \\
H_y(z_1^{<})
\end{bmatrix}
= \begin{bmatrix}
cos(\delta_1) & i\gamma_1\sin(\delta_1) \\
i\gamma_1\sin(\delta_1) & \cos(\delta_1)
\end{bmatrix}
\begin{bmatrix}
E_x(z_1^{<}) \\
H_y(z_1^{<})
\end{bmatrix}
$$

(2.37)

and equation (2.33) becomes:

$$M_1 = \begin{bmatrix} m_{11} & m_{14} \\ m_{41} & m_{44} \end{bmatrix}$$

Such that, $\gamma_1$ and $\beta_1$ as in equation (2.18).
2.3 Reflectance and Transmittance

To calculate reflection and transmission coefficients in more than one layer we have a long mathematical way to do that. Using the transfer matrix method, this manipulation is highly abbreviated. This section explains the calculation of both reflection and transmission coefficients in Hybrid, TE and TM modes through transfer matrix method.

2.3.1 Reflectance and Transmittance for Hybrid mode

The reflection coefficients \( r \) for Hybrid mode are:

\[
\begin{align*}
    r_E &= \left| \frac{E_r}{E_o} \right| = \sqrt{\frac{E_{r,x}^2 + E_{r,y}^2 + E_{r,z}^2}{E_{o,x}^2 + E_{o,y}^2 + E_{o,z}^2}} \\
    r_M &= \left| \frac{H_r}{H_o} \right| = \sqrt{\frac{H_{r,x}^2 + H_{r,y}^2 + H_{r,z}^2}{H_{o,x}^2 + H_{o,y}^2 + H_{o,z}^2}}
\end{align*}
\]

The transmission coefficients \( t \) for Hybrid mode are:

\[
\begin{align*}
    t_E &= \left| \frac{E_t}{E_o} \right| = \sqrt{\frac{E_{t,x}^2 + E_{t,y}^2 + E_{t,z}^2}{E_{o,x}^2 + E_{o,y}^2 + E_{o,z}^2}} \\
    t_M &= \left| \frac{H_t}{H_o} \right| = \sqrt{\frac{H_{t,x}^2 + H_{t,y}^2 + H_{t,z}^2}{H_{o,x}^2 + H_{o,y}^2 + H_{o,z}^2}}
\end{align*}
\]

The reflectance \( R \) for Hybrid mode are given by:

\[
R_E = \left| r_E \right|^2 = \left| \frac{E_r}{E_o} \right|^2 = \left( \frac{\sqrt{E_{r,x}^2 + E_{r,y}^2 + E_{r,z}^2}}{\sqrt{E_{o,x}^2 + E_{o,y}^2 + E_{o,z}^2}} \right)^2
\]
The transmittance (T) for Hybrid mode are given by:

\[ T_E = |t_E|^2 = \frac{|E_{\bar{t}_2}|^2}{|E_o|^2} = \frac{E_{t_{x2}}^2 + E_{t_{y2}}^2 + E_{t_{z2}}^2}{H_{0x}^2 + H_{0y}^2 + H_{0z}^2} \]

\[ T_M = |t_M|^2 = \frac{|H_{\bar{t}_1}|^2}{|H_o|^2} = \frac{H_{t_{x1}}^2 + H_{t_{y1}}^2 + H_{t_{z1}}^2}{H_{0x}^2 + H_{0y}^2 + H_{0z}^2} \]

### 2.3.2 Reflectance and Transmittance for TE and TM modes (special case)

In finding the transfer matrix, parts of equations from (2.1) to (2.4) and from (2.14) to (2.17) were ignored. Those remaining equations are:

\[ E_y(z_1) = E_0 + E_{r1} \]  \hspace{1cm} (2.46)

\[ E_y(z_1) = E_{t2} \]  \hspace{1cm} (2.47)

\[ E_x(z_1) = \gamma_0(H_0 - H_{r1}) \]  \hspace{1cm} (2.48)

\[ E_x(z_1) = \gamma_s H_{t2} \]  \hspace{1cm} (2.49)

\[ H_y(z_1) = H_0 + H_{r1} \]  \hspace{1cm} (2.50)

\[ H_y(z_1) = H_{t2} \]  \hspace{1cm} (2.51)

\[ H_x(z_1) = \beta_0(E_0 - E_{r1}) \]  \hspace{1cm} (2.52)

\[ H_x(z_1) = \beta_s E_{t2} \]  \hspace{1cm} (2.53)

From equations (2.46) to (2.53) can be substituted into (2.32), along with (2.33) for the transfer matrix, to yield:
The matrix system in (2.54) is equivalent to the following four equations:

\[
\begin{aligned}
\gamma_0 (H_0 - H_{r1}) & = m_{11} \gamma_s H_{t2} + m_{14} H_{t2} \\
E_0 + E_{r1} & = m_{22} E_{t2} + m_{23} \beta_s E_{t2} \\
\beta_0 (E_0 - E_{r1}) & = m_{32} E_{t2} + m_{33} \beta_s E_{t2} \\
H_0 + H_{r1} & = m_{41} \gamma_s H_{t2} + m_{44} H_{t2}
\end{aligned}
\] (2.54)

Dividing these equations by \(E_0\) and \(H_0\) and making use of the reflection and transmission coefficients defined as \(r_{TE} = \frac{E_{r1}}{E_0}\), \(r_{TM} = \frac{H_{r1}}{H_0}\) and \(t_{TE} = \frac{E_{t2}}{E_0}\), \(t_{TM} = \frac{H_{t2}}{H_0}\) results in:

\[
\begin{aligned}
\gamma_0 (1 - r_{TM}) & = m_{11} \gamma_s t_{TM} + m_{14} t_{TM} & (2.55) \\
1 + r_{TE} & = m_{22} t_{TE} + m_{23} \beta_s t_{TE} & (2.56) \\
\beta_0 (1 - r_{TE}) & = m_{32} t_{TE} + m_{33} \beta_s t_{TE} & (2.57) \\
1 + r_{TM} & = m_{41} \gamma_s t_{TM} + m_{44} t_{TM} & (2.58)
\end{aligned}
\]

Equations (2.55) with (2.58) and (2.56) with (2.57) can be solved separately for the reflection and transmission coefficients as:

\[
\begin{aligned}
r_{TE} & = \frac{m_{22} \beta_0 + m_{23} \beta_0 \beta_s - m_{32} - m_{33} \beta_s}{m_{22} \beta_0 + m_{23} \beta_0 \beta_s + m_{32} + m_{33} \beta_s} & (2.59) \\
t_{TE} & = \frac{2 \beta_0}{m_{22} \beta_0 + m_{23} \beta_0 \beta_s + m_{32} + m_{33} \beta_s} & (2.60) \\
r_{TM} & = \frac{m_{41} \gamma_0 \gamma_s + m_{44} \gamma_0 - m_{11} \gamma_s - m_{14}}{m_{41} \gamma_0 \gamma_s + m_{44} \gamma_0 + m_{11} \gamma_s + m_{14}} & (2.61)
\end{aligned}
\]
\[ t_{TM} = \frac{2Y_0}{m_{41}Y_0 + m_{44}Y_0 + m_{11}Y_5 + m_{14}} \]  

Substituting about equations (2.18) and (2.33) in equations from (2.59) to (2.62) and simplifying results as:

\[ r_{TE} = \frac{[b_0 \cos(\theta_o) - b_x \cos(\theta_s)] + i \left[ \frac{b_0b_x}{b_1} \frac{\cos(\theta_o) \cos(\theta_s)}{\cos(\theta_i)} - b_1 \cos(\theta_1) \right] \tan(\delta_1)}{[b_0 \cos(\theta_o) + b_x \cos(\theta_s)] + i \left[ \frac{b_0b_x}{b_1} \frac{\cos(\theta_o) \cos(\theta_s)}{\cos(\theta_i)} + b_1 \cos(\theta_1) \right] \tan(\delta_1)} \]  

\[ r_{TM} = \frac{[a_0 \cos(\theta_o) - a_x \cos(\theta_s)] + i \left[ \frac{a_0a_x}{a_1} \frac{\cos(\theta_o) \cos(\theta_s)}{\cos(\theta_i)} - a_1 \cos(\theta_1) \right] \tan(\delta_1)}{[a_0 \cos(\theta_o) + a_x \cos(\theta_s)] + i \left[ \frac{a_0a_x}{a_1} \frac{\cos(\theta_o) \cos(\theta_s)}{\cos(\theta_i)} + a_1 \cos(\theta_1) \right] \tan(\delta_1)} \]  

\[ t_{TE} = \frac{2b_0 \cos(\theta_o)}{[b_0 \cos(\theta_o) + b_x \cos(\theta_s)] + i \left[ \frac{b_0b_x}{b_1} \frac{\cos(\theta_o) \cos(\theta_s)}{\cos(\theta_i)} + b_1 \cos(\theta_1) \right] \tan(\delta_1)} \]  

\[ t_{TM} = \frac{2a_0 \cos(\theta_o)}{[a_0 \cos(\theta_o) + a_x \cos(\theta_s)] + i \left[ \frac{a_0a_x}{a_1} \frac{\cos(\theta_o) \cos(\theta_s)}{\cos(\theta_i)} + a_1 \cos(\theta_1) \right] \tan(\delta_1)} \]  

The reflectance is then given by:

\[ R_{TE} = \frac{[b_0 \cos(\theta_o) - b_x \cos(\theta_s)]^2 + \left[ \frac{b_0b_x}{b_1} \frac{\cos(\theta_o) \cos(\theta_s)}{\cos(\theta_i)} - b_1 \cos(\theta_1) \right]^2 \tan^2(\delta_1)}{[b_0 \cos(\theta_o) + b_x \cos(\theta_s)]^2 + \left[ \frac{b_0b_x}{b_1} \frac{\cos(\theta_o) \cos(\theta_s)}{\cos(\theta_i)} + b_1 \cos(\theta_1) \right]^2 \tan^2(\delta_1)} \]  

\[ R_{TM} = \frac{[a_0 \cos(\theta_o) - a_x \cos(\theta_s)]^2 + \left[ \frac{a_0a_x}{a_1} \frac{\cos(\theta_o) \cos(\theta_s)}{\cos(\theta_i)} - a_1 \cos(\theta_1) \right]^2 \tan^2(\delta_1)}{[a_0 \cos(\theta_o) + a_x \cos(\theta_s)]^2 + \left[ \frac{a_0a_x}{a_1} \frac{\cos(\theta_o) \cos(\theta_s)}{\cos(\theta_i)} + a_1 \cos(\theta_1) \right]^2 \tan^2(\delta_1)} \]  

where,

\[ R = |r|^2 = r \cdot r^*, \quad r^* \text{ is the complex conjugate of } r \]  

\[ \text{31} \]
The transmittance is then given by:

\[ T_{TE} = \frac{4b_0 \cos(\theta_0)b_s \cos(\theta_s)}{[b_0 \cos(\theta_0) + b_s \cos(\theta_s)]^2 + \left[ \frac{b_0b_s}{b_1} \cdot \frac{\cos(\theta_0) \cos(\theta_s)}{\cos(\theta_1)} + b_1 \cos(\theta_1) \right]^2 \tan^2(\delta_1)} \tag{2.70} \]

\[ T_{TM} = \frac{4a_0 \cos(\theta_0)a_s \cos(\theta_s)}{[a_0 \cos(\theta_0) + a_s \cos(\theta_s)]^2 + \left[ \frac{a_0a_s}{a_1} \cdot \frac{\cos(\theta_0) \cos(\theta_s)}{\cos(\theta_1)} + a_1 \cos(\theta_1) \right]^2 \tan^2(\delta_1)} \tag{2.71} \]

where,

\[ T_{TE} = \frac{\beta_s}{\beta_0} |t|^2 = \frac{\beta_s}{\beta_0} (t \cdot t^*), \quad T_{TM} = \frac{\gamma_s}{\gamma_0} |t|^2 = \frac{\gamma_s}{\gamma_0} (t \cdot t^*) \tag{2.72} \]

\[ t^* \text{ is the complex conjugate of } t, \quad \text{(Markoš and Soukoulis, 2008)} \]

### 2.3.3 Average for TE and TM modes

The reflectance average and the transmittance average for TE and TM as:

\[ R_{average} = \frac{R_{TE} + R_{TM}}{2} \tag{2.73} \]

\[ T_{average} = \frac{T_{TE} + T_{TM}}{2} \tag{2.74} \]

These coefficients, with the elements of the overall transfer matrix, can be used to determine the reflective and transmission properties of one or many thin films represented by the matrix.

### 2.4 Four-Layer Slab Waveguide Solar Cell:

In this section, we discuss the method of finding the transfer matrix for four layers. Then we use it to find the Reflectance (R) and Transmittance (T) for TE, TM, and hybrid modes depending on the reflection coefficient (r) and the transmission coefficient (t) for four layers as appears below.
Figure (2.2): A schematic diagram of four-layer slab waveguide solar cell.

Figure (2.2) explains a waveguide structure consisting of four layers. When the light radiates on the waveguide structure at oblique incidence, the incidence angle on each interface are $\theta_0, \theta_1, \theta_2, \theta_s$.

These angles are related by Snell's law as:

$$n_0 \sin(\theta_0) = n_1 \sin(\theta_1) = n_2 \sin(\theta_2) = n_s \sin(\theta_s)$$

Where,

$$n_i = \sqrt{\varepsilon_i \mu_i}, \quad i = 0, 1, 2, s$$

Are the refractive indices of different media.

Now, from the relation $n = \frac{k}{\omega c}$, Snell's law becomes:

$$k_0 \sin(\theta_0) = k_1 \sin(\theta_1) = k_2 \sin(\theta_2) = k_s \sin(\theta_s)$$  \hspace{1cm} (2.75)

Such that, $k_0, k_1, k_2$ and $k_3$ are the wave number in different materials.

### 2.4.1 Transfer Matrix for Four layers

#### 2.4.1.1 In Hybrid Mode

The total transfer matrix of N thin films tangled between semi-infinite mediums is the product of the individual transfer matrices as follows in Equation (2.35). But, the structure in this study consists of two thin films, so the total transfer matrix can be written as:
\[ M = M_1 M_2 \]

Such that, the matrix \( M_1 \) is the equation (2.33), and the matrix \( M_2 \) is a similar matrix of the matrix \( M_1 \) but for the second layer as:

\[
M_2 = \begin{bmatrix}
\cos(\delta_2) & 0 & 0 & i \gamma_2 \sin(\delta_2) \\
0 & \cos(\delta_2) & i \sin(\delta_2) / \beta_2 & 0 \\
i \sin(\delta_2) / \gamma_2 & 0 & 0 & \cos(\delta_2)
\end{bmatrix}
\]

and the equation (2.34) can be written as the form

\[
\begin{bmatrix}
E_x(z_1^i) \\
E_y(z_1^i) \\
H_x(z_1^i) \\
H_y(z_1^i)
\end{bmatrix} = M_1 M_2
\begin{bmatrix}
E_x(z_2^i) \\
E_y(z_2^i) \\
H_x(z_2^i) \\
H_y(z_2^i)
\end{bmatrix}
\] (2.76)

So, the transfer matrix for the hybrid mode in four layers as:

\[
M = M_1 M_2 = \begin{bmatrix}
m_{11} & 0 & 0 & m_{14} \\
0 & m_{22} & m_{23} & 0 \\
0 & m_{32} & m_{33} & 0 \\
m_{41} & 0 & 0 & m_{44}
\end{bmatrix}
\] (2.77)

Such that,

\[
m_{11} = \cos(\delta_1) \cos(\delta_2) - \frac{a_1 \cos(\theta_1)}{a_2 \cos(\theta_2)} \sin(\delta_1) \sin(\delta_2)
\]

\[
m_{14} = i a_2 \cos(\theta_2) \cos(\delta_1) \sin(\delta_2) + i a_1 \cos(\theta_1) \sin(\delta_1) \cos(\delta_2)
\]

\[
m_{22} = \cos(\delta_1) \cos(\delta_2) - \frac{b_2 \cos(\theta_2)}{b_1 \cos(\theta_1)} \sin(\delta_1) \sin(\delta_2)
\]

\[
m_{23} = \frac{i \cos(\delta_1) \sin(\delta_2)}{b_2 \cos(\theta_2)} + \frac{i \sin(\delta_1) \cos(\delta_2)}{b_1 \cos(\theta_1)}
\]

\[
m_{32} = i b_1 \cos(\theta_1) \sin(\delta_1) \cos(\delta_2) + i b_2 \cos(\theta_2) \cos(\delta_1) \sin(\delta_2)
\]

\[
m_{33} = \cos(\delta_1) \cos(\delta_2) - \frac{b_1 \cos(\theta_1)}{b_2 \cos(\theta_2)} \sin(\delta_1) \sin(\delta_2)
\]
\[ m_{41} = \frac{i \sin(\delta_1) \cos(\delta_2)}{a_1 \cos(\theta_1)} + \frac{i \cos(\delta_1) \sin(\delta_2)}{a_2 \cos(\theta_2)} \]

\[ m_{44} = \cos(\delta_1) \cos(\delta_2) - \frac{a_2 \cos(\theta_2)}{a_1 \cos(\theta_1)} \sin(\delta_1) \sin(\delta_2) \]

(2.78)

2.4.1.2 In TE Mode

We can get the transfer matrix \( M = M_1 M_2 \) in TE polarization for four layers as in equations (2.36) as:

\[
\begin{bmatrix}
E_y(z_1^-) \\
H_x(z_1^-)
\end{bmatrix} =
\begin{bmatrix}
\cos(\delta_1) & i \sin(\delta_1) \\
\beta_1 \sin(\delta_1) & \cos(\delta_1)
\end{bmatrix}
\begin{bmatrix}
\cos(\delta_2) & i \sin(\delta_2) \\
\beta_2 \sin(\delta_2) & \cos(\delta_2)
\end{bmatrix}
\begin{bmatrix}
E_y(z_1^+) \\
H_x(z_1^+)
\end{bmatrix}
\]

Then,

\[
M = 
\begin{bmatrix}
\cos(\delta_1) & i \sin(\delta_1) \\
ib_1 \cos(\theta_1) \sin(\delta_1) & \cos(\delta_1)
\end{bmatrix}
\begin{bmatrix}
\cos(\delta_2) & i \sin(\delta_2) \\
ib_2 \cos(\theta_2) \sin(\delta_2) & \cos(\delta_2)
\end{bmatrix}
\]

\[
= \begin{bmatrix}
m_{22} & m_{23} \\
m_{32} & m_{33}
\end{bmatrix}
\]

(2.79)

Where, the elements of matrix \( M \) are the elements of equation (2.78).

2.4.1.3 In TM Mode

Similarly, for TM polarization, we can get the transfer matrix \( M = M_1 M_2 \) in four layers as in equations (2.37) as:

\[
\begin{bmatrix}
E_x(z_1^-) \\
H_y(z_1^-)
\end{bmatrix} =
\begin{bmatrix}
\cos(\delta_1) & iy_1 \sin(\delta_1) \\
\gamma_1 i \sin(\delta_1) & \cos(\delta_1)
\end{bmatrix}
\begin{bmatrix}
\cos(\delta_2) & iy_2 \sin(\delta_2) \\
\gamma_2 i \sin(\delta_2) & \cos(\delta_2)
\end{bmatrix}
\begin{bmatrix}
E_x(z_1^+) \\
H_y(z_1^+)
\end{bmatrix}
\]

Then,

\[
M =
\begin{bmatrix}
\cos(\delta_1) & ia_1 \cos(\theta_1) \sin(\delta_1) \\
\gamma_1 i \sin(\delta_1) & \cos(\delta_1)
\end{bmatrix}
\begin{bmatrix}
\cos(\delta_2) & ia_2 \cos(\theta_2) \sin(\delta_2) \\
\gamma_2 i \sin(\delta_2) & \cos(\delta_2)
\end{bmatrix}
\]

\[
= \begin{bmatrix}
m_{22} & m_{23} \\
m_{32} & m_{33}
\end{bmatrix}
\]

35
\[
\begin{bmatrix}
m_{11} & m_{14} \\
m_{41} & m_{44}
\end{bmatrix}
\]
(2.80)

Where, the element of matrix \( M \) is the elements of equation (2.78).

### 2.4.2 Reflectance and Transmittance for Four Layers

#### 2.4.2.1 In TE Mode

The reflection coefficient \( r_{TE} \) as appear in equation (2.59) is:

\[
r_{TE} = \frac{m_{22}b_0 \cos(\theta_0) + m_{23}b_0 \cos(\theta_0) b_s \cos(\theta_s) - m_{32} - m_{33}b_s \cos(\theta_s)}{m_{22}b_0 \cos(\theta_0) + m_{23}b_0 \cos(\theta_0) b_s \cos(\theta_s) + m_{32} + m_{33}b_s \cos(\theta_s)}
\]
(2.81)

Substituting the transfer matrix elements from Eq.(2.78) in Eq. (2.81), we get:

\[
r_{TE} = \frac{A + iB - iC - D}{A + iB + iC + D}
\]
(2.82)

where,

\[
A = b_0 \cos(\theta_0) \left( \cos(\delta_1) \cos(\delta_2) - \frac{b_2 \cos(\theta_2)}{b_1 \cos(\theta_1)} \sin(\delta_1) \sin(\delta_2) \right)
\]
\[
B = b_0 \cos(\theta_0) b_s \cos(\theta_s) \left( \frac{\cos(\delta_1) \sin(\delta_2)}{b_2 \cos(\theta_2)} + \frac{\sin(\delta_1) \cos(\delta_2)}{b_1 \cos(\theta_1)} \right)
\]
\[
C = [b_1 \cos(\theta_1) \sin(\delta_1) \cos(\delta_2) + b_2 \cos(\theta_2) \cos(\delta_1) \sin(\delta_2)]
\]
\[
D = b_s \cos(\theta_s) \left( \cos(\delta_1) \cos(\delta_2) - \frac{b_1 \cos(\theta_1)}{b_2 \cos(\theta_2)} \sin(\delta_1) \sin(\delta_2) \right)
\]

The reflectance equation \( R_{TE} \) is given from the relation in equation (2.69) as:

\[
R_{TE} = \frac{A^2 + B^2 + C^2 + D^2 - 2(AD + BC)}{A^2 + B^2 + C^2 + D^2 + 2(AD + BC)}
\]
(2.83)

The transmission coefficient \( t_{TE} \) is obtained from the relation as in equation (2.60):

\[
t_{TE} = \frac{2b_0 \cos(\theta_0)}{m_{22}b_0 \cos(\theta_0) + m_{23}b_0 \cos(\theta_0) b_s \cos(\theta_s) + m_{32} + m_{33}b_s \cos(\theta_s)}
\]
(2.84)
Substituting the transfer matrix elements from Eq.(2.78) in Eq. (2.84) we get:

\[ t_{TE} = \frac{2b_0 \cos(\theta_0)}{A + iB + iC + D} \]  

(2.85)

The transmittance equation \( T_{TE} \) is given from the relation in equation (2.72) as :

\[ T_{TE} = \frac{4b_0 \cos(\theta_0)b_s \cos(\theta_s)}{A^2 + B^2 + C^2 + D^2 + 2(AD + BC)} \]  

(2.86)

The absorption \( A_{TE} \) can be calculated from the usual relation as:

\[ A_{TE} = 1 - R_{TE} - T_{TE} \]  

(2.87)

These quantities \( A_{TE}, R_{TE} \) and \( T_{TE} \) will be calculated numerically and plotted in chapter 4.

### 2.4.2.2 Reflectance and Transmittance for TM Mode

The reflection coefficient \( r_{TM} \) is given from the relation as in equation (2.61) as:

\[ r_{TM} = \frac{m_{41}a_0 \cos(\theta_0) \cdot a_s \cos(\theta_s) + m_{44}a_0 \cos(\theta_0) - m_{11}a_s \cos(\theta_s) - m_{14}}{m_{41}a_0 \cos(\theta_0) \cdot a_s \cos(\theta_s) + m_{44}a_0 \cos(\theta_0) + m_{11}a_s \cos(\theta_s) + m_{14}} \]  

(2.88)

Substituting the transfer matrix elements from Eq.(2.78) in Eq. (2.88), we get:

\[ r_{TM} = \frac{\tilde{A} + i\tilde{B} - i\tilde{C} - \tilde{D}}{\tilde{A} + i\tilde{B} + i\tilde{C} + \tilde{D}} \]  

(2.89)

Where \( \tilde{A}, \tilde{B}, \tilde{C} \) and \( \tilde{D} \) are given by.

\[ \tilde{A} = a_0 \cos(\theta_0) \left( \cos(\delta_1) \cos(\delta_2) - \frac{a_2 \cos(\theta_2)}{a_1 \cos(\theta_1)} \sin(\delta_1) \sin(\delta_2) \right) \]

\[ \tilde{B} = a_0 \cos(\theta_0) a_s \cos(\theta_s) \left( \frac{\cos(\delta_1) \sin(\delta_2)}{a_2 \cos(\theta_2)} + \frac{\sin(\delta_1) \cos(\delta_2)}{a_1 \cos(\theta_1)} \right) \]
\[ \tilde{c} = [a_1 \cos(\theta_1) \sin(\delta_1) \cos(\delta_2) + a_2 \cos(\theta_2) \cos(\delta_1) \sin(\delta_2)] \]

\[ \tilde{d} = a_s \cos(\theta_s) \left( \cos(\delta_1) \cos(\delta_2) - \frac{a_1 \cos(\theta_1)}{a_2 \cos(\theta_2)} \sin(\delta_1) \sin(\delta_2) \right) \]

The reflectance equation is given from the relation as in equation (2.69) as:

\[ R_{TM} = \frac{\tilde{A}^2 + \tilde{B}^2 + \tilde{C}^2 + \tilde{D}^2 - 2(\tilde{A}\tilde{D} + \tilde{B}\tilde{C})}{\tilde{A}^2 + \tilde{B}^2 + \tilde{C}^2 + \tilde{D}^2 + 2(\tilde{A}\tilde{D} + \tilde{B}\tilde{C})} \] (2.90)

The transmission coefficient \((t_{TM})\) is obtained from the relation as in equation (2.62):

\[ t_{TM} = \frac{2 a_0 \cos(\theta_0)}{m_{41} a_0 \cos(\theta_0) \cdot a_s \cos(\theta_s) + m_{44} a_0 \cos(\theta_0) + m_{11} a_s \cos(\theta_s) + m_{14}} \] (2.91)

Substituting the transfer matrix elements from Eq.(2.78) in Eq. (2.91) we get:

\[ t_{TM} = \frac{2 a_0 \cos(\theta_0)}{\tilde{A} + i\tilde{B} + i\tilde{C} + \tilde{D}} \] (2.92)

The transmittance equation is given from the relation in equation (2.72) as:

\[ T_{TM} = \frac{4a_0 \cos(\theta_0) a_s \cos(\theta_s)}{\tilde{A}^2 + \tilde{B}^2 + \tilde{C}^2 + \tilde{D}^2 + 2(\tilde{A}\tilde{D} + \tilde{B}\tilde{C})} \] (2.93)

The absorption \((A_{TM})\) can be calculated from the usual relation as:

\[ A_{TM} = 1 - R_{TM} - T_{TM} \] (2.94)

In four layers, the reflectance average and the transmittance average for TE and TM as in section 2.3.3. Also, the reflectance and transmittance for hybrid mode as in section 2.3.1.

These quantities \(A_{TM}, R_{TM}, T_{TM}\) and \(R_{average}, T_{average}\) will be calculated numerically and plotted in chapter 4.
Chapter 3

Hybrid matrix method
Chapter 3

Hybrid matrix method

3.1 Introduction

In this chapter, "Simple and stable analysis of multilayered anisotropic materials for design of absorbers and shields" (Ning, J. 2009) analysis of published paper has intensively and carefully been studied, and discussed. Then the hybrid modes methods were implemented in section (2.1) to find out and obtain the transfer matrix (T), impedance matrix (Z), layer hybrid matrix (H) and Stack hybrid matrix.

3.2 Structure model and Hybrid Transfer matrix

Figure(3.1), depicts the geometry of a planar multilayered structure comprising N homogeneous arbitrarily anisotropic layers stratified in \( \hat{z} \) direction. For each layer \( S (S = 1, 2, \ldots, N) \) of thickness \( l_s \), its upper and lower bounding interfaces are denoted, respectively by \( Z_f^\geq \) and \( Z_f^\leq \) such that \( Z_f^\geq \geq z \) and \( Z_f^\leq \leq z \) for \( z \) within the layer. The homogeneous medium of each layer \( f \) is characterized by permittivity \( \bar{\varepsilon}_f \) and permeability \( \bar{\mu}_f \) tensors as, (Tan, E. L., 2006).

\[
D_f = \bar{\varepsilon}_f \cdot E_f \\
B_f = \bar{\mu}_f \cdot H_f
\]

(3.1)

such that \( \bar{\varepsilon}_f \) and \( \bar{\mu}_f \) are \( 3 \times 3 \) Matrices. Assuming a plane harmonic wave with \( \exp(-i\omega t) \) time dependence, where \( D_f \) is the electric flux density for layer \( f \), and \( B_f \) is the magnetic flux density for layer \( f \).
Figure (3.1): cross-section of a planar N-multilayered structure. The upper and lower bounding interfaces of each layer $f$ are denoted by $Z_f^>$ and $Z_f^<$ respectively.

The tangential components of electric vector $E_f$ and magnetic vector $H_f$ fields satisfy a first–order differential system as,

$$
\begin{align*}
\frac{d}{dz} \begin{bmatrix}
E_x \\
E_y \\
H_x \\
H_y 
\end{bmatrix} = A \cdot \begin{bmatrix}
E_x \\
E_y \\
H_x \\
H_y 
\end{bmatrix}
\end{align*}
$$

(3.2)

Here, $A$ is $4 \times 4$ matrix, whose elements are function of frequency, the spectral variables and the layer's constitutive parameters. Using Maxwell Equation written in section 1.6 in equations (1.13), (1.14), (1.17) and (1.18), we get the following equations:

$$
\begin{align*}
\frac{dE_y}{dz} &= iK_y E_z - i\omega(\mu_{xx}H_x + \mu_{xy}H_y + \mu_{xz}H_z) \\
\frac{dE_x}{dz} &= iK_x E_z + i\omega(\mu_{yx}H_x + \mu_{yy}H_y + \mu_{yz}H_z)
\end{align*}
$$

(3.3)  
(3.4)
\[
\frac{dH_y}{dz} = iK_yH_z + i\omega (\varepsilon_{xx}E_x + \varepsilon_{xy}E_y + \varepsilon_{xz}E_z)
\] (3.5)

\[
\frac{dH_x}{dz} = iK_xH_z - i\omega (\varepsilon_{yx}E_x + \varepsilon_{yy}E_y + \varepsilon_{yz}E_z)
\] (3.6)

Also, as shown in equation (1.20) and (1.21),

\[
H_z = \frac{1}{\omega \mu_{zz}} \left( K_x E_y - K_y E_x - \omega \mu_{zx} H_x - \omega \mu_{zy} H_y \right)
\] (3.7)

\[
E_z = \frac{1}{\omega \varepsilon_{zz}} \left( -K_x H_y + K_y H_x - \omega \varepsilon_{zx} E_x - \omega \varepsilon_{zy} E_y \right)
\] (3.8)

Now, after substitute about the equations (3.7) and (3.8) in equations (3.3) until (3.6), and then compensate the result in equation (3.2), we will get on \( A \) matrix, \( A \) takes the form as:

\[
A = i\omega \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} = i\omega \tilde{A}
\] (3.9)

where:

\[
R_{11} = \begin{bmatrix}
-\mu_{zz} k_x \varepsilon_{xx} - \mu_{yz} k_y \varepsilon_{zz} \\
\omega \mu_{zz} \varepsilon_{zz} \\
-\mu_{zz} k_y \varepsilon_{xx} + \mu_{xz} k_x \varepsilon_{zz} \\
\omega \mu_{zz} \varepsilon_{zz}
\end{bmatrix}
\]

\[
R_{12} = \begin{bmatrix}
\mu_{zz} k_x k_y - \omega^2 \mu_{zz} \mu_{xx} \varepsilon_{xx} + \omega^2 \mu_{zz} \mu_{yx} \varepsilon_{zz} \\
\omega^2 \mu_{zz} \varepsilon_{zz} \\
\mu_{zz} k_y k_x - \omega^2 \mu_{zz} \mu_{zx} \varepsilon_{xx} + \omega^2 \mu_{zz} \mu_{xy} \varepsilon_{zz} \\
\omega^2 \mu_{zz} \varepsilon_{zz}
\end{bmatrix}
\]

\[
R_{21} = \begin{bmatrix}
\omega^2 \mu_{zz} \varepsilon_{xy} k_x \varepsilon_{xx} - k_x k_y \varepsilon_{xx} - \omega^2 \mu_{zz} \varepsilon_{xz} \varepsilon_{yy} \\
\omega^2 \mu_{zz} \varepsilon_{zz} \\
-\omega^2 \mu_{zz} \varepsilon_{xy} k_y \varepsilon_{xx} + k_y k_x \varepsilon_{xx} + \omega^2 \mu_{zz} \varepsilon_{xz} \varepsilon_{yy} \\
\omega^2 \mu_{zz} \varepsilon_{zz}
\end{bmatrix}
\]

\[
R_{22} = \begin{bmatrix}
\mu_{xx} k_y \varepsilon_{xy} - \mu_{zz} k_x \varepsilon_{xz} \\
\omega \mu_{zz} \varepsilon_{zz} \\
\mu_{xx} k_y \varepsilon_{xy} - \mu_{zz} k_x \varepsilon_{xz} \\
\omega \mu_{zz} \varepsilon_{zz}
\end{bmatrix}
\]
Also, we can use Mathematical programs for simplification, A matrix can take the form as:

\[
A = i\omega \Gamma_v \left[ (R_{rc} - \mathcal{I}_{rc}) \cdot \left( \mathcal{I}_{cc}^{-1} (R_{cr} - \mathcal{I}_{cr}) \right) - \mathcal{I}_{rr} \right]
\]

Where:

\[
\Gamma_v = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix},
\quad
\mathcal{R}_{rc} = \begin{bmatrix} 0 & -\kappa_y \\ 0 & \kappa_x \\ \kappa_y & 0 \\ -\kappa_x & 0 \end{bmatrix},
\quad
\mathcal{R}_{rr} = \begin{bmatrix} 0 & 0 & \kappa_y & -\kappa_x \\ -\kappa_y & \kappa_x & 0 & 0 \end{bmatrix},
\quad
\mathcal{I}_{cc} = \begin{bmatrix} \varepsilon_{zz} & 0 \\ 0 & \mu_{zz} \end{bmatrix},
\quad
\mathcal{I}_{rc} = \begin{bmatrix} \varepsilon_{xz} & 0 \\ \varepsilon_{yz} & 0 \\ 0 & \mu_{zz} \\ 0 & \mu_{yz} \end{bmatrix},
\quad
\mathcal{I}_{rr} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & 0 & 0 \\ \varepsilon_{yx} & \varepsilon_{yy} & 0 & 0 \\ 0 & 0 & \mu_{xx} & \mu_{xy} \\ 0 & 0 & \mu_{yx} & \mu_{yy} \end{bmatrix}.
\]

Within a source-free homogeneous medium, \( \mathcal{A} \) is constant 4x4 matrix in the homogeneous differential equation (3.2) this equation admits nontrivial solutions of \( e^{iksz} \) dependence subjected to the dispersion relation, by equation (1.12), (3.2) and (3.9), we get this form

\[
\begin{bmatrix} E_x \\ E_y \\ H_x \\ H_y \end{bmatrix} = i\omega \Lambda \cdot \begin{bmatrix} E_x \\ E_y \\ H_x \\ H_y \end{bmatrix} \Rightarrow \left( \omega \mathcal{A} - k_z \mathcal{I}_4 \right) \cdot \begin{bmatrix} E_x \\ E_y \\ H_x \\ H_y \end{bmatrix} = 0
\]

(3.10)

Now, to get an eigenvalues of equation (3.10), we need to use the determinants method to find it as (Sánchez, D. A. 1968).

\[
\text{det} \left( \omega \mathcal{A} - k_z \mathcal{I}_4 \right) = 0
\]

(3.11)
Therefore, we will get a quartic equation for $k_z$, so there are four eigenvalues and four eigenvectors associated with it. As a result, the general solution for the transverse field vector in the layer $f$ can be written in terms of the superposition of eigenwave solution as, (Tan, E. L., 2006).

$$\begin{bmatrix} E_x \\ E_y \\ H_x \\ H_y \end{bmatrix} = \Psi_f \overline{w}_f(z) = \Psi_f P_f(z) C_f,$$

(3.12)

Where,

$$\Psi_f = \begin{bmatrix} e_{xx}^\uparrow & e_{xy}^\uparrow & e_{xx}^\downarrow & e_{xy}^\downarrow \\ e_{yx}^\uparrow & e_{yy}^\uparrow & e_{yx}^\downarrow & e_{yy}^\downarrow \\ h_{xx}^\uparrow & h_{xy}^\uparrow & h_{xx}^\downarrow & h_{xy}^\downarrow \\ h_{yx}^\uparrow & h_{yy}^\uparrow & h_{yx}^\downarrow & h_{yy}^\downarrow \end{bmatrix}, \overline{w}_f(z) = \begin{bmatrix} \overline{w}_f^\uparrow(z) \\ \overline{w}_f^\downarrow(z) \\ \overline{w}_f^\downarrow(z) \end{bmatrix}$$

$$P_f(z) = \begin{bmatrix} p_x^\uparrow(z) & 0 & 0 & 0 \\ 0 & p_y^\uparrow(z) & 0 & 0 \\ 0 & 0 & p_y^\downarrow(z) & 0 \\ 0 & 0 & 0 & p_y^\downarrow(z) \end{bmatrix}, C_f = \begin{bmatrix} c_x^\uparrow \\ c_y^\uparrow \\ c_x^\downarrow \\ c_y^\downarrow \end{bmatrix}$$

Here, the superscripts "\(>\)" and "\(<\)" stand for "upward-bounded" and "downward-bounded" decomposition, respectively. Where the eigen values $k_{zj} \ (j = 1,2,3,4)$ are included in phase matrix $P_f(z)$. The eigen–submatrices of $\Psi_f$ are designated by $e_f^<\uparrow$ and $h_f^<\uparrow$ accordingly, and $\overline{w}_f^<\uparrow$ are the corresponding amplitude weights of the eigen-waves that conveniently lumps the coefficients and exponential terms together and dependent on $z$, $P_f^<\uparrow(z)$ is a diagonal matrix whose elements consist of the eigenvalues as $[\exp(i k_{jz}), \ j = 1,2,3,4]$ corresponding eigenvectors form the columns of eigenvector matrix $\Psi_f$.

$C_f$ is a four-component coefficient vector containing the unknown constants (independent of $z$), (Tan, E. L., 2000), (Tan, E. L., 2006).

Note that, to get an equation (3.12), it has been assumed that $\overline{A}$ is similar to a diagonal matrix and the inverse of $\Psi_f$ exists. In addition, let us assume in the sequel
that $k_{z1}$ and $k_{z2}$ have positive imaginary parts while $k_{z3}$ and $k_{z4}$ have negative imaginary parts. For lossless medium, we may need to apply the concept of slight loss limit to identify the sign of those imaginary parts (Cottis, P. G., & Kondylis, G. D., 1995). Since these $k_{zj}$ appear in the exponents as $\exp(ik_{zj}z)$, $j = 1, 2, 3, 4$, one can regard the eigenfields $e^>_{xx}, e^>_{yy}, h^>_{xx}, h^>_{yy}$ (for $k_{z1}$) and $e^>_{xy}, e^>_{yy}, h^>_{xy}, h^>_{yy}$ (for $k_{z2}$) as outward-bounded waves which remain bounded as $p \to +\infty$. Likewise, $e^<_{xx}, e^<_{yy}, h^<_{xx}, h^<_{yy}$ (for $k_{z3}$) and $e^<_{xy}, e^<_{yy}, h^<_{xy}, h^<_{yy}$ (for $k_{z4}$) correspond to inward-bounded waves that are still bounded as $p \to -\infty$, (Tan, E. L., 2000).

Note that it is actually not very appropriate to term the waves as ‘outgoing’ and ‘incoming’ or ‘outward-propagating’ and ‘inward-propagating’. This is because for general bianisotropic media, phase propagation direction (according to real parts of $k_{zi}$) may not coincide with energy flow direction (Felsen, L. B., & Marcuvitz, N., 1994). In fact, outward-bounded waves may be incoming at infinity! Therefore, the radiation condition should in general based on bounded solutions which require all waves to be sufficiently bounded, (Tan, E. L., 2000).

### 3.3 Transfer Matrix ($T^f$)

**Note:** To make it simple, we use refractive index symbols for layers $f-1, f, f+1$ as the following order $n_0, n_1, n_2$ respectively, where $n_i = \sqrt{\varepsilon_i\mu_i}$.

Now, to get the transfer matrix ($T^f$), we conduct similar completely steps, which were held on the equations from (2.1) to (2.8) and from (2.13) to (2.19) with the same figure(2.1), but for the layer ($f$) with Refractive index $n_1$, where the symbols $i$, $r$ and $t$ referring to incident, reflection and transmission, respectively as.

\[
E_y(z_f^<) = E_0 + E_{r1} = E_{t1} + E_{i1} \quad (3.13)
\]
\[
H_y(z_f^<) = H_0 + H_{r1} = H_{t1} + H_{i1} \quad (3.14)
\]
\[
E_y(z_f^>) = E_{i2} + E_{r2} = E_{t2} \quad (3.15)
\]
\[
H_y(z_f^>) = H_{i2} + H_{r2} = H_{t2} \quad (2.16)
\]
\[ E_x(z_f^<) = -E_0x + E_r1x = -E_t1x + E_{i1}x \] (3.17)

\[ H_x(z_f^<) = H_0x - H_r1x = H_t1x - H_{i1}x \] (3.18)

\[ E_x(z_f^>) = -E_{i2}x + E_r2x = -E_{t2}x \] (3.19)

\[ H_x(z_f^>) = H_{i2}x - H_r2x = H_{t2}x \] (3.20)

By using Maxwell’s Equations that link electric fields with magnetic fields and magnetic fields with electric fields we can get as:

\[ H = \left( \frac{k_z}{\mu_0\varepsilon_0\omega} \right) E, \quad E = -\left( \frac{k_z}{\varepsilon_0\omega} \right) H \] (3.21)

Compensation for the equation (3.21), Equations (3.17), (3.18), (3.19) and (3.20) becomes:

\[ E_x(z_f^<) = \gamma_0(H_0 - H_{r1}) = \gamma_1(H_{t1} - H_{i1}) \] (3.22)

\[ H_x(z_f^<) = \beta_0(E_0 - E_{r1}) = \beta_1(E_{t1} - E_{i1}) \] (3.23)

\[ E_x(z_f^>) = \gamma_1(H_{i2} - H_{r2}) = \gamma_2H_{t2} \] (3.24)

\[ H_x(z_f^>) = \beta_1(E_{i2} - E_{r2}) = \beta_2E_{t2} \] (3.25)

where the following substitutions have been made:

\[ \gamma_i = \frac{k_{iz}}{\varepsilon_i}, \quad \beta_i = \frac{k_{iz}}{\mu_i}, \quad k_{iz} = k_i\cos(\theta_i) \quad i = 0,1,2 \] (3.26)

\( E_{i2}, E_{t1}, B_{i2} \) and \( B_{t1} \) differ from the phase difference that develops from a single traversal of the film. the optical path length associated with one traversal of film is simply one-half that expression (Pedrotti, 1993):

\[ \Delta_1 = n_1l_1 \cos(\theta_i) \]

The phase difference is the product of the optical path length with the wave number:

\[ \delta = k_0\Delta_1 = \left( \frac{2\pi}{\lambda_0} \right) n_1l_1 \cos(\theta_i) \]
Thus:

\[ E_{i2} = E_{t1} e^{-i\delta}, \quad E_{i1} = E_{r2} e^{-i\delta}, \quad H_{i2} = H_{t1} e^{-i\delta} \quad \text{and} \quad H_{i1} = H_{r2} e^{-i\delta} \]  

(3.27)

\[ E_{i2}, E_{r2}, H_{i2} \text{ and } H_{r2} \] in equations (3.13), (3.14), (3.22) and (3.23) can then be eliminated using equation (3.27):

\[ E_x(z_f^\gamma) = \gamma_1 (H_{i2} e^{i\delta} - H_{r2} e^{-i\delta}) \]  

(3.28)

\[ E_y(z_f^\gamma) = E_{i2} e^{i\delta} + E_{r2} e^{-i\delta} \]  

(3.29)

\[ H_x(z_f^\gamma) = \beta_1 (E_{i2} e^{i\delta} - E_{r2} e^{-i\delta}) \]  

(3.30)

\[ H_y(z_f^\gamma) = H_{i2} e^{i\delta} + H_{r2} e^{-i\delta} \]  

(3.31)

Solving for \( E_{r2}, E_{i2}, H_{r2} \) and \( H_{i2} \) in terms of \( E_y(z_f^\gamma), H_x(z_f^\gamma), E_x(z_f^\gamma) \) and \( H_y(z_f^\gamma) \) yields:

\[ E_{i2} = \left( \frac{\beta_1 E_y(z_f^\gamma) + H_x(z_f^\gamma)}{2\beta_1} \right) e^{-i\delta} \]  

(3.32)

\[ E_{r2} = \left( \frac{\beta_1 E_y(z_f^\gamma) - H_x(z_f^\gamma)}{2\beta_1} \right) e^{i\delta} \]  

(3.33)

\[ H_{i2} = \left( \frac{E_x(z_f^\gamma) + \gamma_1 H_y(z_f^\gamma)}{2\gamma_1} \right) e^{-i\delta} \]  

(3.34)

\[ H_{r2} = \left( \frac{\gamma_1 H_y(z_f^\gamma) - E_x(z_f^\gamma)}{2\gamma_1} \right) e^{i\delta} \]  

(3.35)

With the help of the Euler identities for sine and cosine equations from (3.32) to (3.35) can be substituted into (3.15), (3.16), (3.24) and (3.25):

\[ E_x(z_f^\gamma) = \cos(\delta) E_x(z_f^\gamma) - i \gamma_1 \sin(\delta) H_y(z_f^\gamma) \]  

(3.36)

\[ E_y(z_f^\gamma) = \cos(\delta) E_y(z_f^\gamma) - \frac{i \sin(\delta)}{\beta_1} H_x(z_f^\gamma) \]  

(3.37)

\[ H_x(z_f^\gamma) = -i \beta_1 \sin(\delta) E_y(z_f^\gamma) + \cos(\delta) H_x(z_f^\gamma) \]  

(3.38)
\[ H_y(z_f^>) = -\frac{i\sin(\delta)}{\gamma_1} E_x(z_f^>) + \cos(\delta) H_y(z_f^>) \]  

(3.39)

The last four equations can be written in matrix form:

\[
\begin{bmatrix}
E_x(z_f^>) \\
E_z(z_f^>) \\
H_x(z_f^>) \\
H_z(z_f^>)
\end{bmatrix} = T_f^{f} \begin{bmatrix}
E_x(z_f^>) \\
E_z(z_f^>) \\
H_x(z_f^>) \\
H_z(z_f^>)
\end{bmatrix}
\]

(3.40)

where,

\[
T_f^{f} = \begin{bmatrix}
\cos(\delta_f) & 0 & 0 & -i\gamma_f \sin(\delta_f) \\
0 & \cos(\delta_f) & -i\sin(\delta_f) & 0 \\
0 & -i\beta_f \sin(\delta_f) & \frac{\beta_f}{\cos(\delta_f)} & 0 \\
\gamma_f & 0 & 0 & \cos(\delta_f)
\end{bmatrix}
\]

Also, I have managed to find the Transfer Matrix \((T_f^{f})\) by finding the matrix inverse of the Transfer Matrix \((M_1)\) in Equation (2.32), but for the layer \((f)\).

### 3.4 Impedance Matrix \((Z)\)

Although straightforward and elegant, it is well known that when the layer thickness increases, the transfer matrix will encounter numerical instability. To overcome such problem, an impedance matrix \((Z)\) method was alternatively introduced as follows:

Now, we use to find the impedance matrix \((Z_f)\) similar same completely steps, have been followed on the equations (3.13) to (3.27) and then using that,

\[ E_{i1}, E_{t1}, H_{i1} , H_{t1} \text{ and } E_{i2}, E_{r2}, H_{i2} , H_{r2} \text{ in equations (3.14), (3.16), (3.23) and (3.25) can then be eliminated using equation (3.27) as:}
\]

\[ H_x(z_f^>) = \beta_1(E_{t1} - E_{i1}) = \beta_1 \left( E_{i2} e^{i\delta} - E_{r2} e^{-i\delta} \right) \]

(3.41)
\[ H_y(z_f^-) = H_{t1} + H_{i1} = H_{t2}e^{i\delta} + H_{r2}e^{-i\delta} \]  
\[ H_x(z_f^-) = \beta_1(E_{t2} - E_{r2}) = \beta_1(E_{t1}e^{-i\delta} - E_{i1}e^{i\delta}) \]  
\[ H_y(z_f^-) = H_{i2} + H_{r2} = H_{t1}e^{-i\delta} + H_{i1}e^{i\delta} \]  

Now, we will partition the last four equations into two parts:

**Part one**, choice this equations as:

\[ H_x(z_f^-) = \beta_1(E_{t1} - E_{i1}) \]  
\[ H_y(z_f^-) = H_{t1} + H_{i1} \]  
\[ H_x(z_f^-) = \beta_1(E_{t1}e^{-i\delta} - E_{i1}e^{i\delta}) \]  
\[ H_y(z_f^-) = H_{t1}e^{-i\delta} + H_{i1}e^{i\delta} \]

Solving for \( E_{t1}, E_{i1}, H_{t1} \) and \( H_{i1} \) in terms of \( H_x(z_f^-), H_y(z_f^-), H_x(z_f^+) \) and \( H_y(z_f^+) \) yields:

\[ E_{i1} = \left( \frac{H_x(z_f^-)e^{-i\delta} - H_x(z_f^+)}{2i\beta_1 \sin(\delta)} \right) \]  
\[ E_{t1} = \left( \frac{H_x(z_f^-)e^{i\delta} - H_x(z_f^+)}{2i\beta_1 \sin(\delta)} \right) \]  
\[ H_{t1} = \left( \frac{H_y(z_f^-)e^{i\delta} - H_y(z_f^+)}{2i \sin(\delta)} \right) \]  
\[ H_{i1} = \left( \frac{H_y(z_f^-)e^{-i\delta} - H_y(z_f^+)}{2i \sin(\delta)} \right) \]

where, \((e^{-i\delta} - e^{i\delta}) = -2i \sin(\delta)\).

**The second part**: Choice the remaining equations as:

\[ H_x(z_f^-) = \beta_1(E_{i2}e^{i\delta} - E_{r2}e^{-i\delta}) \]  
\[ H_y(z_f^-) = H_{i2}e^{i\delta} + H_{r2}e^{-i\delta} \]  
\[ H_x(z_f^+) = \beta_1(E_{i2} - E_{r2}) \]
\[ H_y(z_f^*) = H_{i2} + H_{r2} \]  

(3.56)

Also, solving for \( E_{i2}, E_{r2}, H_{i2} \) and \( H_{r2} \) in terms of \( H_x(z_f^*), H_y(z_f^*), H_x(z_f^*) \) and \( H_y(z_f^*) \) yields:

\[
E_{i2} = \left( \frac{H_x(z_f^*) - H_y(z_f^*) e^{-i\delta}}{2i\beta_1 \sin(\delta)} \right) 
\]  

(3.57)

\[
E_{r2} = \left( \frac{H_x(z_f^*) - H_y(z_f^*) e^{i\delta}}{2i\beta_1 \sin(\delta)} \right) 
\]  

(3.58)

\[
H_{i2} = \left( \frac{H_y(z_f^*) - H_y(z_f^*) e^{-i\delta}}{2i\sin(\delta)} \right) 
\]  

(3.59)

\[
H_{r2} = \left( \frac{H_y(z_f^*) e^{i\delta} - H_y(z_f^*)}{2i\sin(\delta)} \right) 
\]  

(3.60)

Now, with the help of the Euler identities for sine and cosine equations from (3.49) to (3.52) and equations from (3.57) to (3.60) can be substituted into (3.13), (3.15), (3.22) and (3.24) to get:

\[
E_x(z_f^*) = \frac{\gamma_1 \cos(\delta)}{i \sin(\delta)} H_y(z_f^*) - \frac{\gamma_1}{i \sin(\delta)} H_y(z_f^*) 
\]  

(3.61)

\[
E_y(z_f^*) = \frac{\cos(\delta)}{i \beta_1 \sin(\delta)} H_x(z_f^*) - \frac{1}{i \beta_1 \sin(\delta)} H_x(z_f^*) 
\]  

(3.62)

\[
E_x(z_f^*) = \frac{\gamma_1}{i \sin(\delta)} H_y(z_f^*) - \frac{\gamma_1 \cos(\delta)}{i \sin(\delta)} H_y(z_f^*) 
\]  

(3.63)

\[
E_y(z_f^*) = \frac{1}{i \beta_1 \sin(\delta)} H_x(z_f^*) - \frac{\cos(\delta)}{i \beta_1 \sin(\delta)} H_x(z_f^*) 
\]  

(3.64)

The last four equations can be written in matrix form:

\[
\begin{bmatrix}
E_x(z_f^*)\\
E_y(z_f^*)\\
E_x(z_f^*)\\
E_y(z_f^*)
\end{bmatrix} = Z^f \begin{bmatrix}
H_x(z_f^*)\\
H_y(z_f^*)\\
H_x(z_f^*)\\
H_y(z_f^*)
\end{bmatrix}
\]

where,
3.5 The layer hybrid matrix (H)

It can be demonstrated that the impedance matrix is free from numerical instability for large thickness. Unfortunately, it is ill-conditioned when the layer thickness becomes too small.

The problem above can be overcome altogether by resorting to the use of layer hybrid matrix.

Now, to get the layer hybrid matrix ($H^f$), we are working on similar steps, as in the impedance matrix ($Z^f$) and then, we have

$$E_{i1}, E_{t1}, H_{i1}, H_{t1} \text{ and } E_{i2}, E_{r2}, H_{i2}, H_{r2} \text{ in equations (3.14), (3.15), (3.23) and (3.24) can then be eliminated using equation (3.27),}$$

$$H_x(z_f^x) = \beta_1(E_{t1} - E_{i1}) = \beta_1(E_{i2}e^{i\delta} - E_{r2}e^{-i\delta})$$  

$$H_y(z_f^y) = H_{t1} + H_{i1} = H_{i2}e^{i\delta} + H_{r2}e^{-i\delta}$$

$$E_x(z_f^x) = \gamma_1(H_{t1}e^{-i\delta} - H_{i1}e^{i\delta}) = \gamma_1(H_{i2} - H_{r2})$$

$$E_y(z_f^y) = E_{i2} + E_{r2} = E_{t1}e^{-i\delta} + E_{i1}e^{i\delta}$$

Now, we will partition the last four equations into two parts:

**Part one**, choice this equations as:

$$H_x(z_f^x) = \beta_1(E_{t1} - E_{i1})$$  

(3.65)
\[ H_y(z_f^<) = H_{t1} + H_{i1} \]  
\[ E_x(z_f^>) = \gamma_1(H_{t1}e^{-i\delta} - H_{i1}e^{i\delta}) \]  
\[ E_y(z_f^>) = E_{t1}e^{-i\delta} + E_{i1}e^{i\delta} \]

Solving for \( E_{t1}, E_{i1}, H_{t1} \) and \( H_{i1} \) in terms of \( H_x(z_f^<), H_y(z_f^<), E_x(z_f^>) \) and \( E_y(z_f^>) \) yields:

\[ E_{t1} = \left( \frac{\beta_1 E_y(z_f^<) - H_x(z_f^<)e^{-i\delta}}{2\beta_1 \cos(\delta)} \right) \]  
\[ E_{i1} = \left( \frac{\beta_1 E_y(z_f^<) + H_x(z_f^<)e^{i\delta}}{2\beta_1 \cos(\delta)} \right) \]  
\[ H_{t1} = \left( \frac{\gamma_1 H_y(z_f^<)e^{i\delta} + E_x(z_f^<)}{2\gamma_1 \cos(\delta)} \right) \]  
\[ H_{i1} = \left( \frac{\gamma_1 H_y(z_f^<)e^{-i\delta} - E_x(z_f^<)}{2\gamma_1 \cos(\delta)} \right) \]

where, \((e^{-i\delta} + e^{i\delta}) = 2\cos(\delta)\).

The second part: Choice the remaining equations as:

\[ H_x(z_f^>) = \beta_1 \left( E_{i2}e^{i\delta} - E_{r2}e^{-i\delta} \right) \]  
\[ H_y(z_f^>) = H_{i2}e^{i\delta} + H_{r2}e^{-i\delta} \]  
\[ E_x(z_f^>) = \gamma_1(H_{i2} - H_{r2}) \]  
\[ E_y(z_f^>) = E_{i2} + E_{r2} \]

Also, Solving for \( E_{i2}, E_{r2}, H_{i2} \) and \( H_{r2} \) in terms of \( H_x(z_f^<), H_y(z_f^<), E_x(z_f^>) \) and \( E_y(z_f^>) \) yields:

\[ E_{i2} = \left( \frac{H_x(z_f^<) + \beta_1 E_y(z_f^<)e^{-i\delta}}{2\beta_1 \cos(\delta)} \right) \]  
\[ E_{r2} = \left( \frac{-H_x(z_f^<) + \beta_1 E_y(z_f^<)e^{i\delta}}{2\beta_1 \cos(\delta)} \right) \]
\[ H_{i2} = \left( \frac{\gamma_1 H_y(z_f^<) + E_x(z_f^>) e^{-i\delta}}{2\gamma_1 \cos(\delta)} \right) \]  
\[ H_{r2} = \left( \frac{\gamma_1 H_y(z_f^>) - E_x(z_f^<) e^{i\delta}}{2\gamma_1 \cos(\delta)} \right) \]  

(3.84) 

(3.85) 

Now, with the help of the Euler identities for sine and cosine equations (3.74) until (3.77) and equations (3.82) until (3.85) can be substituted into Eqs. (3.13),(3.16),(3.22) and (3.25) yields:

\[ E_x(z_f^<) = i \gamma_1 \sin(\delta) H_y(z_f^<) + \frac{1}{\cos(\delta)} E_x(z_f^>) \]  
\[ E_y(z_f^<) = \frac{i \sin(\delta)}{\beta_1 \cos(\delta)} H_x(z_f^<) + \frac{1}{\cos(\delta)} E_y(z_f^>) \]  
\[ H_x(z_f^<) = \frac{1}{\cos(\delta)} H_x(z_f^<) - i \frac{\beta_1 \sin(\delta)}{\cos(\delta)} E_y(z_f^>) \]  
\[ H_y(z_f^<) = \frac{1}{\cos(\delta)} H_y(z_f^<) - i \frac{\sin(\delta)}{\gamma_1 \cos(\delta)} E_x(z_f^>) \]  

(3.86) 

(3.87) 

(3.88) 

(3.89) 

The last four equations can be written in matrix form:

\[
\begin{bmatrix}
E_x(z_f^<) \\
E_y(z_f^<) \\
H_x(z_f^<) \\
H_y(z_f^<)
\end{bmatrix}
= H_f
\begin{bmatrix}
H_x(z_f^<) \\
H_y(z_f^<) \\
E_x(z_f^>) \\
E_y(z_f^>)
\end{bmatrix}
\]

Where,

\[
H_f =
\begin{bmatrix}
0 & i \gamma_f \tan(\delta_f) & \sec(\delta_f) & 0 \\
-i \tan(\delta_f) & 0 & 0 & \sec(\delta_f) \\
\beta_f \sec(\delta_f) & 0 & 0 & -i \beta_f \tan(\delta_f) \\
0 & \sec(\delta_f) & \frac{-i \tan(\delta_f)}{\gamma_f} & 0
\end{bmatrix}
\]

(3.90) 

The pertaining matrix H is called the layer hybrid matrix since its elements are a mixture of impedance \( \frac{E}{H} \) and admittance \( \frac{1}{Z} \) and transfer quantities.
3.6 The Stack Hybrid Matrix

Now, for solving multilayered problem, we also define the stack hybrid matrix as

\[
\begin{bmatrix}
E_x(z_i^l) \\
E_y(z_i^l) \\
H_x(z_i^l) \\
H_y(z_i^l)
\end{bmatrix}
= H^{(l,f)}
\begin{bmatrix}
H_x(z_i^l) \\
H_y(z_i^l) \\
E_x(z_i^l) \\
E_y(z_i^l)
\end{bmatrix}
\]

where,

\[
H^{(l,f)} = H^l H^f
\]

and

\[
H^l = \begin{bmatrix}
0 & i\gamma_i \tan(\delta_i) & \sec(\delta_i) & 0 \\
i\tan(\delta_i) & 0 & 0 & \sec(\delta_i) \\
\beta_i & 0 & 0 & -i\beta_i \tan(\delta_i) \\
0 & \sec(\delta_i) & \frac{-i\tan(\delta_i)}{\gamma_i} & 0
\end{bmatrix}
\]

And

\[
H^f
\]
is the matrix as in equation (3.90). So, the matrix \(H^{(l,f)}\) has the form as,

\[
H^{(l,f)} = \begin{bmatrix}
H_{11}^{(l,f)} & H_{12}^{(l,f)} \\
H_{21}^{(l,f)} & H_{22}^{(l,f)}
\end{bmatrix}
\]

(3.92)

Such that,

\[
H_{11}^{(l,f)} = \begin{bmatrix}
\frac{-\gamma_i}{\beta_f} \tan(\delta_i) \tan(\delta_f) + \sec(\delta_i) \sec(\delta_f) & 0 \\
0 & \frac{-\gamma_f}{\beta_i} \tan(\delta_i) \tan(\delta_f) + \sec(\delta_i) \sec(\delta_f)
\end{bmatrix}
\]

\[
H_{12}^{(l,f)} = \begin{bmatrix}
i\gamma_i \tan(\delta_i) \sec(\delta_f) - i\beta_f \tan(\delta_f) \sec(\delta_i) \\
i\beta_i \tan(\delta_i) \sec(\delta_f) - \frac{i}{\gamma_f} \tan(\delta_i) \sec(\delta_i)
\end{bmatrix}
\]
\[ H^{(l,f)}_{21} = \begin{bmatrix} 0 & i \gamma \tan(\delta_f) \sec(\delta_l) - i \beta \tan(\delta_l) \sec(\delta_f) \\ \frac{i}{\beta} \tan(\delta_f) \sec(\delta_l) - \frac{i}{\gamma} \tan(\delta_l) \sec(\delta_f) & 0 \end{bmatrix} \]

\[ H^{(l,f)}_{22} = \begin{bmatrix} \sec(\delta_l) \sec(\delta_f) - \frac{\beta}{\gamma} \tan(\delta_l) \tan(\delta_f) & 0 \\ 0 & \sec(\delta_l) \sec(\delta_f) - \frac{\beta}{\gamma} \tan(\delta_l) \tan(\delta_f) \end{bmatrix} \]

where,
\[
\delta_i = \left( \frac{2\pi}{\lambda_0} \right) n_i l_i \cos(\theta_i), \quad \gamma_i = \frac{k_{iz}}{\varepsilon_i}, \quad \beta_i = \frac{k_{iz}}{\mu_i}, \quad i = l, f
\]

Where \( H^{(l,f)} \) is the total stack hybrid matrix from layer \( l \) to layer \( f \), we choose \( l < f \)

### 3.7 Relation to Transfer, Impedance and Layer Hybrid Matrices

There are a relation between both of the transfer matrix \( (T^f) \), Impedance matrix and layer hybrid matrix \( (H^f) \) as illustrated below.

The transfer matrix can be related to the hybrid matrix partitions by (Tan, E. L. 2006).

\[
T^f = \begin{bmatrix} (H^f_{12})^{-1} & -(H^f_{12})^{-1} H^f_{11} \\ H^f_{22}(H^f_{12})^{-1} & H^f_{21} - H^f_{22}(H^f_{12})^{-1} H^f_{11} \end{bmatrix}
\]

(3.93)

Also, the hybrid matrix can be related to the transfer matrix partitions by (Tan, E. L. 2006).

\[
H^f = \begin{bmatrix} -(T^f_{11})^{-1} T^f_{12} & (T^f_{11})^{-1} \\ T^f_{22} - T^f_{21}(T^f_{11})^{-1} T^f_{12} & T^f_{21}(T^f_{11})^{-1} \end{bmatrix}
\]

(3.94)
I also managed to find a relation between both the transfer matrix \((T^f)\), impedance matrix \((Z^f)\) and layer hybrid matrix \((H^f)\) as it is shown below:

The hybrid matrix can be related to impedance matrix as,

\[
H^f = \begin{bmatrix}
-Z_{22}^f + Z_{21}^f (Z_{11}^f)^{-1} Z_{12}^f & -(Z_{11}^f)^{-1} Z_{12}^f \\
Z_{21}^f (Z_{11}^f)^{-1} & -(Z_{11}^f)^{-1}
\end{bmatrix}
\] (3.95)

The impedance matrix can be related to the hybrid matrix as,

\[
Z^f = \begin{bmatrix}
-(H_{22}^f)^{-1} & (H_{22}^f)^{-1} H_{12}^f \\
-(H_{22}^f)^{-1} H_{12}^f & (H_{22}^f)^{-1}
\end{bmatrix}
\] (3.96)

Also, the impedance matrix can be related to the transfer matrix as,

\[
Z^f = \begin{bmatrix}
-T_{11}^f (T_{21}^f)^{-1} & T_{11}^f (T_{11}^f, T_{21}^f)^{-1} \\
-T_{11}^f (T_{11}^f, T_{21}^f)^{-1} & T_{11}^f (T_{21}^f)^{-1}
\end{bmatrix}
\] (3.97)

### 3.8 Reflection and transmission coefficients

For the design of an absorber/antireflection layer, it is simple to compute reflection and transmission coefficients via hybrid matrix method. Let the electromagnetic wave be incidents from layer 0, and the reflection \((r_{0,1})\) and transmission \((t_{0,N+1})\) coefficient matrices be defined as

\[
\begin{align*}
\vec{w}^<_0(z_0^>) &= r_{0,1} \vec{w}^<_0(z_0^>) \\
\vec{w}^>_N(z_N^<) &= t_{0,N+1} \vec{w}^>_0(z_0^>)
\end{align*}
\] (3.98)

In case the external layer \(N + 1\) is semi-infinite, the boundary condition reads

\[
\vec{w}^<_N(z_N^<) = 0
\] (3.100)
Then the reflection and transmission coefficients matrices can be solved explicitly in terms of stack hybrid matrix

\[
\begin{align*}
    r_{0,1} &= [H_s \cdot h_0^< - e_0^\triangleright]^{-1} [e_0^\triangleright - H_s \cdot h_0^\triangleright] \\
    t_{0,N+1} &= [h_{N+1}^\triangleright - H_{22}^{(1,N)} \cdot e_{N+1}^\triangleright]^{-1} H_{21}^{(1,N)} \cdot [h_0^\triangleright + h_0^\triangleright r_{0,1}] \\
    H_s &= H_{11}^{(1,N)} + H_{12}^{(1,N)} \cdot [h_{N+1}^\triangleright (e_{N+1}^\triangleright)^{-1} - H_{22}^{(1,N)}]^{-1} \cdot H_{21}^{(1,N)}
\end{align*}
\] (3.101)

Where \(e_0^\triangleright, h_0^\triangleright\), and \(e_{N+1}^\triangleright, h_{N+1}^\triangleright\) are the eigenvectors for the lower and upper external layers (0 and \(n+1\)). In most practical applications, these external layers are air for which the eigen-matrix may take the form as

\[
\Psi_0 = \begin{bmatrix}
e_0^\triangleright & e_0^\triangleright \\
h_0^\triangleright & h_0^\triangleright
\end{bmatrix} = \begin{bmatrix}
    k_z & 0 & -k_z & 0 \\
    0 & -k_z & 0 & 0 \\
    0 & \frac{\varepsilon_0}{\mu_0} k_z & 0 & -\frac{\varepsilon_0}{\mu_0} k_z \\
    \frac{\varepsilon_0}{\mu_0} k_z & 0 & \frac{\varepsilon_0}{\mu_0} & 0
\end{bmatrix}
\] (3.102)

Here we choose to compose the eigen-matrix with the TE and TM eigenwaves. One should notice that both \(r_{0,1}\) and \(t_{0,N+1}\) are 2×2 matrices:

\[
\begin{align*}
    r_{0,1} &= \begin{bmatrix}
    (r_{0,1})_{11} & (r_{0,1})_{12} \\
    (r_{0,1})_{21} & (r_{0,1})_{22}
    \end{bmatrix} \\
    t_{0,N+1} &= \begin{bmatrix}
    (r_{0,N+1})_{11} & (r_{0,N+1})_{12} \\
    (r_{0,N+1})_{21} & (r_{0,N+1})_{22}
    \end{bmatrix}
\end{align*}
\] (3.103) (3.104)

Let \(r_{co}\) and \(t_{co}\) represent the reflection and transmission coefficients for the same polarizations as the incoming wave; \(r_{cross}\) and \(t_{cross}\) correspond to the cross polarization. For TE incident wave, we have:

\[
\begin{align*}
    (r_{0,1})_{11} &= r_{co} , & (r_{0,1})_{21} &= r_{cross} \\
    (r_{0,N+1})_{11} &= t_{co} , & (r_{0,N+1})_{21} &= t_{cross}
\end{align*}
\] (3.105)
And for TM incident wave:

$$(r_{0,1})_{22} = r_{co} \ , \quad (r_{0,1})_{12} = r_{cross}$$

$$(r_{0,N+1})_{22} = t_{co} \ , \quad (r_{0,N+1})_{12} = t_{cross} \quad (3.106)$$

The shielding effectiveness is defined as

$$SE(dB) = -10 \log \left( \frac{P_t}{P_i} \right) \quad (3.107)$$

Here $P_i$ and $P_t$ are the average powers associated with the incident and transmitted waves, respectively. It can be simply deduced as:

$$SE(dB) = -10 \log(|t_{co}|^2 + |t_{cross}|^2) \quad (3.108)$$

In the case of layer $N+1$ is perfect electric conductor (PEC), the boundary condition reads

$$E_t(z_N^+) = E_t(z_{N+1}^-) = 0 \quad (3.109)$$

Then the reflection coefficient can be determined as

$$r_{0,1} = \left[ H_{11}^{(1,N)} \cdot h_0^+ - e_0^+ \right]^{-1} \left[ e_0^+ - H_{11}^{(1,N)} \cdot h_0^- \right] \quad (3.110)$$

**3.9 Stability analysis**

It is well known that the transfer matrix method suffers from the numerical instability when the layer thickness becomes large.

On the contrary, the hybrid matrix method stays stable, which can be easily understood from the expression of layer hybrid matrix in terms of eigenvectors:
\[
H^f = \begin{bmatrix}
    e^\xi_{xx} & e^\xi_{xy} & e^\xi_{xx} P^\xi_{xx}(-h_f) & e^\xi_{yy} P^\xi_{yy}(-h_f) \\
    e^\xi_{yx} & e^\xi_{yy} & e^\xi_{yx} P^\xi_{yx}(-h_f) & e^\xi_{yy} P^\xi_{yy}(-h_f) \\
    h^\xi_{xx} P^\xi_{xx}(h_f) & h^\xi_{xy} P^\xi_{yx}(h_f) & h^\xi_{xx} & h^\xi_{xy} \\
    h^\xi_{yx} P^\xi_{yx}(h_f) & h^\xi_{yy} P^\xi_{yy}(h_f) & h^\xi_{yx} & h^\xi_{yy}
\end{bmatrix}^* \\
\begin{bmatrix}
    h^\xi_{xx} & h^\xi_{xy} & h^\xi_{xx} P^\xi_{xx}(-h_f) & h^\xi_{yy} P^\xi_{yy}(-h_f) \\
    h^\xi_{yx} & h^\xi_{yy} & h^\xi_{yx} P^\xi_{yx}(-h_f) & h^\xi_{yy} P^\xi_{yy}(-h_f) \\
    e^\xi_{xx} P^\xi_{xx}(h_f) & e^\xi_{yx} P^\xi_{yx}(h_f) & e^\xi_{xx} & e^\xi_{yx} \\
    e^\xi_{yy} P^\xi_{yy}(h_f) & e^\xi_{yy} P^\xi_{yy}(h_f) & e^\xi_{yy} & e^\xi_{yy}
\end{bmatrix}^{-1}
\]

(3.111)

Where \( h_f \to \infty \), \( P_f^\xi(-h_f) \) and \( P_f^\xi(h_f) \), tend to zero, the hybrid matrix is reduced to

\[
H^f\big|_{h_f\to\infty} = \begin{bmatrix}
    H^f_{11} & H^f_{12} \\
    H^f_{21} & H^f_{22}
\end{bmatrix}
\]

(3.112)

where:

\[
H^f_{11} = \frac{1}{A_1} \begin{bmatrix}
    -e^\xi_{xx} h^\gamma_{yy} + e^\gamma_{xy} h^\gamma_{yx} & e^\gamma_{xx} h^\gamma_{xy} - e^\gamma_{xy} h^\gamma_{xx}
\end{bmatrix}
\]

\[
H^f_{12} = \begin{bmatrix}
    0 & 0
\end{bmatrix} = H^f_{21}
\]

\[
H^f_{22} = \frac{1}{A_2} \begin{bmatrix}
    h^\gamma_{xx} e^\xi_{yy} - h^\gamma_{xy} e^\xi_{yx} & -h^\gamma_{xx} e^\xi_{xy} + h^\gamma_{xy} e^\xi_{xx}
\end{bmatrix}
\]

where:

\[
A_1 = -h^\gamma_{yy} h^\gamma_{xx} + h^\gamma_{yx} h^\gamma_{xy}
\]

\[
A_2 = e^\xi_{yy} e^\xi_{xx} - e^\xi_{xy} e^\xi_{xy}
\]

In a similar matrix, from the eigensolution expression of layer impedance matrix:

\[
Z^f = \begin{bmatrix}
    e^\zeta_{xx} & e^\zeta_{xy} & e^\zeta_{xx} P^\zeta_{xx}(-h_f) & e^\zeta_{yy} P^\zeta_{yy}(-h_f) \\
    e^\zeta_{yx} & e^\zeta_{yy} & e^\zeta_{yx} P^\zeta_{yx}(-h_f) & e^\zeta_{yy} P^\zeta_{yy}(-h_f) \\
    h^\zeta_{xx} P^\zeta_{xx}(h_f) & h^\zeta_{xy} P^\zeta_{yx}(h_f) & h^\zeta_{xx} & h^\zeta_{xy} \\
    h^\zeta_{yx} P^\zeta_{yx}(h_f) & h^\zeta_{yy} P^\zeta_{yy}(h_f) & h^\zeta_{yx} & h^\zeta_{yy}
\end{bmatrix}^* \\
\begin{bmatrix}
    h^\zeta_{xx} & h^\zeta_{xy} & h^\zeta_{xx} P^\zeta_{xx}(-h_f) & h^\zeta_{yy} P^\zeta_{yy}(-h_f) \\
    h^\zeta_{yx} & h^\zeta_{yy} & h^\zeta_{yx} P^\zeta_{yx}(-h_f) & h^\zeta_{yy} P^\zeta_{yy}(-h_f) \\
    e^\zeta_{xx} P^\zeta_{xx}(h_f) & e^\zeta_{yx} P^\zeta_{yx}(h_f) & e^\zeta_{xx} & e^\zeta_{yx} \\
    e^\zeta_{yy} P^\zeta_{yy}(h_f) & e^\zeta_{yy} P^\zeta_{yy}(h_f) & e^\zeta_{yy} & e^\zeta_{yy}
\end{bmatrix}^{-1}
\]

(3.113)
One can obtain

\[ Z^f \big|_{h_f \to \infty} = \begin{bmatrix} Z_{11}^f & Z_{12}^f \\ Z_{21}^f & Z_{22}^f \end{bmatrix} \]  

(3.114)

where:

\[ Z_{11}^f = \frac{1}{B_1} \begin{bmatrix} e_{xx}^> h_{yy}^> - e_{xy}^> h_{yx}^> - e_{xx}^> h_{xy}^> + e_{xy}^> h_{yx}^> \\ e_{yx}^> h_{yy}^> - e_{yy}^> h_{yx}^> - e_{yx}^> h_{xy}^> + e_{yy}^> h_{yx}^> \end{bmatrix} \]

\[ Z_{12}^f = 0 \]

\[ Z_{22}^f = \frac{1}{B_2} \begin{bmatrix} e_{xx}^< h_{yy}^< - e_{xy}^< h_{yx}^< - e_{xx}^< h_{xy}^< + e_{xy}^< h_{yx}^< \\ e_{yx}^< h_{yy}^< - e_{yy}^< h_{yx}^< - e_{yx}^< h_{xy}^< + e_{yy}^< h_{yx}^< \end{bmatrix} \]

where:

\[ B_1 = h_{yy}^> h_{xx}^> - h_{yx}^> h_{xy}^> \]

\[ B_2 = h_{yy}^< h_{xx}^< - h_{yx}^< h_{xy}^< \]

Therefore it is shown that both hybrid and impedance matrices are still well-conditioned for large thickness.

On the other hand, when the layer thickness tends to zero, the layer hybrid matrix is still stable which is evident from

\[ H^f \big|_{h_f \to 0} = \begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix} \]  

(3.115)

\[ I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

Noting that when \( h_f = 0 \), \( P_f^< (-h_f) = P_f^>(h_f) = 1 \) On the contrary, the impedance matrix takes the form as

\[ Z^f = \begin{bmatrix} e_{xx}^> & e_{xy}^> & e_{yx}^> & e_{yy}^> \\ e_{xx}^< & e_{xy}^< & e_{yx}^< & e_{yy}^< \\ e_{xx}^> & e_{xy}^> & e_{yx}^> & e_{yy}^> \\ e_{xx}^< & e_{xy}^< & e_{yx}^< & e_{yy}^< \end{bmatrix} \begin{bmatrix} h_{xx}^> & h_{xy}^> & h_{yx}^> & h_{yy}^> \\ h_{xx}^< & h_{xy}^< & h_{yx}^< & h_{yy}^< \\ h_{xx}^> & h_{xy}^> & h_{yx}^> & h_{yy}^> \\ h_{xx}^< & h_{xy}^< & h_{yx}^< & h_{yy}^< \end{bmatrix}^{-1} \]  

(3.116)
The matrix on the right is clearly not invertible due to its repeated rows and columns. Therefore calculation accuracy would be affected when dealing with thin-layer modeling using impedance matrices (Z).

### 3.10 Recursion relations

#### 3.10.1 Recursive asymptotic method for layer hybrid matrix

In each individual layer, one can determine $H^f$ by implementing a recursive asymptotic method based on simple self-recursive algorithm. The method will circumvent the need to solve the eigensolution problem. For this purpose we partition the layer $f$ into $n + 1$ sublayers having thicknesses: $d_i = h_f/2^i$ for $i=1,2,\ldots,n$ and $d_{n+1} = h_f/2^n$. The algorithm is initialized with sublayer hybrid matrix $H^{f(n+1)}$ for $(n+1)$th sublayer. With the thickness $d_{n+1}$ is small enough, we can apply asymptotic thin-layer approximation

$$H^{f(n+1)} = \begin{bmatrix} H^{f(n+1)}_{11} & H^{f(n+1)}_{12} \\ H^{f(n+1)}_{21} & H^{f(n+1)}_{22} \end{bmatrix} \approx \begin{bmatrix} 1 + \frac{d_{n+1}}{2} A^{f}_{11} & \frac{d_{n+1}}{2} A^{f}_{12} \\ \frac{d_{n+1}}{2} A^{f}_{21} & -I + \frac{d_{n+1}}{2} A^{f}_{22} \end{bmatrix}^{-1} \begin{bmatrix} -\frac{d_{n+1}}{2} A^{f}_{12} & I - \frac{d_{n+1}}{2} A^{f}_{11} \\ -I - \frac{d_{n+1}}{2} A^{f}_{22} & -\frac{d_{n+1}}{2} A^{f}_{21} \end{bmatrix}$$ (3.117)

Here $I$ is the $2 \times 2$ identical matrix and $A_{mn}$ is obtained from Eq. (3.9). Then starting from $i = n$ and by using the self-recursive relation:

$$H^{f(i)}_{11} = H^{f(i+1)}_{11} + H^{f(i+1)}_{12} \cdot H^{f(i+1)}_{11} \cdot \left[ I - H^{f(i+1)}_{22} \cdot H^{f(i+1)}_{11} \right]^{-1} \cdot H^{f(i+1)}_{21}$$

$$H^{f(i)}_{12} = H^{f(i+1)}_{12} \cdot \left[ I - H^{f(i+1)}_{11} \cdot H^{f(i+1)}_{22} \right]^{-1} \cdot H^{f(i+1)}_{21}$$

$$H^{f(i)}_{21} = H^{f(i+1)}_{21} \cdot \left[ I - H^{f(i+1)}_{22} \cdot H^{f(i+1)}_{11} \right]^{-1} \cdot H^{f(i+1)}_{21}$$

$$H^{f(i)}_{22} = H^{f(i+1)}_{22} + H^{f(i+1)}_{21} \cdot H^{f(i+1)}_{22} \cdot \left[ I - H^{f(i+1)}_{11} \cdot H^{f(i+1)}_{22} \right]^{-1} \cdot H^{f(i+1)}_{12}$$ (3.118)
The algorithm proceeds unit \( i = 1 \) and the layer hybrid matrix can be obtained as \( H^f = H^{f(1)} \). With this method, the hybrid matrix can be calculated stably even for very thick layer. Moreover, the hybrid matrix is free from all intricacies of solving the eigenvalues and eigenvectors, which require complex root searching. Degeneracy treatment and upward/downward eigenvector sorting or selection. To assess the accuracy of recursive asymptotic methods using hybrid and impedance matrices, we investigate the relative error changes with the number of subdivisions.

### 3.10.2 Recursive method for stack hybrid matrix

to obtain the stack hybrid matrix, we consider the recursion relations starting from top layer \( N \). From the definition of layer and stack hybrid matrices in Eqs. (13) and (14) while noting the continuity of the fields

\[
E_t(z_{f-1}^>) = E_t(z_f^<) \\
H_t(z_{f-1}^>) = H_t(z_f^<)
\]

(3.120)

We can obtain the recursion as

\[
H^{(f,N)}_{11} = H^f_{11} + H^f_{12} \cdot H^{(f+1,N)}_{11} \cdot [1 - H^f_{22} \cdot H^{(f+1,N)}_{11}]^{-1} \cdot H^f_{21} \\
H^{(f,N)}_{12} = H^f_{12} \cdot [1 - H^f_{11} \cdot H^f_{22}]^{-1} \cdot H^{(f+1,N)}_{12} \\
H^{(f,N)}_{21} = H^{(f+1,N)}_{21} \cdot [1 - H^f_{11} \cdot H^f_{22}]^{-1} \cdot H^f_{21} \\
H^{(f,N)}_{22} = H^{(f+1,N)}_{22} + H^{(f+1,N)}_{21} \cdot H^f_{22} \cdot [1 - H^f_{11} \cdot H^f_{22}]^{-1} \cdot H^{(f+1,N)}_{12}
\]

(3.121)

In a similar manner, the recursion relations starting from bottom layer can be deduced as

\[
H^{(1,f)}_{11} = H^{(1,f-1)}_{11} + H^{(1,f-1)}_{12} \cdot H^f_{11} \cdot [1 - H^f_{22} \cdot H^{(1,f-1)}_{11}]^{-1} \cdot H^{(1,f-1)}_{21}
\]
\[ H_{12}^{(1,f)} = H_{12}^{(1,f-1)} \cdot \left[ I - H_{11}^f \cdot H_{22}^{(1,f-1)} \right]^{-1} \cdot H_{12}^f \]

\[ H_{21}^{(1,f)} = H_{21}^f \cdot \left[ I - H_{22}^{(1,f-1)} \cdot H_{11}^f \right]^{-1} \cdot H_{21}^{(1,f-1)} \]

\[ H_{22}^{(1,f)} = H_{22}^f + H_{21}^f \cdot H_{22}^{(1,f-1)} \cdot \left[ I - H_{11}^f \cdot H_{22}^{(1,f-1)} \right]^{-1} \cdot H_{12}^f \]

(3.122)

Comparing Eq. (43) with Eq. (44), one can find that they have the similar forms. Therefore the hybrid matrix recursion can be written in a general form:

\[ H_{11} = H_{11}^< + H_{12}^< \cdot H_{11}^> \cdot \left[ I - H_{22}^< \cdot H_{11}^> \right]^{-1} \cdot H_{21}^< \]

\[ H_{12} = H_{12}^< \cdot \left[ I - H_{22}^> \cdot H_{22}^< \right]^{-1} \cdot H_{12}^> \]

\[ H_{21} = H_{21}^> \cdot \left[ I - H_{22}^< \cdot H_{21}^> \right]^{-1} \cdot H_{21}^< \]

\[ H_{22} = H_{22}^> + H_{21}^> \cdot H_{22}^< \cdot \left[ I - H_{11}^> \cdot H_{22}^< \right]^{-1} \cdot H_{12}^> \]

(3.123)

Where \( H^> \) and \( H^< \) stand for the hybrid matrix for upper and lower layer/stack respectively.
Chapter 4

Four Layers Waveguide Structure for Solar Cell
CHAPTER 4

Four Layers Waveguide Structure for Solar Cell

In chapter 2, we derived the reflectance (R), the transmittance (T) and the absorption (A) for 4-layers waveguide structure solar cell (Figure 2.2). In this chapter, some numerical results regarding TE and TM polarizations are presented. A computer program was developed to compute the effect of various quantities especially reflectance, transmittance and absorption then plot these quantities with different physical parameters have been displayed.

4.1 Data and Calculations

In this discussion all the media are considered to be nonmagnetic media where \( \mu = 1 \). The refractive index of the material, as shown in figure (4.1), consists of the following layers: the first layer is Air with \( n_0 \approx 1 \), (Ciddor, P. E., 1996). And the second layer is Aluminium oxynitride (AlON) with \( n_1 \approx 1.7632 \), (Hartnett, T. M., 1998). And the third layer is Iron-indium Gallium arsenide phosphide (Fe-InGaAsP) with \( n_2 \approx 3.3723 - 0.0021 \cdot I \) (Fonseca, D. B., 2017). And the last layer (substrate) is Silicon(Si) with \( n_3 \approx 3.673 \), (Green, M. A., & Keevers, M. J., 1995).

\[
\begin{array}{c}
\text{Air} \quad n_0 = 1 \\
\text{AlON} \quad n_1 = 1.7782 \\
\text{Fe-InGaAsP} \quad n_2 = 3.3723 - 0.0021 \cdot I \\
\text{Si} \quad n_3 = 3.6730 \\
\end{array}
\]

**Figure (4.1):** Showing the distribution of the materials used on the four layers
4.2 TE - Mode

Discussion of the Results

In the solar cell model, we are interested in getting a minimum reflectance and maximum transmittance in the visible light region. In Figure (4.2), the reflectance (R) was derived and then plotted versus the operating wavelength. It can be seen that reflectance increases when \( \lambda \) approaches 306.84 nm and then begins to decrease very fast till reaching its minimum value when \( \lambda = 600 \) nm. In this figure we have three minima according to various three waveguide widths; namely \( l_1 = 68 \) nm, \( l_2 = 82 \) nm and \( l_3 = 90 \) nm. Further, the places of these minima lay inside the area of visible light. Here, \( l_1 = 82 \) nm is the best thickness because the minimum reflectance which concerns this thickness appears at mid the visible range i.e. at \( \lambda = 600 \) nm. Also, this figure shows the minimum reflectance is shifting toward higher wavelengths when the thickness \( l_1 \) increasing.

![Reflectance graph](image)

**Figure (4.2):** Reflectance (R) versus the wavelength (\( \lambda \)) for different values of \( l_1 \) (thickness of AlON), \( l_2 = 20 \) nm (thickness of Fe-InGaAsP), \( n_0 \) (Air) \( \approx 1 \), \( n_1 \) (AlON) \( \approx 1.7632 \), \( n_2 \) (Fe – InGaAsP) \( \approx 3.3723 - 0.0021 \* I \), \( n_3 \) (Si) \( \approx 3.673 \), \( \theta_0 = 0^\circ \).

In Figure (4.3) shows the transmittance (T) as versus the operating wavelength. We can note that transmittance has opposite values of the reflectance. Transmittance was derived to calculate and plot the absorption (A).
The absorption (A) was calculated from the relation $A = 1 - R - T$ and was then plotted against the wavelength in Figure (4.4). In this figure, we notice that the maximum absorption in visible light range is decreasing whenever the Aluminium oxynitride (AlON) thickness increased. Generally, we can see that absorption has high value when decreases AlON layer thickness and vice versa.

**Figure (4.3):** Transmittance (T) versus the wavelength ($\lambda$) for different values of $l_1$ (thickness of AlON), $l_2 = 20$ nm (thickness of Fe-InGaAsP), $n_0$ (Air) $\approx 1$, $n_1$ (AlON) $\approx 1.7632$, $n_2$ (Fe–InGaAsP) $\approx 3.3723 - 0.0021 \cdot I$, $n_3$ (Si) $\approx 3.673$, $\theta_0 = 0^\circ$.

**Figure (4.4):** Absorption (A) versus the wavelength ($\lambda$) for different values of $l_1$ (thickness of AlON), $l_2 = 20$ nm (thickness of Fe-InGaAsP), $n_0$ (Air) $\approx 1$, $n_1$ (AlON) $\approx 1.7632$, $n_2$ (Fe–InGaAsP) $\approx 3.3723 - 0.0021 \cdot I$, $n_3$ (Si) $\approx 3.673$, $\theta_0 = 0^\circ$. 
Figure (4.5) shows a plotting of the reflectance versus wavelength for various three widths of Fe-InGaAsP. we can seen that the minimum reflectance which belongs to \( l_2 = 35 \text{ nm} \) is relatively large so we will ignore it. the minimum reflectance which concerns \( l_2 = 15 \text{ nm} \) is good but the minimum reflectance at \( l_2 = 20 \text{ nm} \) is the best because it is the minimum reflectance.

Figure (4.6) shows the relation of transmittance versus the wavelength. we can seen that the best transmittance when \( l_2 = 20 \text{ nm} \).

Figure (4.7) indicates the absorption versus wavelength with different Fe-InGaAsP thickness i.e. \( l_2 \). In this figure it has been noticed that the maximum absorption is increasing whenever thickness of Fe-InGaAsP is increasing.

![Reflectance Graph](image)

**Figure (4.5):** Reflectance (R) versus the wavelength (\( \lambda \)) for different values of \( l_2 \) (thickness of Fe-InGaAsP), \( l_1 \) =82 nm (thickness of AlON ), \( n_0 \) (Air) \( \approx \) 1, \( n_1 \) (AlON) \( \approx \) 1.7632, \( n_2 \) (Fe − InGaAsP) \( \approx \) 3.3723 − .0021 * I, \( n_3 \) (Si) \( \approx \) 3.673, \( \theta_0 \) = 0°.
Figure (4.6): Transmittance (T) versus the wavelength ($\lambda$) for different values of $l_2$ (thickness of Fe-InGaAsP), $l_1$ = 82 nm (thickness of AlON), $n_0$(Air) $\approx$ 1, $n_1$(AlON) $\approx$ 1.7632, $n_2$(Fe-InGaAsP) $\approx$ 3.3723 $-$ .0021 * I, $n_3$(Si) $\approx$ 3.673, $\theta_0$ = 0°.

Figure (4.7): Absorption (A) versus the wavelength ($\lambda$) for different values of $l_2$ (thickness of Fe-InGaAsP), $l_1$ = 82 nm (thickness of AlON), $n_0$(Air) $\approx$ 1, $n_1$(AlON) $\approx$ 1.7632, $n_2$(Fe-InGaAsP) $\approx$ 3.3723 $-$ .0021 * I, $n_3$(Si) $\approx$ 3.673, $\theta_0$ = 0°.
In Figure (4.8), the reflectance (R) was showed versus the wavelength. In this figure, three minima have been observed according to various three incidence angles. These three minima at \( \lambda \) approaches 600 nm. The minimum reflectance which concerns \( \theta_0 = 60^\circ \) is relatively large so it will be ignored. The minimum reflectance is related to \( \theta_0 = 0^\circ \) and the second minimum reflectance is related to \( \theta_0 = 30^\circ \), which both excellent, but when \( \theta_0 = 0^\circ \) is the best. Thus, using lower angles of incidence would result in reflectance having lower minima.

![Figure (4.8): Reflectance (R) versus the wavelength (\( \lambda \)) for different values of \( \theta_0 \) (incidence angle), \( l_2 = 20 \) nm (thickness of Fe-InGaAsP), \( l_1 = 82 \) nm (thickness of AlON), \( n_0 \) (Air) \( \approx 1 \), \( n_1 \) (AlON) \( \approx 1.7632 \), \( n_2 \) (Fe-InGaAsP) \( \approx 3.3723 - 0.0021 i \), \( n_3 \) (Si) \( \approx 3.673 \), \( \theta_0 = 0^\circ \).](image)

In Figure (4.9), the transmittance was plotted against the wavelength for different values of incidence angle. We can note that the behaviour of transmittance invert the behaviour of reflectance.

In Figure (4.10) show the absorption versus the wavelength for various incidence angle. It can be observed that the maximum absorption in visible light is increasing when incidence angle is decreasing.
**Figure (4.9):** Transmittance \( T \) versus the wavelength \( \lambda \) for different values of incidence angle, \( \theta_0 \), thickness of layers: \( l_2 = 20 \text{ nm} \) (Fe-InGaAsP), \( l_1 = 82 \text{ nm} \) (AlON), refraction indices: \( n_0 \approx 1 \), \( n_1 \approx 1.7632 \), \( n_2 \approx 3.3723 - 0.0021 \cdot I \), \( n_3 \approx 3.673 \), \( \theta_0 = 0^\circ \).

**Figure (4.10):** Absorption \( A \) versus the wavelength \( \lambda \) for different values of incidence angle, \( \theta_0 \), thickness of layers: \( l_2 = 20 \text{ nm} \) (Fe-InGaAsP), \( l_1 = 82 \text{ nm} \) (AlON), refraction indices: \( n_0 \approx 1 \), \( n_1 \approx 1.7632 \), \( n_2 \approx 3.3723 - 0.0021 \cdot I \), \( n_3 \approx 3.673 \), \( \theta_0 = 0^\circ \).
Figure (4.20) show the behaviour of the reflectance, transmittance and absorption at normal incidence for layers $l_1$ and $l_2$ as in figures (a) and (b), respectively.

**Figure (4.11):** Reflectance (R), Transmittance (T), and absorption (A) at normal incidence versus the wavelength for $l_1$ (thickness of AlON) $= 82$ nm, $l_2$ (thickness of Fe-InGaAsP) $= 20$ nm, $n_3(\text{Si}) \approx 3.6730$, $n_0(\text{Air}) = 1$, $n_1(\text{AlON}) \approx 1.7782$, $n_2(\text{Fe – InGaAsP}) \approx 3.3723 - 0.0021 \times I$. 
4.3 TM - Mode

Discussion of the Results

Figures (4.12) and (4.13) show the reflectance (R) and transmittance (T) versus the wavelength for different values of AlON thickness of \( l_1 \). The two figures show that the minimum reflectance and maximum transmittance appear at \( \lambda = 510 \) nm with \( l_1 = 70 \) nm, \( \lambda = 600 \) nm with \( l_1 = 82 \) nm and \( \lambda = 655 \) with \( l_1 = 90 \) nm. It is clear that the position of the minimum reflectance and maximum transmittance is shifted toward the long waves with increasing AlON thickness. In solar cells model, our interest is to get a minimum reflectance and maximum transmittance in a visible light region, especially at 600 nm, therefore the best thickness of AlON is \( l_1 = 82 \) nm. Figure (4.14) shows the variation of the absorption (\( A = 1 - R - T \)) with the operating wavelength for different values of the thickness of AlON layer. It can be seen that the maximum absorption in visible light region is increasing whenever the AlON thickness is decreasing.

\[ \begin{align*}
\epsilon_0 (\text{Air}) & \approx 1, \\
\epsilon_1 (\text{AlON}) & \approx 3.16199, \\
\epsilon_2 (\text{Fe-InGaAsP}) & \approx (3.3723 - .0021 * l)^2, \\
\epsilon_3 (\text{Si}) & \approx 13.4909, \quad \theta_0 = 0^\circ.
\end{align*} \]

Figure (4.12): Reflectance (R) versus the wavelength (\( \lambda \)) for different values of \( l_1 \) (thickness of AlON), \( l_2 = 20 \) nm (thickness of Fe-InGaAsP), \( \epsilon_0 (\text{Air}) \approx 1, \epsilon_1 (\text{AlON}) \approx 3.16199, \epsilon_2 (\text{Fe-InGaAsP}) \approx (3.3723 - .0021 * l)^2, \epsilon_3 (\text{Si}) \approx 13.4909, \theta_0 = 0^\circ. \]
Figure (4.13): Transmittance ($T$) versus the wavelength ($\lambda$) for different values of $l_1$ (thickness of AlON), $l_2 = 20$ nm (thickness of Fe-InGaAsP), $\varepsilon_0$ (Air) $\approx 1$, $\varepsilon_1$ (AlON) $\approx 3.16199$, $\varepsilon_2$ (Fe-InGaAsP) $\approx (3.3723 - .0021 \times I)^2$, $\varepsilon_3$ (Si) $\approx 13.4909$, $\theta_0 = 0^\circ$.

![Transmittance Graph]

Figure (4.14): Absorption ($A$) versus the wavelength ($\lambda$) for different values of $l_1$ (thickness of AlON), $l_2 = 20$ nm (thickness of Fe-InGaAsP), $\varepsilon_0$ (Air) $\approx 1$, $\varepsilon_1$ (AlON) $\approx 3.16199$, $\varepsilon_2$ (Fe-InGaAsP) $\approx (3.3723 - .0021 \times I)^2$, $\varepsilon_3$ (Si) $\approx 13.4909$, $\theta_0 = 0^\circ$.

![Absorption Graph]
The reflectance (R) and transmittance (T) versus the wavelength for different values of Fe-InGaAsP thickness are shown in Figures (4.15) and (4.16), respectively. It can be seen that the minimum reflectance increases as \( l_2 \) increases and the maximum transmittance increases as \( l_2 \) increases. The two figures show the best minimum reflectance and the best maximum transmittance appear at \( \lambda = 600 \text{ nm} \) with \( l_2 = 20 \text{ nm} \), therefore \( l_2 = 20 \text{ nm} \) is the best value of \( l_2 \).

Figure (4.17) shows the behaviour of the absorption versus the wavelength for different values of Fe-InGaAsP thickness. As the Fe-InGaAsP layer gets thicker the maximum absorption gets higher.

**Figure (4.15):** Reflectance (R) versus the wavelength (\( \lambda \)) for different values of \( l_2 \) (thickness of Fe-InGaAsP), \( l_1 = 82 \text{ nm} \) (thickness of AlON), \( \varepsilon_0 (\text{Air}) \approx 1 \), \( \varepsilon_1 (\text{AlON}) \approx 3.16199 \), \( \varepsilon_2 (\text{Fe-InGaAsP}) \approx (3.3723 - 0.0021 * l)^2 \), \( \varepsilon_3 (\text{Si}) \approx 13.4909 \), \( \theta_0 = 0^{\circ} \).
**Figure (4.16):** Transmittance (T) versus the wavelength (\(\lambda\)) for different values of \(l_2\) (thickness of Fe-InGaAsP), \(l_1 = 82\) nm (thickness of AlON), \(\varepsilon_0\) (Air) \(\approx 1\), \(\varepsilon_1\) (AlON) \(\approx 3.16199\), \(\varepsilon_2\) (Fe-InGaAsP) \(\approx (3.3723 - 0.0021 * I)^2\), \(\varepsilon_3\) (Si) \(\approx 13.4909\), \(\theta_0 = 0^\circ\).

**Figure (4.17):** Absorption (A) versus the wavelength (\(\lambda\)) for different values of \(l_2\) (thickness of Fe-InGaAsP), \(l_1 = 82\) nm (thickness of AlON), \(\varepsilon_0\) (Air) \(\approx 1\), \(\varepsilon_1\) (AlON) \(\approx 3.16199\), \(\varepsilon_2\) (Fe-InGaAsP) \(\approx (3.3723 - 0.0021 * I)^2\), \(\varepsilon_3\) (Si) \(\approx 13.4909\), \(\theta_0 = 0^\circ\).
In Figures (4.18) and (4.19) the reflectance (R) and transmittance (T) are plotted versus the wavelength for different incidence angle $\theta_0$. As can be seen from the two figures, the minimum reflectance and maximum transmittance which appear at $\lambda = 600$ nm with $\theta_0 = 0^\circ$, the two angles $\theta_0 = 0^\circ$ and $\theta_0 = 30^\circ$ are shown to be good incidence angles, but the best when $\theta_0 = 0$. The absorption versus the wavelength for different incident angle $\theta_0$ is shown in Figure (4.20). In visional light region, we see that the maximum absorption is increasing with the increasing incidence of the angle $\theta_0$.

![Reflectance graph](image)

**Figure (4.18):** Reflectance (R) versus the wavelength ($\lambda$) for different values of $\theta_0$(incidence angle), $l_2 = 20$ nm (thickness of Fe-InGaAsP), $l_1 = 82$ nm (thickness of AlON ), $\varepsilon_0$(Air) $\approx 1$, $\varepsilon_1$(AlON) $\approx 3.16199$, $\varepsilon_2$(Fe−InGaAsP) $\approx (3.3723 − .0021 * l)^2$, $\varepsilon_3$(Si) $\approx 13.4909$, $\theta_0 = 0^\circ$. 

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Figure (4.19): Transmittance ($T$) versus the wavelength ($\lambda$) for different values of $\theta_0$ (incidence angle), $l_2 = 20$ nm (thickness of Fe-InGaAsP), $l_1 = 82$ nm (thickness of AlON), $\varepsilon_0$ (Air) $\approx 1$, $\varepsilon_1$ (AlON) $\approx 3.16199$, $\varepsilon_2$ (Fe-InGaAsP) $\approx (3.3723 - 0.0021 \cdot I)^2$, $\varepsilon_3$ (Si) $\approx 13.4909$, $\theta_0 = 0^\circ$.

Figure (4.20): Absorption ($A$) versus the wavelength ($\lambda$) for different values of $\theta_0$ (incidence angle), $l_2 = 20$ nm (thickness of Fe-InGaAsP), $l_1 = 82$ nm (thickness of AlON), $\varepsilon_0$ (Air) $\approx 1$, $\varepsilon_1$ (AlON) $\approx 3.16199$, $\varepsilon_2$ (Fe-InGaAsP) $\approx (3.3723 - 0.0021 \cdot I)^2$, $\varepsilon_3$ (Si) $\approx 13.4909$, $\theta_0 = 0^\circ$. 
Figure (4.21) show the behaviour of the reflectance, transmittance and absorption at normal incidence for layers $l_1$ and $l_2$ as in figures (a) and (b), respectively.

**Figure (4.21):** Reflectance (R), Transmittance (T), and absorption (A) at normal incidence versus the wavelength for $l_1$ (thickness of AlON) = 82 nm, $l_2$ (thickness of Fe-InGaAsP) = 20 nm, $n_3$(Si) $\approx$ 3.6730, $n_0$(Air) $\approx$ 1, $n_1$(AlON) $\approx$ 1.7782, $n_2$(Fe − InGaAsP) $\approx$ 3.3723 − 0.0021 $\times$ $l$. 

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4.4 Comparison between TE and TM modes

In this section, we show the comparison between TE and TM configurations. The reflectance versus of the wavelength for \( l_1 = 82 \) nm thickness of AlON and \( l_2 = 20 \) nm thickness of Fe-InGaAsP for TE and TM modes as shown in Figure (4.22).

We noticed that the reflection for \( l_1 \) in TE and TM have the same behavior exactly, and the reflection for \( l_2 \) in TE and TM are equal almost and the transmittance has opposite values of the reflectance.

**Figure (4.22):** Reflectance (R) for layers \( l_1 \) and \( l_2 \) at normal incidence versus the wavelength for \( l_1 \) (thickness of AlON) = 82 nm, \( l_2 \) (thickness of Fe-InGaAsP) = 20 nm, \( n_3(\text{Si}) \approx 3.6730 \), \( n_0(\text{Air}) \approx 1 \), \( n_1(\text{AlON}) \approx 1.7782 \), \( n_2(\text{Fe-InGaAsP}) \approx 3.3723 - 0.0021 * i \).
Figure (4.23), show that the absorption (A) for \( l_1 \) and \( l_2 \) in TE and TM have the same behavior exactly.

**Figure (4.23):** Absorption (A) for layers \( l_1 \) and \( l_2 \) at normal incidence versus the wavelength for \( l_1 \) (thickness of AlON)= 82 nm, \( l_2 \) (thickness of Fe-InGaAsP) = 20 nm, \( n_3(\text{Si}) \approx 3.6730 \), \( n_0(\text{Air}) \approx 1 \), \( n_1(\text{AlON}) \approx 1.7782 \), \( n_2(\text{Fe-InGaAsP}) \approx 3.3723 - 0.0021 \times I \).
4.5 Average of Reflectance, Transmittance and Absorption:

The average reflectance ($R_{av}$), the average transmittance ($T_{av}$) and the average absorption ($A_{av}$) for the optimum values of $l_1$ and $l_2$ are calculated and plotted versus the wavelength for different values of incidence angle.

Where,

$$R_{av} = \frac{R_{TE} + R_{TM}}{2}, \quad T_{av} = \frac{T_{TE} + T_{TM}}{2}, \quad A_{av} = \frac{A_{TE} + A_{TM}}{2}$$

We see that the average reflectance ($R_{av}$), the average transmittance ($T_{av}$) and the average absorption ($A_{av}$) in Figures (2.24) until (2.29) have the same behaviour of the reflectance, the transmittance and the absorption in TE and TM modes, when the thickness of Layer $l_1$ and $l_2$ are changed.

![Average Reflectance](image)

**Figure (4.24):** Average reflectance ($R_{av}$) versus the wavelength ($\lambda$) for different values of $l_1$ (thickness of AlON), $l_2 = 20$ nm (thickness of Fe-InGaAsP), $\varepsilon_0$ (Air) $\approx 1$, $\varepsilon_1$ (AlON) $\approx 3.16199$, $\varepsilon_2$ (Fe-InGaAsP) $\approx (3.3723 - .0021 \cdot l)^2$, $\varepsilon_3$ (Si) $\approx 13.4909$, $\theta_0 = 0^\circ$. 

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Figure (4.25): Average transmittance ($T_{av}$) versus the wavelength ($\lambda$) for different values of $l_1$ (thickness of AlON), $l_2 = 20$ nm (thickness of Fe-InGaAsP), $\varepsilon_0$ (Air) $\approx 1$, $\varepsilon_1$ (AlON) $\approx 3.16199$, $\varepsilon_2$(Fe-InGaAsP) $\approx (3.3723 - .0021 \times I)^2$, $\varepsilon_3$(Si) $\approx 13.4909$, $\theta_0 = 0^\circ$.

Figure (4.26): Average absorption ($A_{av}$) versus the wavelength ($\lambda$) for different values of $l_1$ (thickness of AlON), $l_2 = 20$ nm (thickness of Fe-InGaAsP), $\varepsilon_0$ (Air) $\approx 1$, $\varepsilon_1$ (AlON) $\approx 3.16199$, $\varepsilon_2$(Fe-InGaAsP) $\approx (3.3723 - .0021 \times I)^2$, $\varepsilon_3$(Si) $\approx 13.4909$, $\theta_0 = 0^\circ$. 
Figure (4.27): Average reflectance ($R_{av}$) versus the wavelength ($\lambda$) for different values of $l_2$ (thickness of Fe-InGaAsP), $l_1 = 82$ nm (thickness of AlON), $n_0$ (Air) $\approx 1$, $n_1$ (AlON) $\approx 1.7632$, $n_2$ (Fe–InGaAsP) $\approx 3.3723 - 0.0021 \times I$, $n_3$ (Si) $\approx 3.673$, $\theta_0 = 0^\circ$.

Figure (4.28): Average transmittance ($T_{av}$) versus the wavelength ($\lambda$) for different values of $l_2$ (thickness of Fe-InGaAsP), $l_1 = 82$ nm (thickness of AlON), $n_0$ (Air) $\approx 1$, $n_1$ (AlON) $\approx 1.7632$, $n_2$ (Fe–InGaAsP) $\approx 3.3723 - 0.0021 \times I$, $n_3$ (Si) $\approx 3.673$, $\theta_0 = 0^\circ$. 
Figure (4.29): Average absorption ($A_{av}$) versus the wavelength ($\lambda$) for different values of $l_2$ (thickness of Fe-InGaAsP), $l_1 = 82$ nm (thickness of AlON), $n_0$ (Air) $\approx 1$, $n_1$ (AlON) $\approx 1.7632$, $n_2$ (Fe-InGaAsP) $\approx 3.3723 - 0.0021i$, $n_3$ (Si) $\approx 3.673$, $\theta_0 = 0^\circ$.

In Figures (4.30) and (4.31) the average reflectance ($R_{av}$) and the average transmittance ($T_{av}$) are plotted versus the wavelength for different incidence angle $\theta_0$. As can be seen from the two figures, the minimum average reflectance and maximum average transmittance have been observed at $\lambda = 600$ nm with $\theta_0 = 0^\circ$. The two angles $\theta_0 = 0^\circ$ and $\theta_0 = 30^\circ$ are shown to be good incidence angles, but the best when $\theta_0 = 0^\circ$. Also, we see that the behaviour of the average transmittance invert the behaviour of the average reflectance.

The absorption versus the wavelength for different incident angle $\theta_0$ is shown in Figure (4.20). We see that the maximum average absorption increases with the increasing incidence of the angle $\theta_0$ when the wavelength is limited between (300-530) nm, and the maximum average absorption increasing with decreasing incidence angle $\theta_0$ when the wavelength is limited between (530-1200) nm.
Figure (4.30): Average reflectance ($R_{av}$) versus the wavelength ($\lambda$) for different values of $\theta_0$ (incidence angle), $l_2 = 20$ nm (thickness of Fe-InGaAsP), $l_1 = 82$ nm (thickness of AlON), $n_0$ (Air) $\approx 1$, $n_1$ (AlON) $\approx 1.7632$, $n_2$ (Fe-InGaAsP) $\approx 3.3723 - 0.0021i$, $n_3$ (Si) $\approx 3.673$, $\theta_0 = 0^\circ$.

Figure (4.31): Average transmittance ($T_{av}$) versus the wavelength ($\lambda$) for different values of $\theta_0$ (incidence angle), $l_2 = 20$ nm (thickness of Fe-InGaAsP), $l_1 = 82$ nm (thickness of AlON), $n_0$ (Air) $\approx 1$, $n_1$ (AlON) $\approx 1.7632$, $n_2$ (Fe-InGaAsP) $\approx 3.3723 - 0.0021i$, $n_3$ (Si) $\approx 3.673$, $\theta_0 = 0^\circ$. 
Figure (4.32): Average absorption ($A_{av}$) versus the wavelength ($\lambda$) for different values of $\theta_0$ (incidence angle), $l_2 = 20$ nm (thickness of Fe-InGaAsP), $l_1 = 82$ nm (thickness of AlON), $n_0$ (Air) $\approx 1$, $n_1$ (AlON) $\approx 1.7632$, $n_2$ (Fe-InGaAsP) $\approx 3.3723 - 0.0021 * I$, $n_3$ (Si) $\approx 3.673$, $\theta_0 = 0^\circ$.

4.6 Comparison between TE and TM modes and average

The average reflectance (transmittance) and the reflectance (transmittance) of TE and TM modes versus the wavelength. Figures from (4.33) to (4.36) show that the average reflectance and the reflectance of TE and TM modes for $l_1$ and $l_2$ layers have the same behavior. Also, for average transmittance and the transmittance of TE and TM modes, since the transmittance have opposite values of the reflectance.
Figure (4.33): Average reflectance, TE reflectance and TM reflectance for layer \( l_1 \) versus the wavelength for \( l_1 \) (thickness of AlON) = 82 nm, \( l_2 \) (thickness of Fe-InGaAsP) = 20 nm, \( n_3(\text{Si}) \approx 3.6730, n_0(\text{Air}) \approx 1, n_1(\text{AlON}) \approx 1.7782, n_2(\text{Fe-InGaAsP}) \approx 3.3723 - 0.0021 * 1, \theta_0 = 0^\circ \).

Figure (4.34): Average transmittance, TE transmittance and TM transmittance for layer \( l_1 \) versus the wavelength for \( l_1 \) (thickness of AlON) = 82 nm, \( l_2 \) (thickness of Fe-InGaAsP) = 20 nm, \( n_3(\text{Si}) \approx 3.6730, n_0(\text{Air}) \approx 1, n_1(\text{AlON}) \approx 1.7782, n_2(\text{Fe-InGaAsP}) \approx 3.3723 - 0.0021 * 1, \theta_0 = 0^\circ \).
Figure (4.35): Average reflectance, TE reflectance and TM reflectance for layer $l_2$ versus the wavelength for $l_1$ (thickness of AlON) $= 82$ nm, $l_2$ (thickness of Fe-InGaAsP) $= 20$ nm, $n_3$ (Si) $\approx 3.6730$, $n_0$ (Air) $\approx 1$, $n_1$ (AlON) $\approx 1.7782$, $n_2$ (Fe−InGaAsP) $\approx 3.3723 - 0.0021 \cdot I$, $\theta_0 = 0^\circ$.

Figure (4.36): Average transmittance, TE transmittance and TM transmittance for layer $l_2$ versus the wavelength for $l_1$ (thickness of AlON) $= 82$ nm, $l_2$ (thickness of Fe-InGaAsP) $= 20$ nm, $n_3$ (Si) $\approx 3.6730$, $n_0$ (Air) $\approx 1$, $n_1$ (AlON) $\approx 1.7782$, $n_2$ (Fe−InGaAsP) $\approx 3.3723 - 0.0021 \cdot I$, $\theta_0 = 0^\circ$. 
Chapter 5
Conclusion
Chapter 5

Conclusion

In this thesis, we used and analyzed the hybrid mode method to get the Transfer matrix method (TMM), in order to calculate the reflectance, transmittance and absorption at solar cell model of N-Layers. After that, I worked to find the special case of the hybrid mode method, is the case of getting both transverse electric (TE) and transverse magnetic (TM), and then analyze the extracted numerical result of (TM) and (TE) and then the computations have been carried out.

The four layers solar cell model has been designed and investigated. The waveguide structure under consideration has an ultra-thin film nonmagnetic metal ($\mu = 1$) on (Si) substrate and covered by Fe-InGaAsP layer, Aluminium oxynitride (AlON) layer that is exposed to air directly. The reflectance, transmittance and absorption have been derived by using the transfer matrix method.

Numerical results have been plotted using a computer program (Maple). Moreover, the submitted waveguide structure has been studied for TE-mode and TM-mode as special cases of Hybrid modes.

Fe-InGaAsP layer was used and then the reflectance (R), the transmittance (T) and the absorption (A) were plotted versus the wavelength with different values for AlON layer thickness, Fe-InGaAsP layer thickness and the incidence angle.

The results indicate the following points:

- The minimum value of reflectance and the maximum value of transmittance for the normal incidence of light in TE and TM modes are identical perfectly.
- There are deviations between TE and TM modes when the incidence angle increases and it increases with the increasing of the incidence angle.
- The minimum value of reflectance and the maximum value of transmittance are shifted toward higher wavelengths with increasing AlON layer.
- The absorption decreases with increasing AlON layer thickness
• The minimum reflectance decreases as Fe-InGaAsP thickness increases and the maximum transmittance increases as Fe-InGaAsP thickness increases.
• The minimum value of reflectance and the maximum value of transmittance has the same wavelengths at 600 nm with increasing Fe-InGaAsP layer thickness.
• The absorption increases with the increasing of Fe-InGaAsP layer thickness.
• The minimum reflectance is decreasing and the maximum transmittance is increasing whenever the incidence angle is decreasing.
• In TE-mode, the maximum absorption increases with the decreasing of the incidence angle.
• In TM-mode, the maximum absorption increases when the incidence angle increases.
• In general, the absorption in the proposed waveguide structure is very weak.
Bibliography


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