Periodically Correlated Time Series Models: 
Representation and Identification

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Dedication

To the spirit of my father and my sister

To my mother

To my wife

To my daughter Dana

To all knowledge seekers...
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## Abbreviations

<table>
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<tr>
<th>Abbreviation</th>
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<tr>
<td>PC</td>
<td>Periodically correlated.</td>
</tr>
<tr>
<td>PeACVF</td>
<td>Periodic autocovariance function</td>
</tr>
<tr>
<td>PeACF</td>
<td>Periodic autocorrelation function</td>
</tr>
<tr>
<td>PePACF</td>
<td>Periodic partial autocorrelation function</td>
</tr>
<tr>
<td>PAR</td>
<td>Periodic autoregression process</td>
</tr>
<tr>
<td>PAR(_d(p_1, p_2, \ldots, p_d))</td>
<td>Periodic autoregression model with period (d) and order ((p_1, p_2, \ldots, p_d))</td>
</tr>
<tr>
<td>PMA</td>
<td>Periodic moving average process</td>
</tr>
<tr>
<td>PMA(_d(q_1, q_2, \ldots, q_d))</td>
<td>Periodic moving average model with period (d) and order (q)</td>
</tr>
<tr>
<td>PARMA</td>
<td>Periodic autoregression moving average process</td>
</tr>
<tr>
<td>PARMA(_d(p_1; q_1, \ldots, p_d; q_d))</td>
<td>Periodic autoregression moving average model with period (d) and orders ((p_1; q_1, \ldots, p_d; q_d))</td>
</tr>
<tr>
<td>VQ</td>
<td>Vector of quarters representation of a periodic autoregression model.</td>
</tr>
<tr>
<td>MC</td>
<td>Multi-companion representation of a periodic autoregression model.</td>
</tr>
<tr>
<td>VAR</td>
<td>Vector Autoregression Model</td>
</tr>
<tr>
<td>VMA</td>
<td>Vector Moving Average Model</td>
</tr>
<tr>
<td>VARMA</td>
<td>Vector Autoregressive Moving Average Model</td>
</tr>
<tr>
<td>VAR((P))</td>
<td>Vector Autoregression Model of order (P)</td>
</tr>
<tr>
<td>VMA((Q))</td>
<td>Vector Moving Average Model of order (Q)</td>
</tr>
<tr>
<td>VARMA((P, Q))</td>
<td>Vector Autoregressive Moving Average Model of order ((P, Q))</td>
</tr>
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</table>
Abstract

This thesis is interested with some characteristics of a class periodically correlated (PC) time series models. We study both periodically correlated time series models and multiple models and discuss the relationship between them. Give many examples.

We discuss in this work many representations of PC models and use these representations in order make evidence that PC class and multiple AR models are theoretically the same. In addition, we propose a new representation, the multi-companion (MC) representation. Give an example.

We discuss the relationship between the vector moving average and the periodic moving average, and give an examples to clarify the relationship between them. Also, we study the periodic autoregressive moving-average (PARMA) models and their representations. We show that any PARMA model can be expressed as a vector ARMA model.

We consider the identification of orders of periodic AR (PAR) models by extending well known techniques to periodic time series, we discuss methods for specifying models and for efficiently estimating the parameters in those models. Finally, a detailed simulation study is given to illustrate the procedures.
Introduction

Many time series we come across in the real world exhibit nonstationary and nonlinear properties. It has been found that many meteorological variables such as rainfall, global temperature are nonstationary.

Most estimation techniques depend on the assumption that the series is stationary. A special class of nonstationary time series has been defined by Gladyshev [8], called periodically correlated time series. These time series are nonstationary, but have periodic means and covariances. So they can be considered as a bridge (weak) stationary and nonstationary processes. It have been shown by Boswijk and Franses [7] that aperiodic autoregressive time series model assumes that the autoregressive parameters vary with the season. In special case, the periodic autoregressions can be represented by a multivariate autoregression model for the annual series containing the observations per-season, and a multivariate AR models equivalent to periodic AR models. In the similar way we can defined periodic moving average and periodic autoregressive moving average (PARMA) and discuss some properties. Also, we can demonstrate that PARMA models can be written as equivalent multivariate ARMA models, conversely, multivariate ARMA models can be represented as PARMA models.

This thesis is organized as follows. We start by recalling background of the multivariate time series and its properties in chapter 1. Chapter 2 introduces some of these periodic models such as the periodic autoregressive (PAR) models, periodic moving average (PMA) models, and the periodic autoregressive moving average (PARMA) models and their properties. Chapter 3 gives the representation of a PAR model, we discuss the various representations, namely the L-form, U-form, and I-form. Also introduce the multi-companion (MC) representation, of the PAR process. In Chapter 4, we consider the identification of pure periodic autoregressive (PAR) models. We consider the model estimation by solving the Yule-Walker equations, we conduct diagnostic checking through residuals of the fitted model.
Chapter 1

Multivariate Time Series

1.1 Introduction

Consider $m$ time series variables $\{X_{1t}\}, \{X_{2t}\}, \ldots, \{X_{mt}\}$, with $E(X_{it}^2) < \infty$ for all $t$ and $i$. A multivariate time series is the $(m \times 1)$ vector time series

$$X_t = \begin{pmatrix} X_{1t} \\ X_{2t} \\ \vdots \\ X_{mt} \end{pmatrix}$$

(1.1.1)

later on, we will use the notation $X_t = (X_{1t}, X_{2t}, \ldots, X_{mt})^\prime$, where $(\cdot)^\prime$ means the transpose of a vector.

The second-order properties of the multivariate time series $\{X_t\}$ are then specified by the mean vectors and covariance matrices.

Definition 1.1.1. [11] The mean of $X_t$ is defined as the $(m \times 1)$ vector

$$\mu = (\mu_1, \mu_2, \ldots, \mu_m)^\prime$$

(1.1.2)

where $\mu_i = E(X_{it})$ for $i = 1, 2, \ldots, m$. 

2
Definition 1.1.2. [11] The variance covariance matrix of \( X_t \) is the \( m \times m \) matrix

\[
\text{Var}(X_t) = \Gamma_0 = E[(X_t - \mu)(X_t - \mu)']
\]

\[
= \begin{pmatrix}
\gamma^{(0)}_{11} & \gamma^{(0)}_{12} & \cdots & \gamma^{(0)}_{1m} \\
\gamma^{(0)}_{21} & \gamma^{(0)}_{22} & \cdots & \gamma^{(0)}_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
\gamma^{(0)}_{m1} & \gamma^{(0)}_{m2} & \cdots & \gamma^{(0)}_{mm}
\end{pmatrix}
\]

where \( \gamma^{(0)}_{ij} = \text{cov}(X_{it}, X_{jt}) \).

The autocovariances of \( X_{it} \) with lag \( h \) for \( i = 1, 2, \ldots, m \) are defined as

\[
\gamma^{(h)}_{ii} = \text{Cov}(X_{it}, X_{i,t-h}) 
\] (1.1.3)

The covariance matrix with lag \( h \) is defined as follows

\[
\Gamma_h = \text{Cov}(X_t, X_{t-h})
\]

\[
= E[(X_t - \mu)(X_{t-h} - \mu)']
\]

\[
= \begin{pmatrix}
\text{Cov}(X_{1t}, X_{1,t-h}) & \text{Cov}(X_{1t}, X_{2,t-h}) & \cdots & \text{Cov}(X_{1t}, X_{m,t-h}) \\
\text{Cov}(X_{2t}, X_{1,t-h}) & \text{Cov}(X_{2t}, X_{2,t-h}) & \cdots & \text{Cov}(X_{2t}, X_{m,t-h}) \\
\vdots & \vdots & \ddots & \vdots \\
\text{Cov}(X_{mt}, X_{1,t-h}) & \text{Cov}(X_{mt}, X_{2,t-h}) & \cdots & \text{Cov}(X_{mt}, X_{m,t-h})
\end{pmatrix}. 
\] (1.1.4)

Definition 1.1.3. The \( m \)-variate time series \( \{X_t\} \) is stationary if the mean vector \( \mu \) is independent of \( t \) and covariance matrix \( \Gamma_h \) is independent of \( t \) for each \( h \).

The simplest multivariate time series is multivariate white noise, the definition of which is quite analogous to that of univariate white noise.

Definition 1.1.4. [11] The \( m \)-variate time series \( \{\epsilon_t\} \) is said to be white noise with mean vector \( 0 \) and covariance matrix \( \Sigma \), written
{\epsilon_t} \sim \text{WN}(0, \Sigma),

If and only if \{\epsilon_t\} is (weak) stationary with mean vector \textbf{0} and covariance matrix function,

\[ \Gamma_h = \text{Cov}(\epsilon_t, \epsilon_{t-h}) = \begin{cases} 
\Sigma, & \text{if } h = 0, \\
0, & \text{otherwise}, 
\end{cases} \quad \text{where } \Sigma = \begin{pmatrix} 
\sigma^2 & \cdots \\
\cdots & \sigma^2 
\end{pmatrix} \]

1.2 Vector Autoregression Models (VAR)

The vector autoregression model is one of the most successful, flexible, and easy to use models for analysis of multivariate time series. It is a natural extension of the univariate autoregressive model to dynamic multivariate time series. The vector autoregression model has proven to be especially useful for describing the dynamic behavior of economic and financial time series and for forecasting.

**Definition 1.2.1.** [12] Vector autoregression models

A \(K\)-dimensional vector autoregression model of order \(P\) is denoted by \(\text{VAR}(P)\) and defined as

\[ X_t = \Phi_1 X_{t-1} + \cdots + \Phi_P X_{t-P} + \epsilon_t \quad (1.2.1) \]

where \(X_t = \begin{pmatrix} X_{1t} \\ X_{2t} \\ \vdots \\ X_{Kt} \end{pmatrix}\) is a \((K \times 1)\) random vector, \(\Phi_i\) are fixed \(K \times K\) coefficient matrices. Finally

\[ \epsilon_t = \begin{pmatrix} \epsilon_{1t} \\ \epsilon_{2t} \\ \vdots \\ \epsilon_{Kt} \end{pmatrix} \] is a \(K\)-dimensional that is \(\{\epsilon_t\} \sim \text{WN}(0, \Sigma)\).
The backward shift operator notation

The presentation of time series models is simplified using backward shift operator notation. The backward shift operator $B$ is defined that for any time series $\{X_t\}$, $BX_t = X_{t-1}$. The backward shift operator has the following properties: $B^2X_t = B(BX_t) = BX_{t-1} = X_{t-2}$, and $B^0 = 1$. Thus,

$$B^kX_t = X_{t-k} \tag{1.2.2}$$

We can rewrite Equation (1.2.1) by using the backward shift operator $B$ $(BX_t = X_{t-1})$ such that

$$(I_K - \Phi_1 B - \cdots - \Phi_P B^P)X_t = \epsilon_t. \tag{1.2.3}$$

For simplicity, we can use the short notation to write Equation (1.2.3) as follows

$$\Phi(B)X_t = \epsilon_t, \tag{1.2.4}$$

where $\Phi(B) = I_K - \Phi_1 B - \cdots - \Phi_P B^P$ is a matrix polynomial and $I_K$ is $K \times K$ identity matrix. We call $\Phi(z)$ the characteristic polynomial of VAR($P$) Models

Now consider the bivariate case i.e., $K = 2$, $X_t = (X_{1t}, X_{2t})'$ and $\epsilon_t = (\epsilon_{1t}, \epsilon_{2t})'$. The VAR(1) model consists of the following two equations:

$$X_{1t} = \phi_{11}X_{1,t-1} + \phi_{12}X_{2,t-1} + \epsilon_{1t} \tag{1.2.5}$$

$$X_{2t} = \phi_{21}X_{1,t-1} + \phi_{22}X_{2,t-1} + \epsilon_{2t}. \tag{1.2.6}$$

The Equations (1.2.5) and (1.2.6) can be rewrite as

$$\begin{bmatrix} X_{1t} \\ X_{2t} \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} X_{1,t-1} \\ X_{2,t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1t} \\ \epsilon_{2t} \end{bmatrix}$$

where $\phi_{ij}$ is the $(i, j)$ th elements of $\Phi$. If $\phi_{12} = 0$, then $X_{1t}$ does not depends on $X_{2,t-1}$, and the model show that $X_{1t}$ only depends on its own past. Similarly if $\phi_{21} = 0$, then
Equation (1.2.6) shows that $X_{2t}$ does not depend on $X_{1,t-1}$ when $X_{2,t-1}$ is given. (See, Ruey S. Tsay [21]).

Consider the VAR(1) model. It is easy to derive the representation as

$$X_t = \Phi X_{t-1} + \epsilon_t$$

$$= \Phi(\Phi X_{t-2} + \epsilon_{t-1}) + \epsilon_t$$

$$= \Phi^2 X_{t-2} + \Phi \epsilon_{t-1} + \epsilon_t$$

$$= \epsilon_t + \Phi \epsilon_{t-1} + \Phi^2 X_{t-2}$$

and continuing this process we have

$$X_t = \epsilon_t + \Phi X_{t-1} + \Phi^2 X_{t-2} + \cdots$$

$$= \sum_{j=0}^{\infty} \Phi^j \epsilon_{t-j}, \quad t = 0, \pm 1, \pm 2, \ldots$$

So the VAR(1) model can be expressed as an infinite moving average $\text{MA}(\infty)$. Mean that VAR(1) is stationary.

**Definition 1.2.2. Stationarity of VAR($P$) Models**

The VAR($P$) model $\{X_t\}$ is stationary, if the condition hold

$$|\Phi(z)| \neq 0 \text{ for all } z \in \mathbb{C} \text{ such that } |z| \leq 1. \quad (1.2.7)$$

That means the roots of the characteristic equation are outside the unit circle.

**Example 1.2.1.** Consider the bivariate (two-dimensional) VAR(2) process

$$X_t = \Phi_1 X_{t-1} + \Phi_2 X_{t-2} + \epsilon_t \quad (1.2.8)$$

with $K = 2$, $\Phi_1 = \begin{bmatrix} 0.5 & -0.1 \\ 0.5 & 0.3 \end{bmatrix}$ and $\Phi_2 = \begin{bmatrix} 0.2 & 0.3 \\ 0.1 & 0.2 \end{bmatrix}$. 

6
Let us check for stationary by calculating the roots.

The characteristic polynomial is

\[
|I_{2x2} - \Phi_1z - \Phi_2z^2| = \begin{vmatrix}
1 & 0 & 0.5 & -0.1 \\
0 & 1 & 0.5 & 0.3 \\
0 & 1 & 0.5z & -0.1z \\
0 & 1 & 0.5z & 0.3z
\end{vmatrix}
\]

\[
= \begin{vmatrix}
1 & 0 & 0.2 & 0.3 \\
0 & 1 & 0.1 & 0.2 \\
0 & 1 & 0.2z^2 & 0.3z^2 \\
0 & 1 & 0.1z^2 & 0.2z^2
\end{vmatrix}z^2
\]

\[
= (1 - 0.5z - 0.2z^2)(1 - 0.3z - 0.2z^2) - ((0.1z - 0.3z^2)(-0.5z - 0.1z^2))
\]

\[
= 0.8z - 0.2z^2 + 0.02z^3 + 0.01z^4.
\]

We can easily calculate the roots of this polynomial by using R [17], by typing the following R command

\[
> \text{polyroot(c(1, -0.8, -0.2, 0.02, 0.01))}
\]

\[
[1] 1.02722-0.000000i 	ext{ -3.90177+2.271127i} 	ext{ -3.90177-2.271127i} 	ext{ 4.77632+0.000000i}
\]

Hence, the roots of this polynomial are

\[
z_1 = 1.03, \quad z_2 = -3.9 + 2.27i, \quad z_3 = z_1 = -3.9 + 2.27i, \quad z_4 = 4.77632.
\]

Since all roots are greater than 1 in absolute value. Therefore the model in Equation (1.2.8) is stationary.

**Note that** `polyroot( )` is a built in function in R to find all roots of a polynomial

**Example 1.2.2.** An example of stationary three-dimensional VAR(1) process

\[
X_{1t} = 0.3 \ X_{1,t-1} + \epsilon_{1t}
\]

\[
X_{2t} = 0.1 \ X_{1,t-1} + 0.1 \ X_{2,t-1} + 0.3 \ X_{3,t-1} + \epsilon_{2t}
\]

\[
X_{3t} = 0.1 \ X_{2,t-1} + 0.3 \ X_{3,t-1} + \epsilon_{3t}
\]

where \( \Phi = \begin{bmatrix} 0.3 & 0 & 0 \\ 0.1 & 0.1 & 0.3 \\ 0 & 0.1 & 0.3 \end{bmatrix} \).

The characteristic polynomial is
\[
|I_{3 \times 3} - \Phi z| = \left| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.3 & 0 & 0 \\ 0.1 & 0.1 & 0.3 \\ 0 & 0.1 & 0.3 \end{bmatrix} z \right|
\]
\[
= \left| \begin{bmatrix} 1 - 0.3z & 0 & 0 \\ -0.1z & 1 - 0.1z & -0.3z \\ 0 & -0.1z & 1 - 0.3z \end{bmatrix} \right|
\]
\[
= (1 - 0.3z)(1 - 0.4z)
\]
\[
= 1 - 0.7z + 0.12z^2.
\]

By using the same R command as in Example (1.2.1) that is

\[
z_1 = 2.5, \quad z_2 = 3.333.
\]

So all the roots greater than 1 in absolute value. The model in (1.2.9) is stationary

**Example 1.2.3.** We can construct example for nonstationary. Consider the three-dimensional of VAR(1) process

\[
X_{1t} = 0.5 \ X_{1,t-1} + \epsilon_{1t}
\]
\[
X_{2t} = X_{1,t-1} + X_{2,t-1} + 0.3 \ X_{3,t-1} + \epsilon_{2t}
\]
\[
X_{3t} = 0.2 \ X_{2,t-1} + 0.3 \ X_{3,t-1} + \epsilon_{3t}.
\]

(1.2.10)

The characteristic polynomial is

\[
|I_{3 \times 3} - \Phi z| = \left| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.5 & 0 & 0 \\ 1 & 1 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} z \right|
\]
\[
= \left| \begin{bmatrix} 1 - 0.5z & 0 & 0 \\ -z & 1 - z & -0.3z \\ 0 & -0.2z & 1 - 0.3z \end{bmatrix} \right|
\]
\[
= (1 - 0.5z)(1 - 1.3z + 0.24z^2)
\]
\[
= 1 - 1.7z + 0.89z^2 - 0.12z^3.
\]
The roots of this polynomial are

\[ z_1 = 0.92, \quad z_2 = 2, \quad z_3 = 4.49. \]

The model in (1.2.10) is nonstationary because one of the roots less than one.

### 1.3 Vector Moving Average Models (VMA)

Before we study vector autoregressive moving average model, we will discuss some properties of vector moving average in this section.

**Definition 1.3.1. Vector Moving Average Models**

A \( K \)-dimensional vector moving average model of order \( Q \) is denoted by VMA(\( Q \)) and defined as

\[
X_t = \epsilon_t + \Theta_1 \epsilon_{t-1} + \cdots + \Theta_Q \epsilon_{t-Q}
\]

(1.3.1)

where \( X_t = (X_{1t}, \ldots, X_{Kt})' \), \( \epsilon_t = (\epsilon_{1t}, \ldots, \epsilon_{Kt})' \) is zero mean white noise with nonsingular covariance matrix \( \Sigma \), i.e., \( \epsilon_t \sim \text{WN}(0, \Sigma) \) and \( \Theta_i \) are fixed \((K \times K)\) coefficient matrices.

We can rewrite Equation (1.3.1) by using the backward shift operator \( B \) (i.e., \( B\epsilon_t = \epsilon_{t-1} \)) such that

\[
X_t = (I_K + \Theta_1 + \cdots + \Theta_Q B^Q) \epsilon_t
\]

(1.3.2)

for simplicity, we can use the short notation

\[
X_t = \Theta(B) \epsilon_t
\]

(1.3.3)

where \( \Theta(B) = I_K + \Theta_1 B + \cdots + \Theta_Q B^Q \) is a matrix polynomial and \( I_K \) is \( K \times K \) identity matrix.

Consider a first order moving average process of dimension two [21]:

\[
\begin{align*}
X_{1t} &= \epsilon_{1t} + \theta_{11} \epsilon_{1,t-1} + \theta_{12} \epsilon_{2,t-1} \\
X_{2t} &= \epsilon_{2t} + \theta_{21} \epsilon_{1,t-1} + \theta_{22} \epsilon_{2,t-1}.
\end{align*}
\]

(1.3.4)
The system (1.3.4) may be written in a matrix notation as

\[
\begin{bmatrix}
X_{1t} \\
X_{2t} \\
X_{3t}
\end{bmatrix} =
\begin{bmatrix}
\epsilon_{1t} \\
\epsilon_{2t} \\
\epsilon_{3t}
\end{bmatrix} +
\begin{bmatrix}
\theta_{11} & \theta_{12} \\
\theta_{21} & \theta_{22} \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\epsilon_{1,t-1} \\
\epsilon_{2,t-1} \\
\epsilon_{3,t-1}
\end{bmatrix}.
\]

The characteristic polynomial of a vector moving average VMA(\(Q\)) is denoted by \(\Theta(z)\) and defined as follows:

\[
\Theta(z) = I_K + \Theta_1 z + \cdots + \Theta_Q z^Q.
\]

**Definition 1.3.2.** (Brockwell and Davis [5]) **Invertibility of VMA(\(Q\)) Models**

The VMA(\(Q\)) model \(\{X_t\}\) is said to be invertible if there exist matrices \(\{\Pi_j\}\) with absolutely summable components such that \(\{X_t\}\) can be expressed as a vector infinite autoregressive process

\[
\epsilon_t = \sum_{j=0}^{\infty} \Pi_j X_{t-j} \text{ for all } t.
\]

Invertibility is equivalent to the condition

\[
|\Theta(z)| \neq 0 \text{ for all } z \in \mathbb{C} \text{ such that } |z| \leq 1.
\] (1.3.5)

That means the roots of the characteristic equation are outside the unit circle.

The matrices \(\Pi_j\) are found by the recursively from the equations

\[
\Pi_j = -\Phi_j - \sum_{k=1}^{\infty} \Theta_k \Pi_{j-k}, \quad j = 0, 1, \ldots
\]

where \(\Phi_0 = -I, \Phi_j = 0 \text{ for } j > P, \Theta_j = 0 \text{ for } j > Q, \text{ and } \Pi_j = 0 \text{ for } j < 0.\)

**Example 1.3.1.** An example of an invertible three-dimensional VMA(1) process

\[
\begin{bmatrix}
X_{1t} \\
X_{2t} \\
X_{3t}
\end{bmatrix} =
\begin{bmatrix}
\epsilon_{1t} \\
\epsilon_{2t} \\
\epsilon_{3t}
\end{bmatrix} +
\begin{bmatrix}
0.3 & 0 & 0 \\
0.1 & 0.2 & 0.4 \\
0.3 & 0.1 & 0
\end{bmatrix}
\begin{bmatrix}
\epsilon_{1,t-1} \\
\epsilon_{2,t-1} \\
\epsilon_{3,t-1}
\end{bmatrix}.
\] (1.3.6)
The characteristic polynomial is

\[
|I_{3\times3} + \Theta z| = \begin{vmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{vmatrix} + \begin{vmatrix}
0.3 & 0 & 0 \\
0.1 & 0.2 & 0.4 \\
0.3 & 0.1 & 0
\end{vmatrix} z
= \begin{vmatrix}
1 + 0.3z & 0 & 0 \\
0.1z & 1 + 0.2z & 0.4z \\
0.3z & 0.1z & 1
\end{vmatrix}
= (1 + 0.3z)(1 + 0.2z - 0.04z^2)
\]

The roots are

\[z = -3.09, \quad z = -3.33, \quad z = 8.09.\]

Since all roots greater than 1 in absolute value. The model in (1.3.6) is an invertible.

**Example 1.3.2.** Consider the three-dimensional VMA(1) process

\[
(X_{1t} = \epsilon_{1t} + 0.5 \epsilon_{1,t-1} + 0.3 \epsilon_{2,t-1} + 0.2 \epsilon_{3,t-1})
X_{2t} = \epsilon_{2t} + \epsilon_{2,t-1} + 0.5 \epsilon_{3,t-1}
X_{3t} = \epsilon_{3t} + \epsilon_{2,t-1} + 0.6 \epsilon_{3,t-1}
\]

The characteristic polynomial is

\[
|I_{3\times3} + \Theta z| = \begin{vmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{vmatrix} + \begin{vmatrix}
0.5 & 0.3 & 0.2 \\
0 & 0 & 0.5 \\
0 & 0 & 0.6
\end{vmatrix} z
= \begin{vmatrix}
1 + 0.5z & 0.3z & 0.2z \\
0 & 1 + z & 0.5z \\
0 & z & 1 + 0.6z
\end{vmatrix}
= (1 + 0.5z)(1 + 1.7z + 0.1z^2)
= 1 + 2.2z + 0.9z^2 + 0.5z^3)
\]
The roots are

\[ z_1 = -0.59, \quad z_2 = -2.22, \quad z_3 = -15.19. \]

The model in (1.3.7) is not invertible because one of the roots less than one in absolute value.

### 1.4 Vector Autoregressive Moving Average Models (VARMA)

We have considered pure VAR\((P)\) and pure VMA\((Q)\) processes. We can combine the two from a broader class of models where is both a VMA and VAR components. This is called a VARMA\((P,Q)\) or vector autoregressive moving average.

**Definition 1.4.1.** (Brockwell and Davis [5])

A stationary process, \(\{X_t\}\) is a vector autoregressive moving average processes of order \((P,Q)\), VARMA\((P,Q)\) if for every \(t\),

\[
X_t - \Phi_1 X_{t-1} - \cdots - \Phi_P X_{t-P} = \epsilon_t + \Theta_1 \epsilon_{t-1} + \cdots + \Theta_Q \epsilon_{t-Q} \tag{1.4.1}
\]

where \(\{\epsilon_t\} \sim (0, \Sigma)\).

Equation (1.4.1) can be written in the short notation by using the backward shift operator \(B\)

\[
\Phi(B)X = \Theta(B)\epsilon_t \tag{1.4.2}
\]

where \(\Phi(z) = I_K - \Phi_1 z - \cdots - \Phi_P z^P\) and \(\Theta(z) = I_K + \Theta_1 z + \cdots + \Theta_Q z^Q\) are matrix valued polynomials \(I_K\) is the identity matrix. Each component of the matrices \(\Phi(z), \Theta(z)\) is a polynomial with real coefficients and degree less than or equal to \(P, Q\), respectively.

**Remark 1.4.1.** The polynomials \(I_K - \Phi_1 z - \cdots - \Phi_P z^P\) and \(I_K + \Theta_1 z + \cdots + \Theta_Q z^Q\) have no common roots.

For example consider a VARMA\((1, 1)\) of two dimensions contained in the vector \(X_t = (X_{1t}, X_{2t})\) the bivariate ARMA\((1, 1)\) model is written as

\[
X_t - \Phi_1 X_{t-1} = \epsilon_t + \Theta_1 \epsilon_{t-1} \tag{1.4.3}
\]
where $\Phi_1 = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}$ is the VAR parameter matrix, $\Theta_1 = \begin{bmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{bmatrix}$ is the VMA parameter matrix. Substituting in (1.4.3) we have

$$
\begin{bmatrix}
X_{1t} \\
X_{2t}
\end{bmatrix} - \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} X_{1,t-1} \\
X_{2,t-1} \end{bmatrix} = \begin{bmatrix} \epsilon_{1t} \\
\epsilon_{2t} \end{bmatrix} + \begin{bmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{bmatrix} \begin{bmatrix} \epsilon_{1,t-1} \\
\epsilon_{2,t-1} \end{bmatrix}.
$$

**Definition 1.4.2.** (Brockwell and Davis [5]) **Causality of VARMA($P,Q$) Models**

The VARMA($P,Q$) model $\{X_t\}$ is causal, or a causal function of $\{\epsilon_t\}$, if there exist matrices $\{\Psi_j\}$ with absolutely summable components such that

$$X_t = \sum_{j=0}^{\infty} \Psi_j \epsilon_{t-i} \quad \text{for all } t. \quad (1.4.4)$$

Causality is equivalent to the condition

$$|\Phi(z)| \neq 0 \text{ for all } z \in \mathbb{C} \text{ such that } |z| \leq 1. \quad (1.4.5)$$

That means the roots of the characteristic equation are outside the unit circle.

The matrices $\Psi_j$ are found recursively from the equations

$$\Psi_j = \Theta_j + \sum_{k=1}^{\infty} \Phi_k \Psi_{j-k}, \quad j = 0, 1, \ldots$$

where $\Theta_0 = I$, $\Theta_j = 0$ for $j > Q$, $\Phi_j = 0$ for $j > P$, and $\Psi_j = 0$ for $j < 0$.

**Definition 1.4.3.** (Brockwell and Davis [5]) **Invertibility of VARMA($P,Q$) Models**

The VARMA($P,Q$) model $\{X_t\}$ is said to be invertible if there exist matrices $\{\Pi_j\}$ with absolutely summable components such that, $X_t$ can be expressed as an infinite autoregressive process

$$\epsilon_t = \sum_{j=0}^{\infty} \Pi_j X_{t-j} \text{ for all } t.$$  

Invertibility is equivalent to the condition

$$|\Theta(z)| \neq 0 \text{ for all } z \in \mathbb{C} \text{ such that } |z| \leq 1. \quad (1.4.6)$$

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That means the roots of the characteristic equation are outside the unit circle.

The matrices $\Pi_j$ are found recursively from the equations

$$
\Psi_j = -\Phi_j + \sum_{k=1}^{\infty} \Theta_k \Pi_{j-k}, \quad j = 0, 1, \ldots
$$

where $\Phi_0 = -I$, $\Phi_j = 0$ for $j > P$, $\Theta_j = 0$ for $j > Q$, and $\Pi_j = 0$ for $j < 0$.

**Example 1.4.1.** Consider the bivariate VARMA(2, 1) process

$$
\begin{bmatrix}
X_{1t} \\
X_{2t}
\end{bmatrix} =
\begin{bmatrix}
0.5 & 0.1 \\
0.4 & 0.5
\end{bmatrix}
\begin{bmatrix}
X_{1,t-1} \\
X_{2,t-1}
\end{bmatrix} +
\begin{bmatrix}
0 & 0 \\
0.25 & 0
\end{bmatrix}
\begin{bmatrix}
X_{1,t-2} \\
X_{2,t-2}
\end{bmatrix}
+ \begin{bmatrix}
\epsilon_{1t} \\
\epsilon_{2t}
\end{bmatrix} +
\begin{bmatrix}
0.6 & 0.2 \\
0 & 0.3
\end{bmatrix}
\begin{bmatrix}
\epsilon_{1,t-1} \\
\epsilon_{2,t-1}
\end{bmatrix}.
\tag{1.4.7}
$$

The model $\{X_t\}$ is a causal if,

$$
|\Phi(z)| \neq 0 \quad \text{for all } |z| \leq 1.
$$

The characteristic polynomial is

$$
|\Phi(z)| = |I_{2\times2} - \Phi_1 z - \Phi_2 z^2| = \left| \begin{array}{cc}
1 & 0 \\
0 & 1
\end{array} \right| - \left| \begin{array}{cc}
0.5 & 0.1 \\
0.4 & 0.5
\end{array} \right| z \left| \begin{array}{cc}
0 & 0 \\
0 & 0.25
\end{array} \right| z^2
= \left| \begin{array}{cc}
1 - 0.5z & -z \\
-0.4z & 1 - 0.25z
\end{array} \right|
= (1 - 0.5z)(1 - 0.5z - 0.25z^2)
$$

The roots of this polynomial are

$$
z_1 = 1.13, \quad z_2 = -3.09, \quad z_3 = 2.29.
$$

They obviously all greater than 1 in absolute value. Therefore the process in (1.4.7) is a causal.

Now the model is an invertible if,

$$
|\Theta(z)| \neq 0 \quad \text{for all } |z| \leq 1.
$$
The characteristic polynomial is

\[ |\Phi(z)| = |I_{2 \times 2} + \Theta z| = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{bmatrix} 0.6 & 0.2 \\ 0 & 0.3 \end{bmatrix} z \\
= \begin{vmatrix} 1 + 0.6z & 0.2z \\ 0 & 1 + 0.3z \end{vmatrix} \\
= (1 + 0.6z)(1 + 0.3z) \]

The roots of this polynomial are

\[ z_1 = -1.66, \quad z_2 = -3.33. \]

They obviously all greater than 1 in absolute value. Therefore the process in (1.4.7) is an invertible.
Chapter 2

Periodically Correlated Time Series

2.1 Introduction

A special class of nonstationary time series has been defined by Gladyshev [8], called periodically correlated (PC) time series (also known as cyclostationary time series). These time series are nonstationary, but have periodic means and covariances. They have been known to be very useful in describing many time series (Jones and Brelsford [10], Nematollahi and Subba Rao [14], Boshnakov and Iqelan [3], Pagano [16]).

Definition 2.1.1. [14] The time series \( \{X_t\} \) with finite first and second moments is said to be periodically correlated (PC) with period \( d \), if there exists a positive integer \( d \) such that the mean function \( \mu_t = E(X_t) \), and the covariance function \( \gamma(t, s) = \text{Cov}(X_t, X_s) = E((X_t - \mu_t)(X_s - \mu_s)) \). satisfy

\[
\mu_t = \mu_{t+d} \quad \text{and} \quad \gamma(t, s) = \gamma(t + d, s + d). \tag{2.1.1}
\]

for all integers \( t \) and \( s \), and moreover \( d \) is the smallest number for which (2.1.1) holds.

When \( d = 1 \) the periodically correlated time series (PC) is equivalent to a stationary series.

The periodic mean is defined as

\[
\mu_s = E(X_{s+rd}) \quad s = 1, 2, \ldots, d. \tag{2.1.2}
\]
and the autocovariance function defined

\[ \gamma_s(k) = \text{Cov}(X_{s+r_d}, X_{s+r_d-k}) = E[(X_{s+r_d} - \mu_s)(X_{s+r_d-k} - \mu_{s-k})] \]

(2.1.3)

**Definition 2.1.2.** [11] The process \( \{\epsilon_{s+r_d}\} \) is said to be periodic white noise, denoted by PWN(0, \( \sigma^2 \), d), if and only if it has the following properties

1. \( \mu_s = E(\epsilon_{s+r_d}) = 0, s = 1, \ldots, d, \)
2. \( \sigma_s^2 = E(\epsilon_{s+r_d}^2), s = 1, \ldots, d, \)
3. \( E(\epsilon_{t_1,t_2}) = 0 \) for \( t_1 \neq t_2. \)

### 2.2 Periodic Autoregression Models (PAR)

The periodic autoregression models introduced by Jones and Brelsford [10]. In this section we introduce the definition of a periodic autoregression models and some properties.

**Definition 2.2.1.** a \( d \)-periodic autoregressive model \( \{X_t\} \) with period \( d > 0 \) and order \( p_1, \ldots, p_d \) is denoted by PAR\(_d\)(\( p_1, \ldots, p_d \)), and defined as:

\[ X_t = \sum_{i=1}^{p_t} \phi_{t,i}X_{t-i} + \epsilon_t, \quad t = 1, 2, \ldots, d \]

(2.2.1)

where \( d \) is the number of seasons, \( p_s \) is the model order for season \( s \), \( \phi_{s,i} \) \( i = 1, \ldots, p_t \) are model parameters for season \( s \) (for \( s = 1, \ldots, d \)), and \( \{\epsilon_t\} \) is a periodic white noise sequence with \( \text{Var} \epsilon_t = \sigma_t^2 \), \( \epsilon_t \) denotes a standard white noise process with constant variance, \( \{\epsilon_t\} \sim \text{PWN}(0, \sigma_t^2, d) \).

If \( p_1 = p_2 = \cdots = p_s = p \) then the model (2.2.1) is denoted by PAR\(_d\)(\( p \)).

Consider a univariate time series \( \{X_t\} \) having seasons over a period of \( N \) years, let \( \{X_{s+r_d}\} \) represent a time series observation in the \( r \)th year and \( d \)th season where \( r = 1, 2, \ldots, N \) and \( s = 1, 2, \ldots, d. \)
If we let \( t = s + rd \) in Equation (2.2.1) we have

\[
X_{s+rd} = \sum_{i=1}^{p_s} \phi_{s,i} X_{s+rd-i} + \epsilon_{s+rd}
\]  

(2.2.2)

where \( \phi_{s,1} \) through \( \phi_{s,p_s} \) are autoregressive parameters which may vary with season \( s \), \( s = 1, \ldots, d \).

We can rewrite Equation (2.2.2) in operator form by using the backward shift operator \( B \), \( (B^k X_{s+rd} = X_{s+rd-k}) \)

\[
\phi_s(B) X_{s+rd} = \epsilon_{s+rd},
\]  

(2.2.3)

where

\[
\phi_s(B) = 1 - \phi_{s,1}(B) - \cdots - \phi_{s,p_s} B^{p_s}
\]

is the autoregressive operator of order \( p_s \) for season \( s \).

**Definition 2.2.2.** [11] A PAR(\( p_1, p_2, \ldots, p_d \)) model for the time series \( X_{s+rd} \) given by the equation (2.2.3) is said to be causal, or a causal function of \( X_{s+rd} \), if for each season \( s = 1, 2, \ldots, d \), there exist sequences of \( \{\psi_{s,i}\} \), such that

\[
X_{s+rd} = \sum_{i=1}^{\infty} \psi_{s,i} \epsilon_{s+rd-i}, \quad \text{for each } s = 1, \ldots, d, \quad r = 0, 1, 2, \ldots
\]

(2.2.4)

where \( \psi_{s,0} = 1 \) and all the values \( \psi_{s,i} \) are \( d \)-periodic and satisfy \( \sum_{i=1}^{\infty} |\psi_{s,i}| < \infty \) for each season \( s \), (for \( d=4 \) the values of \( \psi_{s,i} \) are \( \psi_{1,i}, \psi_{2,i}, \psi_{3,i}, \psi_{4,i} \)). The causality of a PAR model is equivalent to the condition we will discussed in section (3.5).

**Example** 2.2.1. Consider a simple periodic PAR\(_d\)(1) model, has the form

\[
\begin{align*}
X_{1+rd} &= \phi_{1,1} X_{1+rd-1} + \epsilon_{1+rd} \\
X_{2+rd} &= \phi_{2,1} X_{1+rd} + \epsilon_{2+rd} \\
&\vdots \\
X_{d+rd} &= \phi_{d,1} X_{(d-1)+rd} + \epsilon_{d+rd}.
\end{align*}
\]

(2.2.5)
Example 2.2.2. Consider a periodic PAR$_3(1, 3, 2)$ model written as

\[
\begin{align*}
X_{1+3r} &= 0.5 \ X_{1+3r-1} + \epsilon_{1+3r} \\
X_{2+3r} &= X_{2+3r-1} + 0.3 \ X_{2+4r-2} + 0.1 \ X_{2+3r-3} + \epsilon_{2+4r} \\
X_{3+3r} &= 0.9 \ X_{3+3r-1} + 0.5 \ X_{3+3r-2} + \epsilon_{3+3r}.
\end{align*}
\] (2.2.6)

2.3 Periodic Moving Average Models (PMA)

Definition 2.3.1. [11] A $d$-periodic moving average model \( \{X_t\} \) with period \( (d > 0) \) and order \( q_1, q_2, \ldots, q_d \) is denoted by PMA$_d(q_1, q_2, \ldots, q_d)$, and defined as:

\[
X_{s+rd} = \epsilon_{s+rd} + \sum_{i=1}^{q_s} \theta_{s,i} \epsilon_{s+rd-i}
\] (2.3.1)

where \( q_s \geq 0, \theta_{s,1} \) through \( \theta_{s,q_s} \) are moving parameters, the season \( s = 1, 2, \ldots, d \) and the \( r \)th year \( r = 1, 2, \ldots, N \).

If \( q = \max(q_1, \ldots, q_s) \) we can rewrite the Equation (2.3.1) as follows

\[
X_{s+rd} = \epsilon_{s+rd} + \sum_{i=1}^{q} \theta_{s,i} \epsilon_{s+rd-i}
\] (2.3.2)

and the model (2.3.2) denoted by PMA$_d(q)$. We can rewrite Equation (2.3.1) using the backward shift operator \( B \), \((B^k\epsilon_{s+rd} = \epsilon_{s+rd-k})\) by

\[
X_{s+rd} = \theta_s(B)\epsilon_{s+rd},
\] (2.3.3)

with

\[
\theta_s(B) = 1 + \theta_{s,1}(B) + \cdots + \theta_{s,q_s}B^{q_s}
\]

is the moving operator of order \( q_s \) for season \( s \).

Definition 2.3.2. [11] A periodic moving-average (PMA$_d(q)$) model for the time series \( X_{s+rd} \) given by the Equation (2.3.3) is said to be invertible, if for each season \( s = 1, 2, \ldots, d \), there exist sequences of \( \{\pi_{s,i}\} \), such that

\[
\epsilon_{s+rd} = \sum_{i=0}^{\infty} \pi_{s,i} X_{s+rd-i}, \quad \text{for each } s = 1, \ldots, d, \quad r = 0, 1, 2, \ldots
\] (2.3.4)
where $\pi_{s,0} = 1$ and all the values $\pi_{s,i}$ are $d$-periodic and satisfy $\sum_{i=1}^{\infty} |\pi_{s,i}| < \infty$ for each season $s$.

The invertibility of a PMA model is equivalent to the condition we will discussed in section (3.5)

**Example 2.3.1.** [7] For simple example consider a PMA$_d(1)$ model, has the form

$$X_t = \epsilon_t + \theta_{s,1}\epsilon_{t-1},$$

(2.3.5)

autocovariances for the PMA$_d(1)$ processes in season $s$,

$$\gamma_k(s) = \text{cov}(X_t, X_{t+k})$$

$$= \text{cov}(\epsilon_t + \theta_{s,1}\epsilon_{t-1}, \epsilon_{t+k} + \theta_{s,1}\epsilon_{t+k-1})$$

$$= E(\epsilon_t + \theta_{s,1}\epsilon_{t-1})(\epsilon_{t+k} + \theta_{s,1}\epsilon_{t+k-1})$$

$$= E(\epsilon_t\epsilon_{t+k}) + \theta_{s,1}E(\epsilon_t\epsilon_{t+k-1}) + \theta_{s,1}E(\epsilon_{t-1}\epsilon_{t+k-1}) + \theta_{s,1}^2E(\epsilon_{t-1}\epsilon_{t+k-1})$$

$$= \begin{cases} 
(1 + \theta_{s,1}^2)\sigma^2, & k = 0; \\
\theta_{s,1}\sigma^2, & k = \pm 1; \\
0, & k = \pm 2, \pm 3, \ldots . 
\end{cases}$$

So, the autocovariance function $\gamma_k(s)$ does not depends on time $t$ means that the process PMA are stationary.

The autocorrelation function $\rho_k(s)$ for the PMA$_d(1)$ processes then

$$= \begin{cases} 
1, & k = 0; \\
\frac{\theta_{s,1}}{1 + \theta_{s,1}^2}, & k = \pm 1; \\
0, & k = \pm 2, \pm 3, \ldots . 
\end{cases}$$

**Definition 2.3.3.** The autocorrelation function $\rho_k(s)$ of periodic moving average of order $q$ cuts off after lag $q$.

**Remark 2.3.1.** In previous example the autocorrelation function equal zero after lag 1.
**Example 2.3.2.** Consider a periodic moving average model PMA_{4}(1),

\begin{align*}
X_{1+4r} &= \epsilon_{1+4r} + 0.5 \epsilon_{1+4(r-1)} \\
X_{2+4r} &= \epsilon_{2+4r} + 0.3 \epsilon_{2+4(r-1)} \\
X_{3+4r} &= \epsilon_{3+4r} - 0.3 \epsilon_{3+4(r-1)} \\
X_{4+4r} &= \epsilon_{4+4r} - 0.5 \epsilon_{4+4(r-1)}.
\end{align*}

(2.3.6)

### 2.4 Periodic Autoregression Moving Average Models (PARMA)

**Definition 2.4.1.** [19] A d-periodic autoregressive moving average model \( \{X_t\} \) of orders \( p_1, q_1; p_2, q_2; \ldots; p_d, q_d \) is denoted by PARMA_{d}(p_1, q_1; p_2, q_2; \ldots; p_d, q_d) with period \( d > 0 \), and defined as:

\[ X_t - \sum_{i=1}^{p_t} \phi_{s,i} X_{t-i} = \sum_{i=1}^{q_t} \theta_{s,i} \epsilon_{t-i} \]  

(2.4.1)

where \( p_t, q_t \geq 0, \phi_{t,1}, \ldots, \phi_{t,p_t} \) and \( \theta_{t,1}, \ldots, \theta_{t,q_t} \) for each season \( t = 1, 2, \ldots, d \), are constants. The process \( \{\epsilon_t\} \) is a periodic white noise \( \sim \text{PWN}(0, \sigma^2, d) \).

If we let \( t = s + rd \), the Equation (2.4.1) can be written as follows,

\[ X_{s+rd} - \sum_{i=1}^{p_s} \phi_{s,i} X_{s+rd-i} = \sum_{i=1}^{q_s} \theta_{s,i} \epsilon_{s+rd-i} \]  

(2.4.2)

where \( \phi_{s,1}, \ldots, \phi_{s,p_s} \) and \( \theta_{s,1}, \ldots, \theta_{s,q_s} \) are the autoregressive and moving average model coefficients, respectively.

The model in (2.4.2) can be rewrite as, which can be expressed Equation (2.4.1)

\[ X_{s+rd} - \sum_{i=1}^{p} \phi_{s,i} X_{s+rd-i} = \sum_{i=1}^{q} \theta_{s,i} \epsilon_{s+rd-i} \]  

(2.4.3)

where \( p = \max_{1 \leq s \leq d} p_s \) and \( q = \max_{1 \leq s \leq d} q_s \).

Again the model in (2.4.2) can be write using the backward shift operator \( B(B^k X_{s+rd} = X_{s+rd-k}) \) as follows

\[ \phi_s(B)X_{s+rd} = \theta_s(B)\epsilon_{s+rd}, \]  

(2.4.4)
where $\phi_s(B) = 1 - \phi_{s,1}B - \phi_{s,2}B^2 - \cdots - \phi_{s,p_s}B^{p_s}$, is the AR operator of order $p_s$ for season $s$, and $\theta_s(B) = 1 + \theta_{s,1}(B) + \cdots + \theta_{s,q_s}B^{q_s}$, is the MA operator of order $q_s$ for season $s$.

**Remark 2.4.1.** When $p = 0$, the process in Equation (2.4.3) becomes a pure periodic moving average (PMA) model, and when $q = 0$, the process in Equation (2.4.3) becomes a pure periodic autoregressive (PAR) model.

**Example 2.4.1.** Consider the periodic autoregressive moving average model PARMA$_4(2, 0; 2, 2; 3, 0; 0, 4)$

\[
\begin{align*}
X_{1+rd} &= \phi_{1,1}X_{s+rd-1} + \phi_{1,2}X_{s+rd-2} + \epsilon_{1+rd} \\
X_{2+rd} &= \phi_{2,1}X_{s+rd-1} + \phi_{2,2}X_{s+rd-2} + \epsilon_{2+rd} + \theta_{2,1}\epsilon_{s+rd-1} + \theta_{2,2}\epsilon_{s+rd-2} \\
X_{3+rd} &= \phi_{3,1}X_{s+rd-1} + \phi_{3,2}X_{s+rd-2} + \phi_{3,3}X_{s+rd-3} + \epsilon_{3+rd} + \theta_{3,1}\epsilon_{s+rd-1} \\
X_{4+rd} &= \epsilon_{4+rd} + \theta_{4,1}\epsilon_{s+rd-1} + \theta_{4,2}\epsilon_{s+rd-2} + \theta_{4,3}\epsilon_{s+rd-3} + \theta_{4,4}\epsilon_{s+rd-4}
\end{align*}
\]

**Definition 2.4.2.** [11] A PARMA model for the time series $X_{s+rd}$ given by the equation (2.4.4) is said to be causal, or a causal function of $X_{s+rd}$, if for each season $s = 1, 2, \ldots, d$, there exist sequences of $\{\psi_{s,i}\}$, such that

\[
X_{s+rd} = \sum_{i=0}^{\infty} \psi_{s+rd-i} \quad s = 1, \ldots, d. \quad (2.4.5)
\]

where $\psi_{s,0} = 1$ and $\sum_{i=1}^{\infty} |\psi_{s,i}| < \infty$ for each season $s$.

**Definition 2.4.3.** [11] A periodic autoregressive moving-average PARMA model for the time series $X_{s+rd}$ given by the Equation (2.3.3) is said to be invertible, if for each season $s = 1, 2, \ldots, d$, there exist sequences of $\{\pi_{s,i}\}$, such that

\[
\epsilon_{s+rd} = \sum_{i=0}^{\infty} \pi_{s,i}X_{s+rd-i} \quad s = 1, \ldots, d. \quad (2.4.6)
\]

where $\pi_{s,0} = 1$ and $\sum_{i=1}^{\infty} |\pi_{s,i}| < \infty$ for each season $s$. 

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Chapter 3

Representation

3.1 Vector Representation of a PAR model

In this section we give a brief overview of periodic autoregressions and we consider notation and representation. Many authors wrote in the vector representation of a PAR model, Boswijk and franses [4], Franses and Paap [7], Boshnakov and Iqelan [3] and Tiao and Grupe [20]. We introduce various form of the VAR representation, namely the L-form, U-form and I-form.

3.1.1 L-Form Representation of a PAR model

Definition 3.1.1. (Iqelan [11], Definition 2.7.1) The multivariate process \( \{ X_t \} \) of a \( d \)-periodically correlated process \( \{ X_{s+rd} \} \) is defined as

\[
X_t = (X_{dt-(d-1)}, X_{dt-(d-2)} + \cdots + X_{dt})'
\]

\[
X_1 = (X_1, X_2, \ldots, X_d); \quad X_2 = (X_{d+1}, X_{d+2}, \ldots, X_{2d}); \quad X_3 = (X_{2d+1}, X_{2d+2}, \ldots, X_{3d})'
\]

and so on.

Consider a univariate time series \( \{ X_{s+rd} \} \) consisting of \( d \) seasons per year, where \( r = 0, 1, 2, \ldots \), is the year index and \( s \) is the season index, \( 1 \leq s \leq d \). The observation of the time series \( \{ X_{s+rd} \} \) are available for precisely \( N \) years, with total sample size \( n = Nd \). The
periodic autoregressive model of order $d$ and lag $p$ in Equation (2.2.1)

$$X_t = \phi_{s,1} X_{t-1} + \cdots + \phi_{s,p} X_{t-p} + \epsilon_t, \quad s=1, 2, \ldots, d. \quad (3.1.2)$$

The PAR\(_d\)(\(_p\)) can be written as the \(_d\)-dimensional vector autoregressive VAR(\(_P\)) model

$$\Phi_0 X_r = \Phi_1 X_{r-1} + \cdots + \Phi_P X_{r-P} + \epsilon_r \quad (3.1.3)$$

where $\Phi_0$ is \((d \times d)\) lower triangular matrix with unit elements on the diagonal such that

$$\Phi_0 = \begin{pmatrix}
1 & 0 & \cdots & 0 & 0 \\
-\phi_{2,1} & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-\phi_{d,d-1} & -\phi_{d,d-2} & \cdots & -\phi_{d,1} & 1
\end{pmatrix}, \quad (3.1.4)$$

$\Phi_i\_s, i = 1, 2, \ldots, P,$ are \((d \times d)\) matrices

$$\Phi_i = \begin{pmatrix}
\phi_{1,id} & \phi_{1,id-1} & \cdots & \phi_{1,(i-1)d+2} & \phi_{1,(i-1)d+1} \\
\phi_{2,id+1} & \phi_{2,id} & \cdots & \phi_{2,(i-1)d+3} & \phi_{2,(i-1)d+2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\phi_{d,(i+1)d-1} & \phi_{d,(i+1)d-2} & \cdots & \phi_{d,(id+1)} & \phi_{d,1}
\end{pmatrix}, \quad (3.1.5)$$

and $\epsilon_t$ is a \((d \times 1)\) vector white noise, $\epsilon_t = (\epsilon_{1+d(r-1)}, \epsilon_{2+d(r-1)}, \ldots, \epsilon_{d+d(r-1)})^T.$ Now a PAR\(_d\)(\(_p\)) process corresponds to a VAR(\(_P\)) process with

$$P = [(p - 1)/d] + 1 \quad (3.1.6)$$

where [·] denotes integer part.

**Example** 3.1.1. Consider a periodic autoregressive of order 1 with period $d$, as follows

$$X_{1+rd} = \phi_{1,1} X_{1+rd-1} + \epsilon_{1+rd}$$

$$X_{2+rd} = \phi_{2,1} X_{1+rd} + \epsilon_{2+rd}$$

$$\vdots$$

$$X_{d+rd} = \phi_{d,1} X_{(d-1)+rd} + \epsilon_{d+rd} \quad (3.1.7)$$
The system (3.1.7) can be written in a matrix notation as follows,

\[
\begin{pmatrix}
1 & 0 & \cdots & 0 \\
-\phi_{2.1} & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix}
\begin{pmatrix}
X_{1+rd} \\
X_{2+rd} \\
\vdots \\
X_{d+rd}
\end{pmatrix}
= 
\begin{pmatrix}
0 & 0 & \cdots & \phi_{1.1} \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{pmatrix}
\begin{pmatrix}
X_{1+(r-1)d} \\
X_{2+(r-1)d} \\
\vdots \\
X_{d+(r-1)d}
\end{pmatrix}
+ 
\begin{pmatrix}
\epsilon_{1+rd} \\
\epsilon_{2+rd} \\
\vdots \\
\epsilon_{d+rd}
\end{pmatrix}
\]

(3.1.8)

The model (3.1.8) is a vector representation of order 1, VAR(1). Which can be written as

\[
\Phi_0 X_t = \Phi_1 X_{t-1} + \epsilon_t.
\]  

(3.1.9)

**Remark** 3.1.1. When \( p \) exceed the number of season \( d \) we compute the value of \( P \) from Equation (3.1.6).

**Example** 3.1.2. (Working days measurements)

Consider a time series \( \{X_t\}, t = 1, 2, \ldots, n \) which is available for a sample size \( N = \frac{n}{6} \) weeks corresponding to \( n \) days. The 6-periodic autoregressive of order \( p \) (\( \text{PAR}_6(p) \)) can be written in the general form

\[
X_t = \phi_{w,1} X_{t-1} + \cdots + \phi_{w,p} X_{t-p} + \epsilon_t
\]

(3.1.10)

where the \( \phi_{w,i} \) are the autoregressive parameters up to order \( p \), and season \( w, w = 1, 2, 3, 4, 5, 6 \). When \( p = 17 \) then the \( \text{PAR}_6(17) \) model can be represented by a \( \text{VAR}(P) \) model with order \( P = [(17-1)/6]+1 \). Then

\[
\Phi_0 X_r = \Phi_1 X_{r-1} + \Phi_2 X_{r-2} + \Phi_3 X_{r-3} + \epsilon_r.
\]

(3.1.11)

where \( X_r = (X_{1r}, X_{2r}, X_{3r}, X_{4r}, X_{5r}, X_{6r}) \) and \( X_{wr} \) is the observation in day \( w \) of week \( r = 1, 2, \ldots, N \) and \( w = 1,2,3,4,5,6 \). The matrices \( \Phi_0, \Phi_1, \Phi_2, \Phi_3 \) are \( 6 \times 6 \) parameter matrices.
as follows

\[
\Phi_0 = \begin{pmatrix}
-\phi_{2,1} & 1 & 0 & 0 & 0 & 0 \\
-\phi_{3,2} & -\phi_{3,1} & 1 & 0 & 0 & 0 \\
-\phi_{4,3} & -\phi_{4,2} & -\phi_{4,1} & 1 & 0 & 0 \\
-\phi_{5,4} & -\phi_{5,3} & -\phi_{5,2} & -\phi_{5,1} & 1 & 0 \\
-\phi_{6,5} & -\phi_{6,4} & -\phi_{6,3} & -\phi_{6,2} & -\phi_{6,1} & 1 \\
\end{pmatrix}, \quad (3.1.12)
\]

\[
\Phi_i = \begin{pmatrix}
\phi_{1,6i} & \phi_{1,6i-1} & \phi_{1,6i-2} & \phi_{1,6i-3} & \phi_{1,6i-4} & \phi_{1,6i-5} \\
\phi_{2,6i+1} & \phi_{2,6i} & \phi_{2,6i-1} & \phi_{2,6i-2} & \phi_{2,6i-3} & \phi_{2,6i-4} \\
\phi_{3,6i+2} & \phi_{3,6i+1} & \phi_{3,6i} & \phi_{3,6i-1} & \phi_{3,6i-2} & \phi_{3,6i-3} \\
\phi_{4,6i+3} & \phi_{4,6i+2} & \phi_{4,6i+1} & \phi_{4,6i} & \phi_{4,6i-1} & \phi_{4,6i-2} \\
\phi_{5,6i+4} & \phi_{5,6i+3} & \phi_{5,6i+2} & \phi_{5,6i+1} & \phi_{5,6i} & \phi_{5,6i-1} \\
\phi_{6,6i+5} & \phi_{6,6i+4} & \phi_{6,6i+3} & \phi_{6,6i+2} & \phi_{6,6i+1} & \phi_{6,6i} \\
\end{pmatrix}, \quad i = 1, 2, 3. \quad (3.1.13)
\]

**Vector of Quarters Representation**

In a particular case consider a univariate time series \( \{X_t\} \) which is observed quartely for \( N \) years. Then the PAR\(_4\)(p) model of order \( p \), can be written as an AR\((P)\) model for the 4-dimensional vector process \( \mathbf{X}_r = (X_{1r}, X_{2r}, X_{3r}, X_{4r}) \), \( r = 1, 2, \ldots, N \), where \( \mathbf{X}_r \) is the observations in season \( s \) in year \( r \), \( s = 1, 2, 3, 4 \). The vector of quarters representation is the same in Equation (3.1.2). The vector representation in this case called the vector of quarters denoted by VQ\((P)\) as referred by Franses and Boswijk [4], where \( P = [(p + 4)/4] + 1 \). The matrices \( \Phi_0 \) and \( \Phi_i, \quad i = 1, 2, \ldots \), are defined as in similar way in Equation (3.1.12) and Equation (3.1.13) with \( d \) equal 4.

**Example** 3.1.3. Consider a PAR\(_4\)(3) model,

\[
X_{s+4r} = \phi_{s,1}X_{s+4r} + \phi_{s,2}X_{s+4r-2} + \phi_{s,3}X_{s+4r-3} \epsilon_{s+4r}, \quad s=1, 2, 3, 4. \quad (3.1.14)
\]
can be written in expanded form as

\[
\begin{align*}
X_{1+4r} &= \phi_{1,1}X_{4+4(r-1)} + \phi_{1,2}X_{3+4(r-1)} + \phi_{1,3}X_{2+4(r-1)} + \epsilon_{1+4r} \\
X_{2+4r} &= \phi_{2,1}X_{1+4r} + \phi_{2,2}X_{4+4(r-1)} + \phi_{2,3}X_{3+4(r-1)} + \epsilon_{2+4r} \\
X_{3+4r} &= \phi_{3,1}X_{2+4r} + \phi_{3,2}X_{1+4r} + \phi_{3,3}X_{4+4(r-1)} + \epsilon_{3+4r} \\
X_{4+4r} &= \phi_{4,1}X_{3+4r} + \phi_{4,2}X_{2+4r} + \phi_{4,3}X_{2+4r} + \epsilon_{4+4r}.
\end{align*}
\]  

(3.1.15)

The system (3.1.15) can be written in a matrix representation as follows

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
-\phi_{2,1} & 1 & 0 & 0 \\
-\phi_{3,2} & -\phi_{3,1} & 1 & 0 \\
-\phi_{4,3} & -\phi_{4,2} & -\phi_{4,1} & 1
\end{pmatrix}
\begin{pmatrix}
X_{1+4r} \\
X_{2+4r} \\
X_{3+4r} \\
X_{4+4r}
\end{pmatrix}
= 
\begin{pmatrix}
0 & \phi_{1,3} & \phi_{1,2} & \phi_{1,1} \\
0 & 0 & \phi_{2,3} & \phi_{2,2} \\
0 & 0 & 0 & \phi_{3,3} \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
X_{1+4(r-1)} \\
X_{2+4(r-1)} \\
X_{3+4(r-1)} \\
X_{4+4(r-1)}
\end{pmatrix}
\]

\[+ \begin{pmatrix}
X_{1+4r} \\
X_{2+4r} \\
X_{3+4r} \\
X_{4+4r}
\end{pmatrix}.
\]  

(3.1.16)

### 3.1.2 U-Form Representation of the PAR Model

We will discuss the second form of the PAR model. This form is denoted by U-form. The U-form is based on the vector obtained by re-order of the observations of the vector \( X_t \) in Equation (3.1.1) (starting from the most recent), ([Iqelan [11]]) that is

\[
U_t = (X_{dt}, X_{dt-1}, \ldots, X_{d-(d-1)})'.
\]  

(3.1.17)

Now we can construct some vector,

\[
U_1 = (X_d, \ldots, X_1)', \quad U_2 = (X_{2d}, \ldots, X_{d+1})', \quad U_3 = (X_{3d}, \ldots, X_{d+2})', \text{ and so on.}
\]

In the same way you can construct the U-form of the PAR has also been constructing the L-form of the PAR model in the previous subsection. The U-form can be written as the \( d \)-dimensional vector autoregressive \( \text{VAR}(P) \) model

\[
A_0 U_r = A_1 U_{r-1} + \cdots + A_P U_{r-P} + W_r
\]  

(3.1.18)

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where $\mathbf{W}_r$ is a $(d \times 1)$ vector white noise, $\mathbf{W}_r = (\epsilon_{dt}, \epsilon_{dt-1}, \ldots, \epsilon_{dt-(d-1)})^t$, and the $\mathbf{A}_0$ matrix is a $(d \times d)$ upper triangular matrix (with unit elements on the diagonal) such that

$$
\mathbf{A}_0 = \begin{pmatrix}
1 & -\phi_{d,2} & \cdots & -\phi_{d,d-2} & -\phi_{d,d-1} \\
0 & 1 & \cdots & -\phi_{d-1,d-3} & -\phi_{d-1,d-2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & -\phi_{2,1} \\
0 & 0 & \cdots & 0 & 1
\end{pmatrix} \tag{3.1.19}
$$

and the matrices $\mathbf{A}_i$s, $i = 1, 2, \ldots, P$, are real $(d \times d)$

$$
\mathbf{A}_i = \begin{pmatrix}
\phi_{d,\text{id}} & \phi_{d,\text{id}+1} & \cdots & \phi_{d,(i+1)d-2} & \phi_{d,(i+1)d-1} \\
\phi_{d-1,\text{id}-1} & \phi_{d-1,\text{id}} & \cdots & \phi_{d-1,(i-1)d-3} & \phi_{d-1,(i+1)d-2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\phi_{1,(i-1)d+1} & \phi_{1,(i-1)d+2} & \cdots & \phi_{1,\text{id}-1} & \phi_{1,\text{id}}
\end{pmatrix} \tag{3.1.20}
$$

One can conclude that the relationship between the Equation (3.1.2) and Equation (3.1.18),

$$
\mathbf{U}_r = \mathbf{P}\mathbf{X}_r, \quad \mathbf{W}_r = \mathbf{P}\mathbf{\epsilon}_r, \quad r = 1, 2, \ldots, \tag{3.1.21}
$$

and

$$
\mathbf{A}_i = \mathbf{P}\Phi_i\mathbf{P} \tag{3.1.22}
$$

where $\mathbf{P}$ is the $(d \times d)$ permutation matrix,

$$
\mathbf{P} = \begin{pmatrix}
0 & 0 & \cdots & 0 & 1 \\
0 & 0 & \cdots & 1 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 1 & \cdots & 0 & 0 \\
1 & 0 & \cdots & 0 & 0
\end{pmatrix} \tag{3.1.23}
$$
Example 3.1.4. Consider the 5-periodic autoregression model PAR$_{5}(2)$,

\[ X_{s+5r} = \phi_{s,1}X_{s+5r} + \phi_{s,2}X_{s+5r-2} + \epsilon_{s+5r}, \quad s=1, 2, 3, 4, 5. \]

The model above written as follows

\[
\begin{align*}
X_{5+5r} &= \phi_{5,1}X_{4+5r} + \phi_{5,2}X_{3+5r} + \epsilon_{5+5r} \\
X_{4+5r} &= \phi_{4,1}X_{3+5r} + \phi_{4,2}X_{2+5r} + \epsilon_{4+5r} \\
X_{3+5r} &= \phi_{3,1}X_{2+5r} + \phi_{3,2}X_{1+5r} + \epsilon_{3+5r} \quad \text{(3.1.24)} \\
X_{2+5r} &= \phi_{2,1}X_{1+5r} + \phi_{2,2}X_{5+5(r-1)} + \epsilon_{2+5r} \\
X_{1+5r} &= \phi_{1,1}X_{5+5(r-1)} + \phi_{1,2}X_{1+5(r-1)} + \epsilon_{1+4r}
\end{align*}
\]

The system (3.1.24) can be written in a matrix representation as follows

\[
\begin{pmatrix}
1 & -\phi_{5,1} & -\phi_{5,2} & 0 & 0 \\
0 & 1 & -\phi_{4,1} & -\phi_{4,2} & 0 \\
0 & 0 & 1 & -\phi_{3,1} & -\phi_{3,2} \\
0 & 0 & 0 & 1 & -\phi_{2,1} \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
X_{5+5r} \\
X_{4+5r} \\
X_{3+5r} \\
X_{2+5r} \\
X_{1+5r}
\end{pmatrix}
= 
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\phi_{2,2} & 0 & 0 & 0 & 0 \\
\phi_{1,1} & \phi_{1,2} & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
X_{5+5(r-1)} \\
X_{4+5(r-1)} \\
X_{3+5(r-1)} \\
X_{2+5(r-1)} \\
X_{1+5(r-1)}
\end{pmatrix}
+ 
\begin{pmatrix}
\epsilon_{5+5r} \\
\epsilon_{4+5r} \\
\epsilon_{3+5r} \\
\epsilon_{2+5r} \\
\epsilon_{1+5r}
\end{pmatrix}
\tag{3.1.25}
\]

The model in Equation (3.1.25) is a vector autoregressive model of order 1, VAR(1). For simplicity written as

\[ A_0 U_r = A_1 U_{r-1} + W_r. \tag{3.1.26} \]

### 3.1.3 I-Form Representation of the PAR Model

The I-form is the another form of U-form in (3.1.18). Since $A_0$ is upper triangular matrix with unit elements on the diagonal, $\det(A_0) \neq 0$ implies that $A_0$ is non-singular. To obtain
the I-form of the PAR model pre-multiplying both sides of Equation by $A_0^{-1}$, the inverse matrix of the upper triangular matrix given in (3.1.19). Then

$$U_r = Y_1 U_{r-1} + \cdots + Y_p U_{r-p} + V_r$$  \hspace{1cm} (3.1.27)

with $Y_i = A_0^{-1} A_i$, for $i = 1, 2, \ldots P$ and $V_r = A_0^{-1} W_r$.

In the same way we can find the I-form representation from the L-form. Pre-multiplying both sides of Equation (3.1.3) by $\Phi^{-1}$ to obtain the I-form,

$$X_r = \Psi_1 X_{r-1} + \cdots + \Psi_p X_{r-p} + E_r$$  \hspace{1cm} (3.1.28)

where $\Psi_i = \Phi^{-1} \Phi_i$, for $i = 1, 2, \ldots, P$ and $E_r = \Phi^{-1} \epsilon_r$.

**Example 3.1.5.** (Numerical Example)

Consider the 4-periodic autoregressive model PAR4(2),

$$X_{s+rd} = \phi_{s,1} X_{s+rd-1} + \phi_{s,2} X_{s+rd-2} + \epsilon_{s+rd}$$  \hspace{1cm} (3.1.29)

with coefficients

<table>
<thead>
<tr>
<th></th>
<th>$\phi_{s,1}$</th>
<th>$\phi_{s,2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.8</td>
<td>0.1</td>
</tr>
<tr>
<td>2</td>
<td>0.2</td>
<td>0.7</td>
</tr>
<tr>
<td>3</td>
<td>-0.2</td>
<td>0.7</td>
</tr>
<tr>
<td>4</td>
<td>-0.8</td>
<td>0.1</td>
</tr>
</tbody>
</table>

The model in Equation (3.1.29) can be written as a vector representation of order 1, VAR(1) as follows

$$\Phi_0 X_r = \phi_1 X_{r-1} + \epsilon_r$$  \hspace{1cm} (3.1.30)

with

$$\Phi_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -0.2 & 1 & 0 & 0 \\ -0.7 & 0.2 & 1 & 0 \\ -0.5 & -0.1 & 0.8 & 1 \end{pmatrix}, \quad \phi_1 = \begin{pmatrix} 0 & 0 & 0.1 & 0.8 \\ 0 & 0 & 0 & 0.7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
It is easy to find $\Phi_0^{-1}$ and $\Phi_0^{-1}\Phi_1$

$\Phi_0^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.2 & 1 & 0 & 0 \\ 0.66 & -0.2 & 1 & 0 \\ -0.508 & 0.26 & -0.8 & 1 \end{pmatrix}$, \[ \Phi_0^{-1}\Phi_1 = \begin{pmatrix} 0 & 0 & 0.1 & 0.8 \\ 0 & 0 & 0.2 & 0.86 \\ 0 & 0 & 0.066 & 0.388 \\ 0 & 0 & -0.0508 & -0.224 \end{pmatrix} \]

Note that for any PAR model $\Phi_0^{-1}$ is again a lower triangular matrix, and Equation (3.1.30) can be written as I-form

\[ X_r = \Psi_1 X_{r-1} + E_r \]  \hspace{1cm} (3.1.31)

where $\Psi_1 = \Phi_0^{-1}\Phi_1$, and $E_r = \Phi_0^{-1}\epsilon_r$. 
3.2 Multi-companion Matrices

In this section we defined the multi-companion matrix, we study more materials and explain some important properties of multi-companion matrices, and explained how to write a PAR model as multi-companion representation.

**Definition 3.2.1.** (Boshnakov [2], definition 1.1) Let the $p \times p$ matrix $M$

\[
M = \begin{pmatrix}
  f_{1,1} & f_{1,2} & \cdots & f_{1,p} & f_{1,p-d} & f_{1,p-d+1} & \cdots & f_{1,p}
  \\
  : & : & \cdots & : & : & \vdots & \ddots & : \\
  f_{d,1} & f_{d,2} & \cdots & f_{d,p} & f_{d,p-d} & f_{d,p-d+1} & \cdots & f_{d,p}
  \\
  1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
  0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
  : & : & \cdots & : & : & \vdots & \ddots & : \\
  0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0
\end{pmatrix}
\]  

(3.2.1)

The matrix $M$ is said to be multi-companion of order $d$ (or $d$ companion) if

1. the first $d$ rows of $M$ are arbitrary;
2. the $d$th sub-diagonal of $M$ consists of ones;
3. all other elements of $M$ are zero;
4. $1 \leq d < p$.

There is another version of the multi-companion matrix $M$ defined in (Boshnakov and Iqelan [3]) as follows

\[
M = \begin{pmatrix}
  A & B \\
  I & 0
\end{pmatrix}
\]

(3.2.2)

where $A$, $B$, $I$ and $0$ are matrices of size $d \times (p - d), d \times d, (p - d) \times (p - d), (p - d) \times d$, respectively, for some integer $d$, $1 \leq d \leq p$. 

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In important particular case when \( p \) is multiple of \( d \), multi-companion matrices are often called block-companion (or companion).

The abbreviation \( C[a_1, a_2, \ldots, a_p] \) is used for the companion matrix, corresponding to the bracketed variable,

\[
C[a_1, a_2, \ldots, a_p] = \begin{pmatrix}
a_1 & a_2 & \cdots & a_{p-1} & a_p \\
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0
\end{pmatrix}
\] (3.2.3)

According to the definition mentioned in (3.2.1) a 1-companion matrix is companion matrix. Moreover, we can be consider that the identity matrix as being a 0-companion matrix.

Before beginning to discuss the information on the matrix \( M \), we have to discuss some important properties of multiplication by multi-companion matrices and certain about factorization of multi-companion. Some properties in the following theorems, and its corollaries.

**Theorem 3.2.1.** (Boshnakov [2], Theorem 3.1). Let \( M \) be a \( d \)-companion, and \( B \) and arbitrary matrix.

1. Left multiplication of any matrix \( B \) by a \( d \)-companion matrix \( M \) moves the first \( p-d \) rows of \( B \) \( d \) rows downwards without change. The first \( d \) rows of the product \( D=MB \) have the usual general form.

2. Right multiplication of any matrix \( B \) by the transposed of a \( d \)-companion matrix \( M \) moves the first \( p-d \) columns of \( B \) \( d \) columns downwards without change. The first \( d \) columns of the product \( R=BM' \) have the usual general form.
Corollary 3.2.2. Let $H$ and $B$ be $p \times p$ multi-companion matrices of order $d$ and $l$ respectively with $d+l < p$, let $M=BH$. Then

1. $M$ is a multi-companion matrix of order $d+l$;
2. the non-trivial rows of $H$ occupy the last non-trivial rows of $M$.

The second corollary on the theorem above demonstrates the idea of the factorization of a multi-companion matrix of order $d$ into products of companion matrices.

Corollary 3.2.3. (Iqelan [11], Corollary 5.3.3) The product $A_d \ldots A_1$ of companion matrices

$$A_i = C[a_{i1}, a_{i2}, \ldots, a_{ip}], \quad i = 1, 2, \ldots, d, \quad d < p,$$

is multi-companion of order $d$. This product is non-singular if and only if $a_{dp}, \ldots, a_{1p} \neq 0$.

The inverse is not true even in the non-singular case. This mean that for any $d$-companion matrix its not always possible to represent it as a product of companion matrices.

### 3.2.1 Multi-companion Representation of a PAR Model

In this section we will discuss another kinds of multivariate stationary process representation of the periodic model, namely the multi-companion (MC).

The $d$-periodic autoregressive model of order $p_1, p_2, \ldots, p_d$

$$X_t = \sum_{i=1}^{p_d} \phi_{t,i} X_{t-i} + \epsilon_t, \quad t = 1, \ldots, d.$$ (3.2.4)

The PAR$_d$(p) model with $p = \max\{p_1, p_2, \ldots, p_d\}$ can be written in a matrix representation form as follows

$$
\begin{pmatrix}
X_t \\
X_{t-1} \\
X_{t-2} \\
\vdots \\
X_{t-p+1}
\end{pmatrix}
= 
\begin{pmatrix}
\phi_{t,1} & \phi_{t,2} & \ldots & \phi_{t,p-1} & \phi_{t,p} \\
1 & 0 & \ldots & 0 & 0 \\
0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & 0
\end{pmatrix}
\begin{pmatrix}
X_{t-1} \\
X_{t-2} \\
X_{t-3} \\
\vdots \\
X_{t-p}
\end{pmatrix} + 
\begin{pmatrix}
\epsilon_t \\
0 \\
0 \\
\vdots \\
0
\end{pmatrix}
$$ (3.2.5)
The Equation (3.2.5) is called a companion form of a PAR\(_d(p)\) model, the first row of the Equation (3.2.5) will give us the Equation (3.2.4) and the other rows give us the identity \(X_{t-j} = X_{t-j}\) for \(j = 1, 2, \ldots, p - 1\). The \(p \times p\) matrix

\[
C_t = \begin{pmatrix}
\phi_{t,1} & \phi_{t,2} & \ldots & \phi_{t,p-1} & \phi_{t,p} \\
1 & 0 & \ldots & 0 & 0 \\
0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & 0
\end{pmatrix}
\] (3.2.6)

is called a companion matrix.

The vector representation of the companion matrices from Equation (3.2.5) can be written as

\[
Z_t = C_t Z_{t-1} + E_t
\] (3.2.7)

where \(E_t\) is uncorrelated to \(Z_s\) for \(s < t\), and \(\{Z_t\}\) is \(p\)-dimensional process defined as a vector

\[
Z_t = \begin{pmatrix}
X_t \\
X_{t-1} \\
X_{t-2} \\
\vdots \\
X_{t-p+1}
\end{pmatrix}, \quad Z_{t-1} = \begin{pmatrix}
X_{t-1} \\
X_{t-2} \\
X_{t-3} \\
\vdots \\
X_{t-p}
\end{pmatrix}
\] (3.2.8)

Proposition 3.2.4. (Iqelan [11], proposition 2.7.1) The \(p\)-dimensional process \(\{Z_t\}\) of a \(d\)-periodic autoregressive process \(\{X_t\}\) has the following property

\[
Z_t = M_t Z_{t-d} + W_t
\] (3.2.9)

where \(M_t = C_t C_{t-1} \ldots C_{t-d+1}\), \(W_t\) is uncorrelated to \(Z_s\) for \(s \leq t - d\), and is given by the formula

\[
W_t = (C_t C_{t-1} \ldots C_{t-d+2}) E_{t-d+1} + \ldots + C_t E_{t-1} + E_t
\]
proof. Since \( Z_{t-1} = C_{t-1}Z_{t-2} + E_{t-1} \). Substitute in Equation (3.2.7), we have

\[
Z_t = C_tZ_{t-1} + E_t
\]

\[
= C_t(C_{t-1}Z_{t-2} + E_{t-1}) + E_t
\]

\[
= C_tC_{t-1}Z_{t-2} + C_tE_{t-1} + E_t
\]

Continuity this process to obtain

\[
Z_t = C_tC_{t-1} \ldots C_{t-d+1}Z_{t-d} + C_tC_{t-1} \ldots C_{t-d+2}E_{t-d+1} + \ldots + C_tE_{t-1} + E_t
\]

\[
= M_tZ_{t-d} + W_t. \quad \Box
\]

From above the matrix \( M_t \) is the product of \( d \)-companion matrices such that

\[
M_d = C_dC_{d-1} \ldots C_1
\]

(3.2.11)

Replacing now \( M_t \) by \( M_d \) in Equation (3.2.9) to obtain:

\[
Z_t = M_dZ_{t-d} + W_t
\]

(3.2.12)

The matrix \( M_d \) is an \((p \times p)\) multi-companion of order \( d \) since it \( d \) companion matrices, each companion matrix in the product in Equation (3.2.11) has the parameters of the corresponding season of the PAR model as its elements in the first row.

The matrix \( M_d \) is an \((p \times p)\) multi-companion has a special structure

\[
M_d = \begin{pmatrix}
    f_{1,1} & f_{1,2} & \ldots & f_{1,p-d} & f_{1,p-d+1} & \ldots & f_{1,p} \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    f_{d,1} & f_{d,2} & \ldots & f_{d,p-d} & f_{d,p-d+1} & \ldots & f_{d,p} \\
    1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\
    0 & 1 & \ldots & 0 & 0 & \ldots & 0 \\
    \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \ldots & 1 & 0 & \ldots & 0 
\end{pmatrix}
\]

(3.2.13)

The first rows of \( M_d \) are arbitrary. The matrix \( M_d \) is called multi-companion matrices of order \( d \), or simply \( d \)-companion (See [2])
**Example 3.2.1.** Consider \( \{X_t\} \) be a 2-periodic autoregressive model of order \((p_1, p_2) = (5, 5)\). We can obtain the companion form of \( \{X_t\} \) as follows

\[
\begin{pmatrix}
X_t \\
X_{t-1} \\
X_{t-2} \\
X_{t-3} \\
X_{t-4}
\end{pmatrix} = \begin{pmatrix}
\phi_{2,1} & \phi_{2,2} & \phi_{2,3} & \phi_{2,4} & \phi_{2,5} \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix} \begin{pmatrix}
\phi_{1,1} & \phi_{1,2} & \phi_{1,3} & \phi_{1,4} & \phi_{1,5} \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix} \begin{pmatrix}
X_{t-2} \\
X_{t-3} \\
X_{t-4} \\
X_{t-5} \\
X_{t-6}
\end{pmatrix} + W_t
\]

The system above can be written as a multivariate autoregressive model

\[
Z_t = M_2 Z_{t-2} + W_t. \tag{3.2.14}
\]

where \( M_2 \) is a 2-companion matrix of dimension \( 5 \times 5 \)

\[
M_2 = \begin{pmatrix}
\phi_{2,1} & \phi_{2,2} & \phi_{2,3} & \phi_{2,4} & \phi_{2,5} \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix} \begin{pmatrix}
\phi_{1,1} & \phi_{1,2} & \phi_{1,3} & \phi_{1,4} & \phi_{1,5} \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix}
\]

The second row of \( M_2 \) are the parameters of the first season of the PAR model.

**Example 3.2.2.** (Numerical example)

Consider \( \{X_t\} \) be a 2-periodic autoregressive model of order \((p_1, p_2) = (4, 4)\). The companion
form of \( \{X_t\} \) can be written as follows
\[
\begin{pmatrix}
X_t \\
X_{t-1} \\
X_{t-2} \\
X_{t-3}
\end{pmatrix}
= \begin{pmatrix}
1.9 & 0.2 & 1.2 & 0.3 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
0.9 & 1.2 & 0.7 & 1.3 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
X_{t-2} \\
X_{t-3} \\
X_{t-4} \\
X_{t-5}
\end{pmatrix}
+ W_t
\]

The system above can be rewritten as a multivariate autoregressive model
\[
Z_t = M_2 Z_{t-2} + W_t. \tag{3.2.15}
\]
where \( M_2 \) is a 2-companion matrix of dimension \( 4 \times 4 \)
\[
M_2 = \begin{pmatrix}
1.9 & 0.2 & 1.2 & 0.3 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
0.9 & 1.2 & 0.7 & 1.3 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}
= \begin{pmatrix}
1.91 & 3.48 & 1.63 & 2.47 \\
0.90 & 1.20 & 0.70 & 1.30 \\
1.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 1.0 & 0.0 & 0.0
\end{pmatrix}
\]

Clearly that The second row of \( M_2 \) are the parameters of the first season of the PAR model.

### 3.3 Vector Representation of a PMA Model

In this section we discuss the relationship between the vector moving average and the periodic moving average, also give an examples to clarify the relationship between them.

**Definition 3.3.1.** [1] Let \( \{X_t\} \) be \( d \)-periodically correlated with the representation of Equation (2.3.2)
\[
X_{s+rd} = \epsilon_{s+rd} + \sum_{i=1}^{q} \theta_{s,i}\epsilon_{s+rd-i}
\]
We can write \( \{X_t\} \) as a \( d \)-variate moving average denoted by \( (\text{VMA}_d(Q)) \)

\[
X_t = \Theta_0 E_t + \sum_{i=1}^{Q} \Theta_i E_{t-i} \tag{3.3.1}
\]

where \( Q = [(q-1)/d] + 1 \), \( \Theta_0 \) is lower diagonal

\[
\Theta_0 = \begin{pmatrix}
1 & 0 & \cdots & 0 & 0 \\
\theta_{2,1} & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\theta_{d,d-1} & \theta_{d,d-2} & \cdots & \theta_{d,1} & 1
\end{pmatrix}. \tag{3.3.2}
\]

and \( \Theta_i's, i = 1, 2, \ldots, Q, \) are \((d \times d)\) matrices, such that

\[
\Theta_i = \begin{pmatrix}
\theta_{1,id} & \theta_{1,id-1} & \cdots & \theta_{1,(i-1)d+2} & \theta_{1,(i-1)d+1} \\
\theta_{2,id+1} & \theta_{2,id} & \cdots & \theta_{2,(i-1)d+3} & \theta_{2,(i-1)d+2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\theta_{d,(i+1)d-1} & \theta_{d,(i+1)d-2} & \cdots & \theta_{d,(id+1)} & \theta_{d,id}
\end{pmatrix}. \tag{3.3.3}
\]

The vector moving average \( \{X_t\} \) of a \( d \)-periodic moving average process is defined as

\[
X_t = (X_{dt-(d-1)}, X_{dt-(d-2)} + \cdots + X_{dt})'. \tag{3.3.4}
\]

Also the \( \{\epsilon_t\} \) of a \( d \)-periodic moving average process is defined as

\[
E_t = (\epsilon_{dt-(d-1)}, \epsilon_{dt-(d-2)} + \cdots + \epsilon_{dt})'. \tag{3.3.5}
\]

For example, we construct some vector of \( E_t \) as follows,

\[
E_1 = \begin{pmatrix}
\epsilon_1 \\
\epsilon_2 \\
\vdots \\
\epsilon_d
\end{pmatrix}, \quad E_2 = \begin{pmatrix}
\epsilon_{d+1} \\
\epsilon_{d+2} \\
\vdots \\
\epsilon_{2d}
\end{pmatrix}, \quad E_2 = \begin{pmatrix}
\epsilon_{2d+1} \\
\epsilon_{2d+2} \\
\vdots \\
\epsilon_{3d}
\end{pmatrix}
\]

and so on.

Also we can obtain the U-form representation of the \( \text{VMA}_d(Q) \) in Equation (3.3.1),

\[
Z_t = \Psi_0 U_t + \sum_{i=1}^{Q} \Psi_i U_{t-i} \tag{3.3.6}
\]
where \( Z_t = (X_{dt} + \cdots + X_{dt-(d-2)} + X_{dt-(d-1)}) \), and \( U_t = (\epsilon_{dt} + \cdots + \epsilon_{dt-(d-2)} + \epsilon_{dt-(d-1)}) \),

which are the reserve order of the observations. The matrices \( \Psi_0 \) and \( \Psi_i \) for \( i = 1, 2, \ldots, Q \) in Equation (3.3.6) defined as follows

\[
\Psi_0 = \begin{pmatrix}
1 & \theta_{d,1} & \cdots & \theta_{d,d-2} & \theta_{d,d-1} \\
0 & 1 & \cdots & \theta_{d-1,d-3} & \theta_{d-1,d-2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & \theta_{2,1} \\
0 & 0 & \cdots & 0 & 1
\end{pmatrix},
\] (3.3.7)

\[
\Psi_i = \begin{pmatrix}
\theta_{d, id} & \theta_{d, id+1} & \cdots & \theta_{1,(i+1)d-1} & \theta_{1,(i+1)d-2} \\
\theta_{d-1, id-1} & \theta_{d-1, id} & \cdots & \theta_{d-i,(i+1)d-3} & \theta_{d-i,(i+1)d-2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\theta_{1,((i-1)d+1)} & \theta_{1,((i-1)d+2)} & \cdots & \theta_{1, id-1} & \theta_{1, id}
\end{pmatrix},
\] (3.3.8)

For \( \Psi_i \)’s, \( i = 1, 2, \ldots, Q \), are \((d \times d)\) matrices, such that

Example 3.3.1. We can construct an example. Consider the 5-periodic moving average model \( \text{PMA}_5(2) \)

\[
X_{1+5r} = \theta_{1,1} \epsilon_{5+5(r-1)} + \theta_{1,2} \epsilon_{4+5(r-1)} + \epsilon_{1+5r}
\]
\[
X_{2+5r} = \theta_{2,1} \epsilon_{1+5r} + \theta_{2,2} \epsilon_{5+5(r-1)} + \epsilon_{2+5r}
\]
\[
X_{3+5r} = \theta_{3,1} \epsilon_{2+5r} + \theta_{3,2} \epsilon_{1+5r} + \epsilon_{3+5r}
\]
\[
X_{4+5r} = \theta_{4,1} \epsilon_{3+5r} + \theta_{4,2} \epsilon_{2+5r} + \epsilon_{4+5r}
\]
\[
X_{5+5r} = \theta_{5,1} \epsilon_{4+5r} + \theta_{5,2} \epsilon_{3+5r} + \epsilon_{5+5r}
\] (3.3.9)

The system above can be written in a matrix representation as follows,

\[
\begin{pmatrix}
X_{1+5r} \\
X_{2+5r} \\
X_{3+5r} \\
X_{4+5r} \\
X_{5+5r}
\end{pmatrix} =
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
\theta_{2,1} & 1 & 0 & 0 & 0 \\
\theta_{3,2} & \theta_{3,1} & 1 & 0 & 0 \\
0 & \theta_{4,2} & \theta_{4,1} & 1 & 0 \\
0 & 0 & \theta_{5,2} & \theta_{5,1} & 1
\end{pmatrix}
\begin{pmatrix}
\epsilon_{1+5r} \\
\epsilon_{2+5r} \\
\epsilon_{3+5r} \\
\epsilon_{4+5r} \\
\epsilon_{5+5r}
\end{pmatrix}
\]
The model (3.3.10) is a vector moving average of order 1, VMA\(_4(1)\). For simplicity written as

\[
X_t = \Theta_1 E_t + \Theta_2 E_{t-1}.
\]  

(3.3.11)

### 3.4 Vector Representation of a PARMA Model

Let \(\{X_t\}\) be a \(d\)-periodic autoregressive moving average model of Equation (2.4.3). We can written Equation (2.4.3) in the \(d\)-variate form

\[
\Phi_0 X_t - \sum_{i=1}^{P} \Phi_i X_{t-i} = \Theta_0 E_t + \sum_{i=1}^{Q} \Theta_i E_{t-i}
\]  

(3.4.1)

where \(P = [(p - 1)/d] + 1\) and \(Q = [(q - 1)/d] + 1\), \(\Phi_0\) defined in Equation (3.1.4), the matrices \(\Phi_i, \ i = 1, \ldots, P\) defined in Equation (3.1.5), \(\Theta_0\) defined in Equation (3.3.2) and the matrices \(\Theta_i, \ i = 1, \ldots, Q\) defined in Equation (3.3.3).

**Example** 3.4.1. Consider the 4-periodic autoregressive moving average model PARMA\(_4(2, 1)\)

\[
\begin{align*}
X_{1+4r} &= \phi_{1,1}X_{s+4r-1} + \phi_{1,2}X_{s+4r-2} + \theta_{1,1}\epsilon_{1+4r} + \epsilon_{1+4r} \\
X_{2+4r} &= \phi_{2,1}X_{s+4r-1} + \phi_{2,2}X_{s+4r-2} + \theta_{2,1}\epsilon_{1+4r} + \epsilon_{1+4r} \\
X_{3+4r} &= \phi_{3,1}X_{s+4r-1} + \phi_{3,2}X_{s+4r-2} + \theta_{3,1}\epsilon_{1+4r} + \epsilon_{1+4r} \\
X_{4+4r} &= \phi_{4,1}X_{s+4r-1} + \phi_{4,2}X_{s+4r-2} + \theta_{4,1}\epsilon_{1+4r} + \epsilon_{1+4r}
\end{align*}
\]  

(3.4.2)

The system can be written as a vector autoregression moving average VARMA\(_d(1, 1)\) as
follows,
\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
-\phi_{2,1} & 1 & 0 & 0 \\
-\phi_{3,2} & -\phi_{3,1} & 1 & 0 \\
0 & -\phi_{4,2} & -\phi_{4,1} & 1
\end{pmatrix}
\begin{pmatrix}
X_{1+4r} \\
X_{2+4r} \\
X_{3+4r} \\
X_{4+4r}
\end{pmatrix}
- 
\begin{pmatrix}
0 & 0 & \phi_{1,2} & \phi_{1,1} \\
0 & 0 & 0 & \phi_{2,2} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
X_{1+4(r-1)} \\
X_{2+4(r-1)} \\
X_{3+4(r-1)} \\
X_{4+4(r-1)}
\end{pmatrix}
\]
= 
\begin{pmatrix}
1 & 0 & 0 & 0 \\
\theta_{2,1} & 1 & 0 & 0 \\
0 & \theta_{3,1} & 1 & 0 \\
0 & 0 & \theta_{4,1} & 1
\end{pmatrix}
\begin{pmatrix}
\epsilon_{1+4r} \\
\epsilon_{2+4r} \\
\epsilon_{3+4r} \\
\epsilon_{4+4r}
\end{pmatrix}
+ 
\begin{pmatrix}
0 & 0 & 0 & \theta_{1,1} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\epsilon_{1+4(r-1)} \\
\epsilon_{2+4(r-1)} \\
\epsilon_{3+4(r-1)} \\
\epsilon_{4+4(r-1)}
\end{pmatrix}
\tag{3.4.3}
\]
or
\[
\Phi_0 X_t - \Phi_1 X_{t-1} = \Theta_0 E_t + \Theta_1 E_{t-1}
\tag{3.4.4}
\]

3.5 Causality, Invertibility and Stationarity of a PARMA model

Causality and invertibility of a PARMA model has been defined in Previous section (2.3). But now we want to be defined more generally by Iqelan [11]

**Definition 3.5.1.** [11] A PARMA model for the time series \(X_{s+rd}\) given by the equation (2.4.4) is said to be causal, or a causal function of \(X_{s+rd}\), if for each season \(s = 1, 2, \ldots, d\), there exist sequences of \(\{\psi_{s,i}\}\), such that
\[
X_{s+rd} = \sum_{i=0}^{\infty} \psi_{s+rd-i} \quad s = 1, \ldots, d.
\tag{3.5.1}
\]
where \(\psi_{s,0} = 1\) and \(\sum_{i=1}^{\infty} |\psi_{s,i}| < \infty\) for each season \(s\).

Causality is equivalent to the condition
\[
\det(\Phi_0 - \sum_{i=1}^{P} \Phi_i z^i) \neq 0 \text{ for all } |z| \leq 1.
\tag{3.5.2}
\]
where \(\Phi_0\) and \(\Phi_i\) are the matrices given in Equations (3.1.4) and (3.1.5).
Definition 3.5.2. [11] A periodic autoregressive moving-average PARMA model for the time series $X_{s+rd}$ given by the Equation (2.3.3) is said to be invertible, if for each season $s = 1, 2, \ldots, d$, there exist sequences of $\{\pi_{s,i}\}$, such that

$$X_{s+rd} = \sum_{i=0}^{\infty} \psi_{s,i} \epsilon_s + rd - i \quad s = 1, \ldots, d.$$  \hspace{1cm} (3.5.3)

where $\pi_{s,0} = 1$ and $\sum_{i=1}^{\infty} |\pi_{s,i}| < \infty$ for each season $s$.

Invertibility is equivalent to the condition

$$\det(\Theta_0 - \sum_{i=1}^{P} \Theta_i z^i) \neq 0 \text{ for all } |z| \leq 1.$$  \hspace{1cm} (3.5.4)

where $\Theta_0$ and $\Theta_i$ are the matrices given in Equations (3.3.2) and (3.3.3).

Definition 3.5.3. The PARMA model in Equation (2.4.1) can be equivalently written as a special case of the general multivariate ARMA model. Since the stationarity and invertibility conditions for the general multivariate ARMA model are available, also known for the PARMA model.

3.6 Examples

Example 3.6.1. Consider the following PAR$_4(1)$ with

$$\phi_{1,1} = 0.9, \ \phi_{2,1} = -0.6, \ \phi_{3,1} = 0.5, \ \phi_{4,1} = 1.2$$

For $s = 1, 2, 3, 4$, where

$$X_{1+4r} = 0.9X_{1+4r-1} + \epsilon_{1+4r}$$
$$X_{2+4r} = -0.6X_{2+4r-1} + \epsilon_{2+4r}$$
$$X_{3+4r} = 0.5X_{3+4r-1} + \epsilon_{3+4r}$$
$$X_{1+4r} = 1.2X_{4+4r-1} + \epsilon_{4+4r}$$  \hspace{1cm} (3.6.1)

This model can be written as the 4-dimensional vector autoregressive (VAR(1)) model.

$$\Phi_0 X_t = \Phi_1 X_{t-1} + \epsilon_t$$  \hspace{1cm} (3.6.2)
where \( \Theta_0 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0.6 & 1 & 0 & 0 \\
0 & -0.5 & 1 & 0 \\
0 & 0 & -0.12 & 1
\end{pmatrix} \), and \( \Theta_1 = \begin{pmatrix}
0 & 0 & 0 & 0.9 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \).

The causality and stationarity is equivalent to the condition

\[ |\Phi_0 + \Phi_1 z| \neq 0 \text{ for all } |z| \leq 1 \]

Since

\[
|\Phi_0 + \Phi_1 z| = \begin{vmatrix}
1 & 0 & 0 & -0.9z \\
0.6 & 1 & 0 & 0 \\
0 & -0.5 & 1 & 0 \\
0 & 0 & -0.12 & 1
\end{vmatrix} = 1 + 0.324z = 0
\]

which has the root \( z = |-3.086| > 1 \), the VAR(1) model in (3.6.2) is stationary.

**Example 3.6.2.** Recall example (2.2.2). Consider a periodic PAR\(_3\)(1, 3, 2) model

\[
\begin{align*}
X_{1+3r} &= 0.5X_{1+3r-1} + \epsilon_{1+3r} \\
X_{2+3r} &= X_{2+3r-1} + 0.3X_{2+3r-2} + 0.1X_{2+3r-3} + \epsilon_{2+3r} \\
X_{3+3r} &= 0.9X_{3+3r-1} + 0.5X_{3+3r-2} + \epsilon_{3+3r}
\end{align*}
\]  
(3.6.3)

Now if \( r=1 \)

\[
\begin{align*}
X_4 &= 0.5X_3 + \epsilon_4 \\
X_5 - X_4 &= 0.3X_3 + 0.1X_2 + \epsilon_5 \\
X_6 - 0.9X_5 - 0.5X_4 &= \epsilon_6
\end{align*}
\]

if \( r=2 \)

\[
\begin{align*}
X_7 &= 0.5X_6 + \epsilon_7 \\
X_8 - X_7 &= 0.3X_6 + 0.1X_5 + \epsilon_8 \\
X_9 - 0.9X_8 - 0.5X_7 &= \epsilon_9
\end{align*}
\]
The system above in (3.6.3) can be written as a VAR model as follows

\[
\begin{pmatrix}
1 & 0 & 0 \\
-1 & 1 & 0 \\
-0.5 & -0.9 & 1 \\
\end{pmatrix}
\begin{pmatrix}
X_{1+3r} \\
X_{2+3r} \\
X_{3+3r} \\
\end{pmatrix}
= \begin{pmatrix}
0 & 0 & 0.5 \\
0 & 1 & 0.3 \\
0 & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
X_{1+3(r-1)} \\
X_{2+3(r-1)} \\
X_{3+3(r-1)} \\
\end{pmatrix}
+ \begin{pmatrix}
\epsilon_{1+3r} \\
\epsilon_{2+3r} \\
\epsilon_{3+3r} \\
\end{pmatrix}
\] (3.6.4)

For simplicity the system above written as a vector autoregressive model as follows

\[
\Phi_0 X_t = \Phi_1 X_{t-1} + \epsilon_t
\] (3.6.5)

where \(\Phi_0 = \begin{pmatrix}
1 & 0 & 0 \\
-1 & 1 & 0 \\
-0.5 & -0.9 & 1 \\
\end{pmatrix}\), and \(\Phi_1 = \begin{pmatrix}
0 & 0 & 0.5 \\
0 & 1 & 0.3 \\
0 & 0 & 0 \\
\end{pmatrix}\) (3.6.6)

The causality and stationarity for Equation (3.6.5) is equivalent the condition

\[|\Phi_0 - \Phi_1 z| \neq 0 \text{ for all } |z| \leq 1\]

Since

\[
|\Phi_0 - \Phi_1 z| = \begin{vmatrix}
1 & 0 & 0 & | & 0 & 0 & 0.5 & | & 0 & 0 & 0.5 \\
-1 & 1 & 0 & | & 0 & 0 & 1 & | & 0 & 0 & 0.3 \\
-0.5 & -0.9 & 1 & | & 0 & 0 & 0 & | & 0 & 0 & 0 \\
\end{vmatrix}
\]

\[
= \begin{vmatrix}
1 & 0 & 0.5z \\
-1 & 1 & -0.1z & -0.3z \\
-0.5 & -0.9 & 1 \\
\end{vmatrix}
\]

\[
= 1 - 0.37z + 0.5z(1.4 - 0.05z)
\]

\[
= 1 + 0.33z - 0.025z^2.
\]

\[|z_1| = |1 - 4.713| > 1, \ z_2 = |1 - 8.486| > 1, \text{ the VAR(1) model (3.6.5) is causal.}\]

**Example** 3.6.3. Consider an example of an invertible periodic moving average model PMA₄(1),

\[
\begin{align*}
X_{1+4r} &= \epsilon_{1+4r} + 0.5\epsilon_{1+4(r-1)} \\
X_{2+4r} &= \epsilon_{2+4r} + 0.3\epsilon_{2+4(r-1)} \\
X_{3+4r} &= \epsilon_{3+4r} - 0.3\epsilon_{3+4(r-1)} \\
X_{4+4r} &= \epsilon_{4+4r} - 0.5\epsilon_{4+4(r-1)}
\end{align*}
\] (3.6.7)
To find an invertible property convert the system above to a VAR model and calculating the root.

\[
\begin{pmatrix}
X_{1+4r} \\
X_{2+4r} \\
X_{3+4r} \\
X_{4+4r}
\end{pmatrix} =
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0.3 & 1 & 0 & 0 \\
0 & -0.3 & 1 & 0 \\
0 & 0 & -0.5 & 1
\end{pmatrix}
\begin{pmatrix}
\epsilon_{1+4r} \\
\epsilon_{2+4r} \\
\epsilon_{3+4r} \\
\epsilon_{4+4r}
\end{pmatrix} +
\begin{pmatrix}
0 & 0 & 0 & 0.5 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\epsilon_{1+4(r-1)} \\
\epsilon_{2+4(r-1)} \\
\epsilon_{3+4(r-1)} \\
\epsilon_{4+4(r-1)}
\end{pmatrix}
\]

where \( \Theta_0 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0.3 & 1 & 0 & 0 \\
0 & -0.3 & 1 & 0 \\
0 & 0 & -0.5 & 1
\end{pmatrix} \), and \( \Theta_1 = \begin{pmatrix}
0 & 0 & 0 & 0.5 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \).

Invertibility is equivalent to the condition

\[ |\Theta_0 + \Theta_1 z| \neq 0 \text{ for all } |z| \leq 1 \]

Since the characteristic polynomial is

\[
\begin{vmatrix}
1 & 0 & 0 & 0.5z \\
0.3 & 1 & 0 & 0 \\
0 & -0.3 & 1 & 0 \\
0 & 0 & -0.5 & 1
\end{vmatrix} = 1 + 0.075z
\]

which has the root \( z = | - \frac{1}{0.075} | = | -13.33 | > 1 \), mean that the model is an invertible.

**Example 3.6.4.** Consider the periodic autoregressive model PAR\(_4\)(2,1)

\[ X_{s+rd} = \phi_{s,1}X_{s+rd-1} - \phi_{s,2}X_{s+rd-2} = \epsilon_{s+rd} + \epsilon_{s+rd-1} \]  \( (3.6.8) \)

with coefficients

<table>
<thead>
<tr>
<th>( s )</th>
<th>( \phi_{s,1} )</th>
<th>( \phi_{s,2} )</th>
<th>( \theta_{s,1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.8</td>
<td>0.1</td>
<td>0.5</td>
</tr>
<tr>
<td>2</td>
<td>0.2</td>
<td>0.7</td>
<td>0.3</td>
</tr>
<tr>
<td>3</td>
<td>-0.2</td>
<td>0.7</td>
<td>-0.3</td>
</tr>
<tr>
<td>4</td>
<td>-0.8</td>
<td>0.1</td>
<td>-0.5</td>
</tr>
</tbody>
</table>
It can be verified that these parameters induce a causal model and an invertible model.

The model in Equation (3.6.8) can be written in a scalar matrix as follows

\[
\begin{align*}
X_{1+rd} &= 0.8X_{s+rd-1} + 0.1X_{s+rd-2} + 0.5\epsilon_{1+rd} + \epsilon_{1+rd} \\
X_{2+rd} &= 0.2X_{s+rd-1} + 0.7X_{s+rd-2} + 0.3\epsilon_{1+rd} + \epsilon_{1+rd} \\
X_{3+rd} &= -0.2X_{s+rd-1} + 0.7X_{s+rd-2} - 0.3\epsilon_{1+rd} + \epsilon_{1+rd} \\
X_{4+rd} &= -0.8X_{s+rd-1} + 0.1X_{s+rd-2} - 0.5\epsilon_{1+rd} + \epsilon_{1+rd}
\end{align*}
\] (3.6.9)

The VARMA representation is

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
-0.2 & 1 & 0 & 0 \\
-0.7 & 0.2 & 1 & 0 \\
0 & -0.1 & 0.8 & 1
\end{pmatrix}
\begin{pmatrix}
X_{1+4r} \\
X_{2+4r} \\
X_{3+4r} \\
X_{4+4r}
\end{pmatrix}
= 
\begin{pmatrix}
0 & 0 & 0.1 & 0.8 \\
0 & 0 & 0 & 0.7 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
X_{1+4(r-1)} \\
X_{2+4(r-1)} \\
X_{3+4(r-1)} \\
X_{4+4(r-1)}
\end{pmatrix}
+ 
\begin{pmatrix}
0 & 0 & 0 & 0.5 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\epsilon_{1+4r} \\
\epsilon_{2+4r} \\
\epsilon_{3+4r} \\
\epsilon_{4+4r}
\end{pmatrix}
\] (3.6.10)

Simplicity can be written as follows

\[
\Phi_0 X_t - \Phi_1 X_{t-1} = \Theta_0 \epsilon_t + \Theta_1 \epsilon_{t-1}
\]

Now we can check the causality. The characteristic polynomial is

\[
|\Phi_0 - \Phi_1 z| = \begin{vmatrix}
1 & 0 & -0.1z & -0.8z \\
-0.2 & 1 & 0 & -0.7z \\
-0.7 & 0.2 & 1 & 0 \\
0 & -0.1 & 0.8 & 1
\end{vmatrix} = 1 + 0.5224z + 0.0049z^2
\]

The roots are |z_1| = |−1.9| > 1 and |z_2| = |−104.7| > 1, the model in (3.6.9) has a causal.
Chapter 4

Identification and Estimation of PAR Models

There are three stages for fitting a model to a time series data, these stages involve model identification, parameters estimation and model diagnostics.

Model identification is the identification of a possible model based on an available realization, i.e., determining the type of the model with appropriate orders. Parameter estimation is the estimation of the model parameters. At this stage, the orders of the model may be further reduced by significance tests on parameters. Diagnostic checks are directed to the residuals of the fitted model to verify the assumptions on the white noise terms such as independence and normality. If verifications fail, model identification stage is to be repeated leading to a new possible model.

PeACF and PePACF functions can be used for model identification.

The identification of PAR model from the PeACF and PePACF plots is difficult and requires experience, Sakai (1982) in [18] had extended the identification to the periodic case and employed the cut-off of the periodic partial autocorrelation (PePACF) for the identification of the periodic autoregressive (PAR) model.
4.1 Periodic Autocorrelation Function

In this section we introduce the definition of the periodic autocorrelation function and some important properties.

**Definition 4.1.1.** The periodic autocovariance function (PeACVF) of \( \{X_{s+rd}\} \) is defined as

\[
\gamma_s(k) = \text{Cov}(X_{s+rd}, X_{s+rd-k}) = E[(X_{s+rd} - \mu_s)(X_{s+rd-k} - \mu_{s-k})]
\]  

(4.1.1)

For lag \( k \) and season \( s = 1, 2, \ldots, d \), \( \mu_s \) and \( \mu_{s-k} \) are means for season \( s \) and \( s-k \), respectively.

When \( k = 0 \) the periodic autocovariance is the variance of the time series at season \( s \), namely

\[
\gamma_s(0) = E(X_{s+rd} - \mu_s)^2.
\]  

(4.1.2)

The periodic autocovariance function of \( \gamma_s(k) \) has property is not symmetry in \( k \), that is,

\[
\gamma_s(-k) = \gamma_{s+k}(k)
\]  

(4.1.3)

for \( s = 1, 2, \ldots, d \). This property can be proved as follows

\[
\gamma_s(-k) = \text{Cov}(X_{s+rd}, X_{s+rd-(-k)}) = \text{Cov}(X_{s+rd+k}, X_{s+rd+k-(-k)}) = \gamma_{s+k}(k)
\]

**Definition 4.1.2.** The (theoretical) periodic autocorrelation function of \( \{X_{s+rd}\} \) is defined as

\[
\rho_s(k) = E \left[ \left( \frac{X_{s+rd} - \mu_s}{\sqrt{\gamma_s(0)}} \right) \left( \frac{X_{s+rd-k} - \mu_{s-k}}{\sqrt{\gamma_{s-k}(0)}} \right) \right]
\]

\[
\rho_s(k) = \frac{\gamma_s(k)}{\sqrt{\gamma_s(0)\gamma_{s-k}(0)}} \quad k \geq 0.
\]  

(4.1.4)

The possible value of \( \rho_s(k) \) range from -1 to 1, and \( \rho_s(k) \) has a magnitude of unity at lag zero.
4.1.1 Sample Periodic Autocorrelation Function

The theoretical periodic autocorrelation functions are useful for describing the properties of certain models, most of the analysis must be performed using sample data. In this section, we introduce the sample autocovariance function and the sample autocorrelation function. These sample functions are used to estimate the theoretical ones in the case that the model is unknown.

Consider a sample \( X_1, X_2, \ldots, X_n \) from a periodic time series \( \{X_t\} \) with period \( d \). Let \( N = nd \), represent the number of years of data. The time index \( t = s + rd \), satisfies \( 1 \leq s + rd \leq Nd \), where \( r = 0, 1, \ldots, N \) and \( s = 1, 2, \ldots, d \). For the \( s^{th} \) season, the theoretical mean \( \mu_s \) is estimated by sample mean function defined as

**Definition 4.1.3.** The sample mean function is defined as

\[
\hat{\mu}_s = \frac{1}{N} \sum_{r=0}^{N-1} X_{s+rd} \quad \text{for season } s. \tag{4.1.5}
\]

The theoretical periodic autocovariance function, \( \gamma_s(k) \), in Equation (4.1.1) is estimated by the sample periodic autocovariance function.

**Definition 4.1.4.** The sample periodic autocovariance function for season \( s \) and lag \( k \) is defined as

\[
\hat{\gamma}_s(k) = \frac{1}{N} \sum_{r=0}^{N-1} (X_{s+rd} - \hat{\mu}_s)(X_{s+rd-k} - \hat{\mu}_{s-k}) \tag{4.1.6}
\]

When the lag is equal to zero, one can obtains the estimate of the variance of the observation in season \( s \), which is given as

\[
\hat{\gamma}_s(0) = \frac{1}{N} \sum_{r=0}^{N-1} (X_{s+rd} - \hat{\mu}_s)^2, \quad s = 1, 2, \ldots, d. \tag{4.1.7}
\]

Analogously to Equation (4.1.4), the sample periodic autocorrelation function is defined as follows.

**Definition 4.1.5.** The sample periodic autocorrelation function for season \( s \) and lag \( k \) is defined as

\[
\hat{\rho}_s(k) = \frac{\hat{\mu}_s(k)}{\sqrt{\hat{\gamma}_s(0)\hat{\gamma}_{s-k}(0)}} \tag{4.1.8}
\]
4.1.2 Periodic Partial Autocorrelation Function

In this subsection we will discuss one more properties of periodic processes, namely the periodic partial autocorrelation function, (PePACF) which will help us to distinguish one model from another. We say that the partial autocorrelation function for an AR($p$) model cuts off after lag exceeds the order of model. Like the periodic autocorrelation function it also depends only on the second order properties of the process.

The partial autocorrelation (PACF) of a $d$-periodically correlated time series $\{X_t\}$, with finite second order moment, and $E(X_t) = 0$ for any $t \in \mathbb{Z}$. At first we introduce some important notations and basics.

**Definition 4.1.6.** The conditional expectation of $X_t$ given $X_s$, $s < t$ is denoted by

$$E(X_t|X_s, s < t)$$

and defined as follows

$$E(X_t|X_s, s < t) = a_1X_{t-1} + a_2X_{t-2} + \cdots$$

The conditional expectation is replaced by the linear projection. The best linear predictor (BLP) of $X_t$, given $X_s$, $s < t$ is denoted by $\text{pred}(X_t|X_s, s < t)$. (See Iqelan [11])

Consider the discrete time series $X_t$,

$$X_{t-k}, X_{t-k+1}, X_{t-k+2}, \ldots, X_t-2, X_{t-1}, X_t, \ldots$$

For any $k > 1$, The forward linear predictor of $X_t$ from $X_{t-k+1}$ to $X_{t-1}$ is denoted by $\hat{X}_t^{k-1}$.

We can define $\hat{X}_t^{k-1}$ as follows

$$\hat{X}_t^{k-1} = \text{pred}(X_t|X_{t-1}, \ldots, X_{t-k+1})$$

$$= \sum_{i=1}^{k-1} a_{k-1,i}X_{t-k+i} \quad (4.1.9)$$

where $a_{k-1,i}, i = 1, \ldots, k - 1$ are the forward coefficients of the prediction.
The forward residual, $\epsilon_{t}^{k-1}$, of $X_t$ is defined as

\[
\epsilon_{t}^{k-1} = X_t - \text{pred}(X_t|X_{t-1}, \ldots, X_{t-k+1}) = X_t - \hat{X}_{t}^{k-1}
\]

and

\[
\text{Var}(\epsilon_{t}^{k-1}) = \sigma_t^f(k - 1)
\]

where $\sigma_t^f(k - 1)$ is the variance of the forward residuals and is called predictor error.

By reversing the time series index we obtain the backward linear predictor of $X_{t-h}$ from $X_{t-h+1}$ to $X_{t-1}$ denoted by $\tilde{X}_{t-k}^{k-1}$ and defined by

\[
\tilde{X}_{t-k}^{k-1} = \text{pred}(X_{t-k}|X_{t-k+1}, \ldots, X_{t-1}) = \sum_{i=1}^{k-1} b_{k-1,i}X_{t-i}
\]

where $b_{k-1,i}, i = 1, \ldots, k - 1$ are the backward coefficients of the prediction.

The forward residual, $\zeta_{t-k}^{k-1}$, of $X_{t-k}$ is defined as

\[
\zeta_{t-k}^{k-1} = X_{t-k} - \tilde{X}_{t-k}^{k-1}
\]

and

\[
\text{Var}(\zeta_{t-k}^{k-1}) = \sigma_t^b(k - 1)
\]

**Definition 4.1.7.** [11] The partial autocorrelation function $\beta(t, t - k)$ is the partial correlation coefficient between $X_t$ and $X_{t-k}$ after removing the dependence on the intermediate $X$'s, $\{X_{t-1}, \ldots, X_{t-(k-1)}\}$. Thus it is defined on $\mathbb{Z} \times \mathbb{Z}$ by

\[
\beta(t, t - k) = \begin{cases} 
\text{Var}(X_t), & \text{if } k = 0 \\
\text{Corr}(X_t, X_{t-1}), & \text{if } k = 1 \\
\text{Corr}(X_t - \hat{X}_t^{k-1}, X_{t-1} - \tilde{X}_{t-k}^{k-1}), & \text{if } k \geq 2.
\end{cases}
\]
From the above definition, the PePACF function \( \beta(t, t - k) \)

\[
\beta(t, t - k) = \text{Corr}(X_t - \hat{X}^{k-1}_t, X_{t-1} - \hat{X}^{k-1}_{t-k})
= \text{Corr}(\epsilon^{k-1}_t, \zeta^{k-1}_{t-k})
= \frac{\text{Cov}(\epsilon^{k-1}_t, \zeta^{k-1}_{t-k})}{\sqrt{\text{Var}(\epsilon^{k-1}_t)}\sqrt{\text{Var}(\zeta^{k-1}_{t-k})}}
= \frac{\text{Cov}(\epsilon^{k-1}_t, \zeta^{k-1}_{t-k})}{\sqrt{\sigma^t_f(k-1)}\sqrt{\sigma^t_b(k-1)}}
\tag{4.1.15}
\]

### 4.1.3 Periodic Yule-Walker Equations

One can find the theoretical Periodic Yule-Walker Equations for a PAR model, by multiplying Equation (2.2.2)

\[
X_{s+rd} = \phi_{s,1}X_{s+rd-1} + \cdots + \phi_{s,p_s}X_{s+rd-p_s} + \epsilon_{s+rd}
\]

by \( X_{s+rd-k} \) and take expected values to obtain

\[
E(X_{s+rd}X_{s+rd-k}) = \phi_{s,1}E(X_{s+rd-1}X_{s+rd-k}) + \cdots + \phi_{s,p_s}E(X_{s+rd-p_s}X_{s+rd-k}) + E(\epsilon_{s+rd}X_{s+rd-k})
\]

\[
\gamma_s(k) = \gamma_{s-1}(k-1) + \cdots + \phi_{s,p_s}\gamma_{s-p_s}(k-p_s) + 0
\tag{4.1.16}
\]

for \( k \geq 0 \) and \( s = 1, 2, \ldots, d \).

The last term \( E(\epsilon_{s+rd-k}\epsilon_{s+rd}) \) is zero for \( k > 0 \) because the innovation \( \{\epsilon_{s+rd}\} \) are uncorrelated with \( X_{s+rd-k} \) for all \( k > 0 \). Hence the Equation (4.1.16) becomes

\[
\gamma_s(k) = \phi_{s,1}\gamma_{s-1}(k-1) + \cdots + \phi_{s,p_s}\gamma_{s-p_s}(k-p_s)
\tag{4.1.17}
\]

Now by using the periodic AR operator in Equation(2.2.3), we can rewrite Equation (4.1.17) for season \( s \) as follows,

\[
\phi_s(B)\gamma_s(k) = 0 \quad \text{for } k > 0
\tag{4.1.18}
\]

where \( B \) operators on the subscript \( k \) and the subscripts in \( \gamma_s(k) \).
One can find the theoretical Yule-Walker equations for a PAR model. By setting \( k = 1,2,\ldots,p_s \), in Equation (4.1.17), one can obtain the periodic Yule-Walker equations for season \( s \) as:

\[
\begin{align*}
\gamma_s(1) &= \phi_{s,1}\gamma_{s-1}(0) + \phi_{s,2}\gamma_{s-2}(1) + \cdots + \phi_{s,p_s}\gamma_{s-p_s}(p_s - 1) \\
\gamma_s(2) &= \phi_{s,2}\gamma_{s-1}(1) + \phi_{s,2}\gamma_{s-2}(0) + \cdots + \phi_{s,p_s}\gamma_{s-p_s}(p_s - 2) \\
& \quad \vdots \quad \vdots \quad \vdots \\
\gamma_s(p_s) &= \phi_{s,1}\gamma_{s-1}(p_s - 1) + \phi_{s,2}\gamma_{s-2}(p_s - 2) + \cdots + \phi_{s,p_s}\gamma_{s-p_s}(0)
\end{align*}
\]

(4.1.19)

The periodic Yule-Walker equations in (4.1.19) can be written in a matrix form,

\[
\Gamma_{s,p_s} \Phi_s = \gamma_{s,p_s},
\]

where

\[
\Gamma_{s,p_s} = \begin{pmatrix}
\gamma_{s-1}(0) & \gamma_{s-2}(1) & \cdots & \gamma_{s-p_s}(p_s - 1) \\
\gamma_{s-1}(1) & \gamma_{s-2}(0) & \cdots & \gamma_{s-p_s}(p_s - 2) \\
& \vdots & \ddots & \vdots \\
\gamma_{s-1}(p_s - 1) & \gamma_{s-2}(p_s - 2) & \cdots & \gamma_{s-p_s}(0)
\end{pmatrix},
\]

\[
\Phi_s = \begin{pmatrix}
\phi_{s,1} \\
\phi_{s,2} \\
\vdots \\
\phi_{s,p_s}
\end{pmatrix}, \quad \gamma_{s,p_s} = \begin{pmatrix}
\gamma_s(1) \\
\gamma_s(2) \\
\vdots \\
\gamma_s(p_s)
\end{pmatrix}.
\]

**Theorem 4.1.1.** [11] If \( \{X_{s+rd}\} \) is a causal periodic autoregression, then for season \( s=1, \ldots, d \) the autocovariance function \( \gamma_s(k) \) of \( \{X_{s+rd}\} \) satisfied the equations

\[
\gamma_s(k) = \sum_{i=1}^{p_s} \phi_{s,i}\gamma_{s-i}(k - i) \quad k = 1,2,\ldots
\]

(4.1.21)

By setting \( k = 0 \) in Equation (4.1.16), and using non-symmetry property (4.1.3) we have

\[
\gamma_s(0) = \phi_{s,1}\gamma_{s}(1) + \cdots + \phi_{s,p_s}\gamma_{s}(p_s) + E(X_{s+rd}\epsilon_{s+rd})
\]

\[
\gamma_s(0) = \sum_{i=1}^{p_s} \phi_{s,i}\gamma_{s}(i) + \sigma^2_s
\]

(4.1.22)

\[
\sigma^2_s = \gamma_s(0) - \sum_{i=1}^{p_s} \phi_{s,i}\gamma_{s}(i)
\]
4.2 PAR Estimation

There are many methods used for estimation process. These methods are the method of moments, the least square method and the maximum likelihood method. For the estimation of PAR model, the method of moments is sufficient and suitable but for the other methods like PMA or PARMA it may be use the least squares method or the maximum likelihood method. Here we are going to discuss the method of moments and the other methods are discussed in the appendix [A.1].

The method of moments

The method of moments is frequently one of the easiest, if not the most efficient, methods for obtaining parameter estimates. The method consists of equating sample moments to corresponding theoretical moments and solving the resulting equations to obtain estimates of any unknown parameters. The method of moments good for all AR models but inefficient for models contains MA terms. Let $\Phi_s = (\phi_{s,1}, \phi_{s,2}, \ldots, \phi_{s,p_s})$ denote the vector autoregressive parameters for period s, and $\hat{\Phi}_s = (\hat{\phi}_{s,1}, \hat{\phi}_{s,2}, \ldots, \hat{\phi}_{s,p_s})$, stand for the vector estimated parameters. To compute the estimated $\hat{\Phi}_s$ for each $s = 1, 2, \ldots, d$, we solve Yule-Walker estimates for the AR parameters for season s in the PAR model in Equation (2.2.2), simply replace each theoretical ACVF, $\gamma_s(k)$ with their corresponding sample autocovariance $\hat{\gamma}_s(k)$ in Equation (4.1.19), the periodic Yule-Walker equations for each season s

\[
\hat{\gamma}_s(1) = \hat{\phi}_{s,1}\hat{\gamma}_{s-1}(0) + \hat{\phi}_{s,2}\hat{\gamma}_{s-2}(1) + \cdots + \hat{\phi}_{s,p_s}\hat{\gamma}_{s-p_s}(p_s - 1)
\]

\[
\hat{\gamma}_s(2) = \hat{\phi}_{s,1}\hat{\gamma}_{s-1}(1) + \hat{\phi}_{s,2}\hat{\gamma}_{s-2}(2) + \cdots + \hat{\phi}_{s,p_s}\hat{\gamma}_{s-p_s}(p_s - 2)
\]

\[\vdots \quad \vdots \quad \vdots\]

\[
\hat{\gamma}_s(p_s) = \hat{\phi}_{s,1}\hat{\gamma}_{s-1}(p_s - 1) + \hat{\phi}_{s,2}\hat{\gamma}_{s-2}(p_s - 2) + \cdots + \hat{\phi}_{s,p_s}\hat{\gamma}_{s-p_s}(0)
\]

The Equations in (4.2.1) can be written in a matrix form as follows

\[
\hat{\Phi}_s = \hat{\Gamma}_s^{-1}\hat{\gamma}_{s,p_s}
\]
where,

\[
\hat{\Phi}_s = \begin{pmatrix}
\hat{\phi}_{s,1} \\
\hat{\phi}_{s,2} \\
\vdots \\
\hat{\phi}_{s,p_s}
\end{pmatrix}, \quad \hat{\gamma}_{s,p_s} = \begin{pmatrix}
\hat{\gamma}_s(1) \\
\hat{\gamma}_s(2) \\
\vdots \\
\hat{\gamma}_s(p_s)
\end{pmatrix}, \quad \text{and}
\]

\[
\hat{\Gamma}_s = \begin{pmatrix}
\hat{\gamma}_{s-1}(0) & \hat{\gamma}_{s-2}(1) & \cdots & \hat{\gamma}_{s-p_s}(p_s-1) \\
\hat{\gamma}_{s-1}(1) & \hat{\gamma}_{s-2}(0) & \cdots & \hat{\gamma}_{s-p_s}(p_s-2) \\
\vdots & \vdots & \ddots & \vdots \\
\hat{\gamma}_{s-1}(p_s-1) & \hat{\gamma}_{s-2}(p_s-2) & \cdots & \hat{\gamma}_{s-p_s}(0)
\end{pmatrix}.
\]

The estimated residuals denoted by \( \hat{\epsilon} \) are calculated from model (2.2.1) as

\[
\hat{\epsilon}_t = X_t - \sum_{i=1}^{p_s} \hat{\phi}_{s,i} X_{t-i}
\]

The residual variances may be estimated by

\[
\hat{\sigma}^2_s = \hat{\gamma}_s(0) - \hat{\phi}_{s,1} \hat{\gamma}_s(1) - \cdots - \hat{\phi}_{s,p_s} \hat{\gamma}_s(p_s), \quad s = 1, 2, \ldots, d.
\]

**Theorem 4.2.1.** [11] The distribution of the Yule-walker estimators \( \hat{\Phi}_s \) of the model parameters of a causal PAR\((p_1, \cdots, p_d) \) process in Equation (2.2.5) is asymptotically (as \( n \to \infty \)) normal, with mean zero and covariances \( \frac{1}{N} \Gamma_s^{-1} \)

\[
\sqrt{N}(\hat{\Phi}_s - \Phi_s) \Rightarrow \mathcal{N}(0, \frac{1}{N} \Gamma_s^{-1}),
\]

where

\[
I_s = \frac{1}{\sigma^2_s} \gamma_s(i-j))_{i,j=1,2,\ldots,p_s}
\]

\[
= \begin{pmatrix}
\gamma_s(0) & \gamma_s(1) & \gamma_s(2) & \cdots & \gamma_s(p_s-1) \\
\gamma_{s+1}(1) & \gamma_s(0) & \gamma_s(1) & \cdots & \gamma_s(p_s-2) \\
\gamma_{s+1}(2) & \gamma_{s+1}(1) & \gamma_s(0) & \cdots & \gamma_s(p_s-3) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\gamma_{s+p_s-1}(p_s-1) & \gamma_{s+p_s-2}(p_s-2) & \gamma_{s+p_s-3}(p_s-3) & \cdots & \gamma_s(0)
\end{pmatrix}
\]

56
In practice, an estimate $\hat{I}_s$ of $I_s$ is simply obtained by replacing each $\gamma_s(k)$ in Equation in (4.1.1) by its estimate $\hat{\gamma}_s(k)$ in Equation (4.1.6) and $\sigma^2_s$ by its estimate in $\hat{\sigma}^2_s$ in Equation (4.2.4). That is, $\hat{I}_s = \frac{1}{\hat{\sigma}^2_s} \hat{\gamma}_s(i - j))_{i,j=1,2,...,p_s}$.

4.2.1 Residual Analysis

After a model has been identified, the parameters of that model are estimated. Then it is recommended to check whether or not the model adequately represents the data.

If any inadequacies are detected, a new model must be identified and the cycle of identification, estimation, and diagnostic checking repeated. The estimated innovations or residuals are assumed to be independent, normally distributed and have constant variance. If the residuals are white, we need to estimate the residual autocorrelation function (see Mcleod [13])

Let $\hat{\epsilon}_{s+rd}$ denote the residuals from a fitted PAR model. Let $\hat{r}_{s,k}(\hat{\epsilon}_{s+rd})$ denote the residual autocorrelation for lag $k$ and season $s$ defined as

$$\hat{r}_{s,k}(\hat{\epsilon}_{s+rd}) = \frac{1}{N} \sum_r \hat{\epsilon}_{s+rd} \hat{\epsilon}_{s+rd-k} \hat{\sigma}_s \hat{\sigma}_{s-k}, \quad k = 1, 2, \ldots \quad (4.2.5)$$

where $\hat{\sigma}^2_s$ is the estimated value of the residual variance for season $s$.

Portmanteau tests

As a single statistics for an overall for whiteness of the residuals, the portmanteau statistics for season $s$ given by

$$Q_{s,L} = N \sum_{k=1}^{L} \hat{r}_{s,k}^2(\hat{\epsilon}_{s+rd}) \quad (4.2.6)$$

This statistic is $\chi^2$ distributed with $L - p_s$ degrees of freedom for large $N$. A significantly large value of $Q_{s,L}$ shows insufficiency of the model for season $s$. Hence, we reject the null hypotheses that the data in season $s$ are white if the calculated value of $Q_{s,L}$ in (4.2.6) is larger than the tabulated $\chi^2$ value at a specified significance level and degree of freedom.

By Mcleod [13], a modified portmanteau test statistic improves the small sample prop-
properties. In particular, the following exact holds for the periodic correlations of white noise

\[
\text{Var}(r_{s,k}(\epsilon_{s+rd})) = \begin{cases} 
\frac{N-k}{N(N+2)} , & k = 0 \text{ mod } d \\
\frac{N-(k+s+d)}{N^2} , & \text{otherwise.}
\end{cases}
\]  

(4.2.7)

where \([\cdot]\) denotes the integer part and \(r_{s,k}(\epsilon_{s+rd})\) is defined in (4.2.5) by replacing the residual \(\hat{\epsilon}_{s+rd}\) by the theoretical innovations \(\epsilon_{s+rd}\). The modified portmanteau statistic is defined as

\[
\hat{Q}_{s,L} = \sum_{k=1}^{L} \frac{\hat{r}_{s,k}^2(\hat{\epsilon}_{s+rd})}{\sqrt{\text{Var}(r_{s,k}(\epsilon_{s+rd}))}}
\]  

(4.2.8)

The \(\chi^2\) distributed with \(L - p_s\) degree of freedom. It is practical to use the same value of the lag \(L\) in Equation (4.2.8) for all seasons \(s, s = 1, 2, \ldots, d\) but it could be chosen \(L\) to be different from one season to the other (Iqelan [11]). The modified statistic in Equation (4.2.8) reduces to that proposed for a nonseasonal ARMA model. So one can demonstrate that

\[
E(\hat{Q}_{s,L}) = L - p_s
\]  

(4.2.9)

and

\[
E(Q_{s,L}) = N \sum_{k=1}^{L} \text{Var}(r_{s,k}(\epsilon_{s+rd})) - p_s
\]  

(4.2.10)

The portmanteau test statistics for different season \(s, s = 1, 2, \ldots, d\), are independent. Consequently, if the residual from one season to the other are white, then the portmanteau test statistic in Equation (4.2.8) is given by

\[
\hat{Q}_L = \sum_{s=1}^{d} Q_{s,L}
\]  

(4.2.11)

where \(\hat{Q}\) is \(\chi^2\) distributed on \(\sum_{s=1}^{d}(L - p_s)\) degree of freedom (see Hiple and Mcleod [9]).

There are many tests methods used The Bayesian information criterion (BIC) and Akaike's (1973) Information Criterion (AIC). Here we are going to discuss the Portmanteau tests and the other methods are discussed in the appendix [A.1].
4.3 Simulation

In this section we conducted some simulation studies to investigate the usefulness in practice of the theoretical results stated earlier and to demonstrate the identification procedure. The R programming language was used in this simulation study in conjunction with the \texttt{pear} package, see [15].

\textbf{Example 4.3.1.} Let us consider the PAR$_4(1)$ of Example 3.6.1, section 3.6

\begin{align*}
X_{1+4r} &= 0.9X_{1+4r-1} + \epsilon_{1+4r} \\
X_{2+4r} &= -0.6X_{2+4r-1} + \epsilon_{2+4r} \\
X_{3+4r} &= 0.5X_{3+4r-1} + \epsilon_{3+4r} \\
X_{1+4r} &= 1.2X_{4+4r-1} + \epsilon_{4+4r} \\
\end{align*}

(4.3.1)

As we have shown earlier, the model (4.3.1) is periodic stationary autoregression.

To process our simulation study an \texttt{R} code was written to generate 1000 independent replicates. Each of these replicates has sizes $N=100$, $N=250$, and $N=500$. In all cases, the white noise terms are independently and normally distributed with mean zero and variance equal to one, see [11]. In Figure 4.1 one of simulated series (of 2000 observations) which corresponds to $N=500$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.1.png}
\caption{Line graph of 2000 observations generated from PAR$_4(1)$ model}
\end{figure}
For each replicate, the sample PeACF $\hat{\rho}_s(h)$ and the sample PePACF $\hat{\beta}_s(h)$ for $s = 1, 2, 3, 4$, and $h = 1, 2, \ldots, 10$ are computed.

By comparing the theoretical values of the autocorrelations, $\rho_s(h)$, with the average estimated values, $\hat{\rho}_s(h)$, one can see that they are quite close. Also it is of interest to see that as $N$ (the sample size) increases the standard deviations decreases. This is true for all seasons as can be seen in Tables A.1, A.2, A.3, A.4.

Table A.5 produces the computations of the average sample periodic partial autocorrelations (PePACF), $\hat{\beta}_s(h)$. As we expect, the values of $\hat{\beta}_s(h)$ show a cut-off after the first lag for each season.

SaKai [18] showed that, for PAR models, if the correct order is $p_s$ for season $s$, $s = 1, \ldots, d$, the estimated variance of the sample PePACF equal to $\frac{1}{N}$ for lags greater than $p_s$. Based on Table A.6, which represents the variance of the average estimated partial autocorrelation, namely $s^2_{\hat{\beta}_s,h}$.

The relative frequencies (R.F.) of $\hat{\beta}_{s,h}$, $h > 2$, which are outside the 95% confidence intervals $(-1.96/\sqrt{N}, 1.96/\sqrt{N})$ for the 3 sample sizes, are given in the last row of Table A.5. In all cases, this table shows that the relative frequencies are close to the nominal 5% value especially with the large sample size. The relative frequency is the percentage of PePACF values falling outside the corresponding confidence interval.

Thus, the PePACF given in Figure 4.2

![Figure 4.2: The PePACF for the 3 sample size data sets (N=100, N=250, N=500).](image)
**Diagnostic checking**

After fitting the model, we should check whether the model is appropriate. As we explained before, the primary tool for model diagnostic checking is residual analysis. To do such analysis. We choose the BIC criterion to fit each of the simulated data. The determination of an appropriate PAR$_4$(1) model to represent a generated periodic stationary time series needs to estimate the parameters $\phi_{s,1}$, and the white noise variance $\sigma^2_s$ ($s = 1, 2, 3, 4$). By using the Yule-Walker equation (4.2.2), we obtain the parameter estimate $\hat{\phi}_{s,1}$ for each season $s = 1, 2, 3, 4$.

Table A.7, shows the results of the average values of the estimates of the PAR$_4$(1) model, $\bar{\hat{\phi}}_s$, together with the empirical standard deviations, namely $s_{\hat{\phi}_s}$, for realizations where $N = 100$, $N = 250$, and $N = 500$. Also shown are the theoretical (true) values alongside each estimates. When comparing the theoretical values with the values of the estimates, one can see that they are fairly close for all seasons.

The sample 95% confidence bounds is normally given by $\bar{\hat{\phi}}_s \pm 1.96s_{\hat{\phi}_s}$ for each $\hat{\phi}_s$, $s = 1, 2, 3, 4$. We found the values of the confidence limits corresponding to $\hat{\phi}_s$, $s = 1, 2, 3, 4$, for the case $N = 500$. These are shown in Table A.8, seen all values $(\phi_1, \phi_2, \phi_3, \phi_4)$ lie well within the 95% confidence interval.

**Residuals Analysis**

The first step in residual analysis is to compute the residuals $\hat{\epsilon}_{s+rd}$ given the model selected in the estimation stage and observed data using Equation

$$\hat{\epsilon}_{s+rd} = X_{s+rd} - \sum_{i=1}^{p_s} \hat{\phi}_{s,i} X_{s+rd-i}$$

The first step in examining the residuals is to plot them as a function of time to see if it seems to be random noise. In Figure 4.3, one of the residuals is displayed. The graph gives no indication of deviation from random white noise. It also shows that most of the residuals are in the range (-0.5,0.5).
Figure 4.3: Line graph of the residuals, $\hat{\epsilon}_{s+rd}$ of one of the fitted PAR$_4$(1) models.

We obtained the residual autocorrelations $\hat{r}_{s+rd}(\hat{\epsilon}_{s+rd})$ for each season $s$, $s = 1, 2, 3, 4$ and for lags up to $h = 15$ to save space (Note that the calculations continue for lags up to $L = \frac{N}{4} = 125$ in case of sample size $N = 500$).

The sample estimated values of the variances, i.e., $s^2_{\hat{r}_{s,h}(\hat{\epsilon}_{s+rd})}$ are calculated and given in Table A.9. It is observed that these values coincide with the theoretical values of the variances, $\sigma^2_{\hat{r}_{s,h}(\hat{\epsilon}_{s+rd})}$ (see Table A.10).

Figure 4.4: Average sample PeACF values of the residuals, $\overline{\hat{r}}_{s,h}(\hat{\epsilon}_{s+rd})$ for different $N=100$, 250, 500).

Figure 4.4 display a plot of the sample PeACF values of residuals. All sample autocorrelations fall inside the 95% confidence bounds indicating the residuals appear to be random.
We can look at the first \( L = 5, 10, \ldots, \frac{N}{d} \) correlation values together using the portmanteau test, based on the statistic \( \tilde{Q}_{s,L} \). Table A.11, Table A.12, and Table A.13 report the values of the mean portmanteau test \( \tilde{Q}_{s,L} \) of the 3 sample size data sets \( N = 100 \), \( N = 250 \), and \( N = 500 \) respectively.

The empirical significance level of a nominal value 5%, which estimated by counting the number of times that the statistic \( \tilde{Q}_{s,L} \) exceed the corresponding chi-squared value at 5% level of significance with \( L - p_s \) degree of freedom, \( \chi^2_{0.05}(L - p_s) \). If the calculated value of \( \tilde{Q}_{s,L} \) is larger than the tabulated \( \chi^2_{0.05}(L - 1) \) value, then the null hypothesis that there is no correlations in the residuals will be rejected.
Conclusion

In this thesis, we have studied the periodically correlated (PC) time series. The basics of PARMA process are explained. Also been studying the relationships between the multivariate autoregressive moving average (VARMA) models and periodic autoregressive moving average. We showed that any PARMA model can be written in a vector ARMA model. Through this property, it is given that the periodic stationarity conditions, and the invertibility conditions of a univariate PARMA process. We defined the vector of season representation of the periodically correlated process and the various representation are explained, namely the L-form, the U-form and the I-form. In addition we defined a new representation, the multi-companion (MC) representation, of the PAR process. The well known Box-Jenkins techniques for the identification of ordinary ARMA processes, utilizing the cut-off property of the autocorrelation function (ACF) and the partial autocorrelation function (PACF) are generalized to a periodic type identification. Similarly for identification of PARMA processes, the periodic autocorrelation function (PeACF) and the partial autocorrelation function (PePACF) played the same role. We considered the model estimation by solving the Yule-Walker equations. Also, we conducted diagnostic checking through residuals of the fitted model. All three stages of model development, identification, estimation and diagnostic checking are illustrated in a simulation study.
Bibliography


Appendix A

Basic Concept

A.1 PARAMETER ESTIMATION

Least Squares Estimation
Because the method of moments is unsatisfactory for many models, we must consider other methods of estimation. We begin with least squares. For autoregressive models, the ideas are quite straightforward. At this point, we introduce a possibly nonzero mean, into our stationary models and treat it as another parameter to be estimated by least squares.

Maximum Likelihood Estimation
The advantage of the method of maximum likelihood is that all of the information in the data is used rather than just the first and second moments, as is the case with least squares. Another advantage is that many large-sample results are known under very general conditions. One disadvantage is that we must for the first time work specifically with the joint probability density function of the process. See Cryer and Chan [6].

The Ljung-Box Test
In addition to looking at residual correlations at individual lags, it is useful to have a test that takes into account their magnitudes as a group. For example, it may be that most of the residual autocorrelations are moderate, some even close to their critical values, but, taken together, they seem excessive. Box and Pierce (1970) proposed the statistic. (Cryer
and chan [6])

\[ Q = N(\hat{r}_1^2 + \cdots + \hat{r}_L^2) \]  

(A.1.1)

to address this possibility. They showed that if the correct ARMA\((p, q)\) model is estimated, then, for large \(N\), \(Q\) has an approximate chi-square distribution with \(L - p - q\) degrees of freedom. Fitting an erroneous model would tend to inflate \(Q\). Thus, a general portmanteau test would reject the ARMA\((p, q)\) model if the observed value of \(Q\) exceeded an appropriate critical value in a chi-square distribution with \(L - p - q\) degrees of freedom.

The chi-square distribution for \(Q\) is based on a limit theorem as , but Ljung and Box (1978) discovered that even for \(N = 100\), the approximation is not satisfactory. By modifying the \(Q\) statistic slightly, they defined a test statistic whose null distribution is much closer to chi-square for typical sample sizes. The modified Box-Pierce, or Ljung-Box, statistic is given by

\[ Q_\star = N(N + 2) \sum_{k=1}^{L} \frac{\hat{r}_k^2}{N - k} \]  

(A.1.2)

The Bayesian information criterion (BIC) and Akaike (1973) Information Criterion (AIC) may be factored to obtain a separate criterion for each period. Thus

\[ BIC = \sum_{s=1}^{d} BIC_s \]  

(A.1.3)

where

\[ BIC_s = \frac{-n}{2} \ln \hat{\sigma}_s^2 + \ln(n) p_s \]  

(A.1.4)

A number of other approaches to model specification have been proposed since Box and Jenkins seminal work. One of the most studied is Akaike (1973) Information Criterion (AIC). This criterion says to select the model that minimizes

\[ AIC = 2 \ln(\text{maximum likelihood}) + 2k \]  

(A.1.5)

where \(k = p + q + 1\) if the model contains an intercept or constant term and \(k = p + q\) otherwise.
By adding another nonstochastic penalty term to the AIC, resulting in the corrected AIC, denoted by $AIC_c$ and defined by the formula

$$AIC_c = AIC + \frac{2(k + 1)(k + 2)}{N - k - 2} \quad (A.1.6)$$

Here $N$ is the (effective) sample size and again $k$ is the total number of parameters as above excluding the noise variance.

**Model Diagnostics**

After fitting the model, we should check whether the model is appropriate. Models that are more general than the proposed model but that contain the proposed model as a special case. See Cryer and Chan [6]

**Plots of the Residuals**

Our first diagnostic check is to inspect a plot of the residuals over time. If the model is adequate, we expect the plot to suggest a rectangular scatter around a zero horizontal level with no trends whatsoever. Also one can draw the 95% confidence interval for each season. If the seasonal residuals are white noise, they should fall within the 95% confidence limits (see Hipel and McLeod [9]).

**Autocorrelation of the Residuals**

To check on the independence of the noise terms in the model, we consider the sample autocorrelation function of the residuals, we know that for true white noise and large $N$, the sample autocorrelations are approximately uncorrelated and normally distributed with zero means and variance $1/N$.

**Normality of the Residuals**

Quantile-quantile plots are an effective tool for assessing normality. Here we apply them to residuals.

A quantile-quantile plot of the residuals

Shapiro-Wilk normality test.
Overfitting the model

Our second basic diagnostic tool is that of overfitting. After specifying and fitting what we believe to be an adequate model, we fit a slightly more general model; that is, a model close by that contains the original model as a special case.

keep in the principle of promising, that is, the model used should require the smallest number of parameters that will adequately represent the time series.
Table A.1: Average sample PeACF of the PAR$_1(1)$ model with theoretical values.

<table>
<thead>
<tr>
<th>Lag (h)</th>
<th>True value of $\rho_{1,h}$</th>
<th>Average of $\hat{\rho}_{1,h}$</th>
<th>Average of $\hat{\rho}_{1,h}$</th>
<th>Average of $\hat{\rho}_{1,h}$</th>
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<td>0.850</td>
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<tr>
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Table A.2: Average sample PeACF of the PAR$_1$(1) model with theoretical values.

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Table A.3: Average sample PeACF of the PAR$_1(1)$ model with theoretical values.

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<th>Lag (h)</th>
<th>N=100</th>
<th>N=250</th>
<th>N=500</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>True value of $\rho_{3,h}$</td>
<td>average of $\hat{\rho}_{3,h}$</td>
<td>average of $\hat{\rho}_{3,h}$</td>
</tr>
<tr>
<td>1</td>
<td>0.600</td>
<td>0.598</td>
<td>0.605</td>
</tr>
<tr>
<td>2</td>
<td>-0.455</td>
<td>-0.450</td>
<td>-0.455</td>
</tr>
<tr>
<td>3</td>
<td>-0.390</td>
<td>-0.386</td>
<td>-0.388</td>
</tr>
<tr>
<td>4</td>
<td>-0.325</td>
<td>-0.321</td>
<td>-0.324</td>
</tr>
<tr>
<td>5</td>
<td>-0.199</td>
<td>-0.195</td>
<td>-0.197</td>
</tr>
<tr>
<td>6</td>
<td>0.147</td>
<td>0.142</td>
<td>0.145</td>
</tr>
<tr>
<td>7</td>
<td>0.124</td>
<td>0.123</td>
<td>0.120</td>
</tr>
<tr>
<td>8</td>
<td>0.010</td>
<td>0.095</td>
<td>0.099</td>
</tr>
<tr>
<td>9</td>
<td>0.065</td>
<td>0.058</td>
<td>0.064</td>
</tr>
<tr>
<td>10</td>
<td>-0.050</td>
<td>-0.047</td>
<td>-0.046</td>
</tr>
</tbody>
</table>
Table A.4: Average sample PeACF of the PAR$_1(1)$ model with theoretical values.

<table>
<thead>
<tr>
<th>Lag (h)</th>
<th>N=100</th>
<th>N=250</th>
<th>N=500</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>True value</td>
<td>average of</td>
<td>average of</td>
</tr>
<tr>
<td>of $\rho_{4,h}$</td>
<td>$\rho_{4,h}$</td>
<td>$\rho_{4,h}$</td>
<td>$\rho_{4,h}$</td>
</tr>
<tr>
<td>1</td>
<td>0.835</td>
<td>0.833</td>
<td>0.833</td>
</tr>
<tr>
<td>2</td>
<td>0.501</td>
<td>0.498</td>
<td>0.505</td>
</tr>
<tr>
<td>3</td>
<td>−0.380</td>
<td>−0.373</td>
<td>−0.379</td>
</tr>
<tr>
<td>4</td>
<td>−0.325</td>
<td>−0.322</td>
<td>−0.324</td>
</tr>
<tr>
<td>5</td>
<td>−0.273</td>
<td>−0.267</td>
<td>−0.270</td>
</tr>
<tr>
<td>6</td>
<td>−0.164</td>
<td>−0.162</td>
<td>−0.165</td>
</tr>
<tr>
<td>7</td>
<td>0.119</td>
<td>0.117</td>
<td>0.121</td>
</tr>
<tr>
<td>8</td>
<td>0.105</td>
<td>0.103</td>
<td>0.100</td>
</tr>
<tr>
<td>9</td>
<td>0.086</td>
<td>0.084</td>
<td>0.084</td>
</tr>
<tr>
<td>10</td>
<td>0.052</td>
<td>0.048</td>
<td>0.054</td>
</tr>
</tbody>
</table>
Table A.5: Average sample PePACF, $\hat{\beta}_{s,h}$, of the PAR$_4$(1) model for lag up to $h = 10$

<table>
<thead>
<tr>
<th>Lag</th>
<th>$N = 100$</th>
<th></th>
<th>$N = 250$</th>
<th></th>
<th>$N = 500$</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>s=1</td>
<td>s=2</td>
<td>s=3</td>
<td>s=4</td>
<td>s=1</td>
<td>s=2</td>
</tr>
<tr>
<td>1</td>
<td>0.846</td>
<td>-0.749</td>
<td>0.598</td>
<td>0.833</td>
<td>0.850</td>
<td>-0.752</td>
</tr>
<tr>
<td>2</td>
<td>-0.000</td>
<td>-0.003</td>
<td>-0.002</td>
<td>-0.004</td>
<td>0.001</td>
<td>-0.002</td>
</tr>
<tr>
<td>3</td>
<td>0.001</td>
<td>-0.006</td>
<td>-0.010</td>
<td>0.005</td>
<td>0.000</td>
<td>0.002</td>
</tr>
<tr>
<td>4</td>
<td>0.004</td>
<td>-0.003</td>
<td>0.006</td>
<td>-0.007</td>
<td>0.002</td>
<td>-0.002</td>
</tr>
<tr>
<td>5</td>
<td>0.004</td>
<td>-0.008</td>
<td>0.001</td>
<td>0.004</td>
<td>0.003</td>
<td>-0.004</td>
</tr>
<tr>
<td>6</td>
<td>0.005</td>
<td>-0.003</td>
<td>-0.008</td>
<td>0.005</td>
<td>0.002</td>
<td>-0.003</td>
</tr>
<tr>
<td>7</td>
<td>0.001</td>
<td>-0.009</td>
<td>-0.000</td>
<td>-0.001</td>
<td>0.002</td>
<td>-0.002</td>
</tr>
<tr>
<td>8</td>
<td>-0.011</td>
<td>-0.017</td>
<td>-0.016</td>
<td>0.001</td>
<td>-0.002</td>
<td>-0.001</td>
</tr>
<tr>
<td>9</td>
<td>0.018</td>
<td>-0.021</td>
<td>0.011</td>
<td>0.027</td>
<td>0.005</td>
<td>-0.009</td>
</tr>
<tr>
<td>10</td>
<td>-0.002</td>
<td>-0.005</td>
<td>-0.006</td>
<td>-0.007</td>
<td>0.003</td>
<td>0.001</td>
</tr>
<tr>
<td>%RF</td>
<td>4.3</td>
<td>4.7</td>
<td>4.9</td>
<td>5.1</td>
<td>4.6</td>
<td>4.8</td>
</tr>
</tbody>
</table>
Table A.6: The estimated variance of the sample PePACF, $s^2_{\beta_{s,h}}$ of the PAR$_4$(1) model for lag up to $h = 10$.

<table>
<thead>
<tr>
<th>Lag</th>
<th>$N = 100$</th>
<th>$N = 250$</th>
<th>$N = 500$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>s=1</td>
<td>s=2</td>
<td>s=3</td>
</tr>
<tr>
<td>1</td>
<td>0.0008</td>
<td>0.0012</td>
<td>0.0029</td>
</tr>
<tr>
<td>2</td>
<td>0.0099</td>
<td>0.0105</td>
<td>0.0101</td>
</tr>
<tr>
<td>3</td>
<td>0.0102</td>
<td>0.0104</td>
<td>0.0095</td>
</tr>
<tr>
<td>4</td>
<td>0.0102</td>
<td>0.0098</td>
<td>0.0104</td>
</tr>
<tr>
<td>5</td>
<td>0.0094</td>
<td>0.0101</td>
<td>0.0107</td>
</tr>
<tr>
<td>6</td>
<td>0.0096</td>
<td>0.0094</td>
<td>0.0094</td>
</tr>
<tr>
<td>7</td>
<td>0.0103</td>
<td>0.0104</td>
<td>0.0104</td>
</tr>
<tr>
<td>8</td>
<td>0.0104</td>
<td>0.0100</td>
<td>0.0111</td>
</tr>
<tr>
<td>9</td>
<td>0.0102</td>
<td>0.0104</td>
<td>0.0109</td>
</tr>
<tr>
<td>10</td>
<td>0.0101</td>
<td>0.0100</td>
<td>0.0105</td>
</tr>
</tbody>
</table>
Table A.7: Average sample Yule-Walker Estimators of PAR₄(1) with the theoretical values and the standard deviations.

<table>
<thead>
<tr>
<th>Season</th>
<th>True value of $\phi_s$</th>
<th>Average of $\hat{\phi}_s$</th>
<th>Std. Dev. of $\hat{\phi}_s$</th>
<th>Average of $\hat{\phi}$</th>
<th>Std. Dev. of $\hat{\phi}$</th>
<th>Average of $s_{\hat{\phi}}$</th>
<th>Std. Dev. of $s_{\hat{\phi}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.9</td>
<td>0.89205</td>
<td>0.00155</td>
<td>0.89632</td>
<td>0.00091</td>
<td>0.89830</td>
<td>0.00064</td>
</tr>
<tr>
<td>2</td>
<td>-0.6</td>
<td>-0.59894</td>
<td>0.00142</td>
<td>-0.59977</td>
<td>0.00111</td>
<td>-0.59951</td>
<td>0.00065</td>
</tr>
<tr>
<td>3</td>
<td>0.5</td>
<td>0.49432</td>
<td>0.00175</td>
<td>0.50114</td>
<td>0.00093</td>
<td>0.50037</td>
<td>0.00073</td>
</tr>
<tr>
<td>4</td>
<td>1.2</td>
<td>1.20596</td>
<td>0.00197</td>
<td>1.20127</td>
<td>0.00126</td>
<td>1.20036</td>
<td>0.00090</td>
</tr>
</tbody>
</table>
Table A.8: The confidence intervals corresponding to the parameters $\phi_1, \phi_2, \phi_3, \phi_3$ to the case $N=500$.

<table>
<thead>
<tr>
<th>Season s</th>
<th>$\phi_s$</th>
<th>$\hat{\phi}_s$</th>
<th>$s_{\hat{\phi}_s}$</th>
<th>Confidence Limits</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.9</td>
<td>0.89830</td>
<td>0.00064</td>
<td>(0.89705,0.89955)</td>
</tr>
<tr>
<td>2</td>
<td>-0.6</td>
<td>-0.59951</td>
<td>0.00065</td>
<td>(-0.60078,-0.59824)</td>
</tr>
<tr>
<td>3</td>
<td>0.5</td>
<td>0.50037</td>
<td>0.00073</td>
<td>(0.49894,0.50180)</td>
</tr>
<tr>
<td>4</td>
<td>1.2</td>
<td>1.20036</td>
<td>0.00090</td>
<td>(1.19860,1.20212)</td>
</tr>
</tbody>
</table>
Table A.9: The estimated variance of the sample PePACF of the residual, \( s^2_{\hat{r}_{h}(\hat{\epsilon}_{s+r})} \) of the PAR\(_4\)(1) model for lag up to \( h = 10 \).

<table>
<thead>
<tr>
<th>Lag</th>
<th>( N = 100 )</th>
<th>( N = 250 )</th>
<th>( N = 500 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( s=1 ) s=2 s=3 s=4</td>
<td>( s=1 ) s=2 s=3 s=4</td>
<td>( s=1 ) s=2 s=3 s=4</td>
</tr>
<tr>
<td>1</td>
<td>0.007 0.008 0.008 0.005</td>
<td>0.003 0.003 0.003 0.002</td>
<td>0.002 0.002 0.002 0.001</td>
</tr>
<tr>
<td>2</td>
<td>0.007 0.007 0.007 0.009</td>
<td>0.003 0.003 0.003 0.003</td>
<td>0.001 0.002 0.002 0.002</td>
</tr>
<tr>
<td>3</td>
<td>0.009 0.007 0.009 0.008</td>
<td>0.003 0.003 0.004 0.004</td>
<td>0.002 0.002 0.002 0.002</td>
</tr>
<tr>
<td>4</td>
<td>0.009 0.009 0.009 0.008</td>
<td>0.004 0.003 0.003 0.003</td>
<td>0.002 0.002 0.002 0.002</td>
</tr>
<tr>
<td>5</td>
<td>0.009 0.009 0.009 0.009</td>
<td>0.004 0.004 0.004 0.003</td>
<td>0.002 0.002 0.002 0.002</td>
</tr>
<tr>
<td>6</td>
<td>0.009 0.010 0.010 0.009</td>
<td>0.003 0.004 0.004 0.004</td>
<td>0.002 0.002 0.002 0.002</td>
</tr>
<tr>
<td>7</td>
<td>0.010 0.009 0.009 0.009</td>
<td>0.004 0.004 0.004 0.004</td>
<td>0.002 0.002 0.002 0.002</td>
</tr>
<tr>
<td>8</td>
<td>0.009 0.010 0.009 0.009</td>
<td>0.003 0.004 0.004 0.004</td>
<td>0.002 0.002 0.002 0.002</td>
</tr>
<tr>
<td>9</td>
<td>0.010 0.009 0.010 0.009</td>
<td>0.004 0.004 0.004 0.004</td>
<td>0.002 0.002 0.002 0.002</td>
</tr>
<tr>
<td>10</td>
<td>0.009 0.010 0.010 0.010</td>
<td>0.004 0.004 0.004 0.004</td>
<td>0.002 0.002 0.002 0.002</td>
</tr>
<tr>
<td>11</td>
<td>0.010 0.008 0.010 0.010</td>
<td>0.004 0.004 0.004 0.004</td>
<td>0.002 0.002 0.002 0.002</td>
</tr>
<tr>
<td>12</td>
<td>0.009 0.010 0.009 0.010</td>
<td>0.004 0.004 0.004 0.004</td>
<td>0.002 0.002 0.002 0.002</td>
</tr>
<tr>
<td>13</td>
<td>0.010 0.010 0.010 0.009</td>
<td>0.004 0.004 0.004 0.004</td>
<td>0.002 0.002 0.002 0.002</td>
</tr>
<tr>
<td>14</td>
<td>0.010 0.010 0.010 0.010</td>
<td>0.004 0.004 0.004 0.004</td>
<td>0.002 0.002 0.002 0.002</td>
</tr>
<tr>
<td>15</td>
<td>0.010 0.010 0.010 0.009</td>
<td>0.004 0.004 0.004 0.004</td>
<td>0.002 0.002 0.002 0.002</td>
</tr>
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</table>
Table A.10: The theoretical value of variance of the residuals, $\sigma^2_{r,s,h}(\epsilon_{s+rd})$ for lags up to $h=15$

<table>
<thead>
<tr>
<th>Lag</th>
<th>$N = 100$</th>
<th>$N = 250$</th>
<th>$N = 500$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>s=1</td>
<td>s=2</td>
<td>s=3</td>
</tr>
<tr>
<td>1</td>
<td>0.0099</td>
<td>0.0099</td>
<td>0.0099</td>
</tr>
<tr>
<td>2</td>
<td>0.0098</td>
<td>0.0099</td>
<td>0.0099</td>
</tr>
<tr>
<td>3</td>
<td>0.0098</td>
<td>0.0098</td>
<td>0.0099</td>
</tr>
<tr>
<td>4</td>
<td>0.0098</td>
<td>0.0098</td>
<td>0.0098</td>
</tr>
<tr>
<td>5</td>
<td>0.0098</td>
<td>0.0098</td>
<td>0.0098</td>
</tr>
<tr>
<td>6</td>
<td>0.0097</td>
<td>0.0098</td>
<td>0.0098</td>
</tr>
<tr>
<td>7</td>
<td>0.0097</td>
<td>0.0097</td>
<td>0.0098</td>
</tr>
<tr>
<td>8</td>
<td>0.0097</td>
<td>0.0097</td>
<td>0.0097</td>
</tr>
<tr>
<td>9</td>
<td>0.0097</td>
<td>0.0097</td>
<td>0.0097</td>
</tr>
<tr>
<td>10</td>
<td>0.0096</td>
<td>0.0097</td>
<td>0.0097</td>
</tr>
<tr>
<td>11</td>
<td>0.0096</td>
<td>0.0096</td>
<td>0.0097</td>
</tr>
<tr>
<td>12</td>
<td>0.0096</td>
<td>0.0096</td>
<td>0.0096</td>
</tr>
<tr>
<td>13</td>
<td>0.0096</td>
<td>0.0096</td>
<td>0.0096</td>
</tr>
<tr>
<td>14</td>
<td>0.0095</td>
<td>0.0096</td>
<td>0.0096</td>
</tr>
<tr>
<td>15</td>
<td>0.0095</td>
<td>0.0095</td>
<td>0.0096</td>
</tr>
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</table>
Table A.11: The portmanteau test, $\tilde{Q}_{s,L}$, using a sample size $N=100$.

<table>
<thead>
<tr>
<th>L</th>
<th>$\tilde{Q}_{1,L}$ Mean</th>
<th>var</th>
<th>Sig. Level</th>
<th>$\tilde{Q}_{2,L}$ Mean</th>
<th>var</th>
<th>Sig. Level</th>
<th>$\tilde{Q}_{3,L}$ Mean</th>
<th>var</th>
<th>Sig. Level</th>
<th>$\tilde{Q}_{4,L}$ Mean</th>
<th>var</th>
<th>Sig. Level</th>
<th>$\chi^2$ at 5% level of sig. with L-1 d.f</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>4.236</td>
<td>7.878</td>
<td>0.061</td>
<td>4.458</td>
<td>9.389</td>
<td>0.064</td>
<td>4.145</td>
<td>7.959</td>
<td>0.051</td>
<td>4.107</td>
<td>8.791</td>
<td>0.054</td>
<td>9.488</td>
</tr>
<tr>
<td>10</td>
<td>8.100</td>
<td>16.713</td>
<td>0.060</td>
<td>9.520</td>
<td>21.837</td>
<td>0.049</td>
<td>8.977</td>
<td>17.743</td>
<td>0.044</td>
<td>9.084</td>
<td>19.071</td>
<td>0.036</td>
<td>16.919</td>
</tr>
<tr>
<td>15</td>
<td>14.09</td>
<td>24.405</td>
<td>0.052</td>
<td>14.632</td>
<td>34.057</td>
<td>0.055</td>
<td>13.861</td>
<td>28.329</td>
<td>0.048</td>
<td>14.189</td>
<td>732.059</td>
<td>0.037</td>
<td>23.685</td>
</tr>
<tr>
<td>20</td>
<td>19.217</td>
<td>37.077</td>
<td>0.054</td>
<td>19.558</td>
<td>43.404</td>
<td>0.059</td>
<td>18.847</td>
<td>36.842</td>
<td>0.046</td>
<td>19.262</td>
<td>45.104</td>
<td>0.036</td>
<td>30.144</td>
</tr>
<tr>
<td>25</td>
<td>24.325</td>
<td>45.519</td>
<td>0.056</td>
<td>24.815</td>
<td>56.153</td>
<td>0.066</td>
<td>24.141</td>
<td>45.132</td>
<td>0.044</td>
<td>24.337</td>
<td>57.234</td>
<td>0.042</td>
<td>36.415</td>
</tr>
</tbody>
</table>
Table A.12: The portmanteau test, $\tilde{Q}_{s,L}$, using a sample size $N=250$

<table>
<thead>
<tr>
<th>L</th>
<th>$\tilde{Q}_{1,L}$</th>
<th>$\tilde{Q}_{2,L}$</th>
<th>$\tilde{Q}_{3,L}$</th>
<th>$\tilde{Q}_{4,L}$</th>
<th>$\chi^2$ at 5%</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>var</td>
<td>Sig.</td>
<td>Mean</td>
<td>var</td>
</tr>
<tr>
<td>5</td>
<td>4.105</td>
<td>7.547</td>
<td>0.052</td>
<td>4.011</td>
<td>6.888</td>
</tr>
<tr>
<td>10</td>
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Table A.13: The portmanteau test, $\tilde{Q}_{s,L}$, using a sample size $N=500$

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